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Braid groups and their applications

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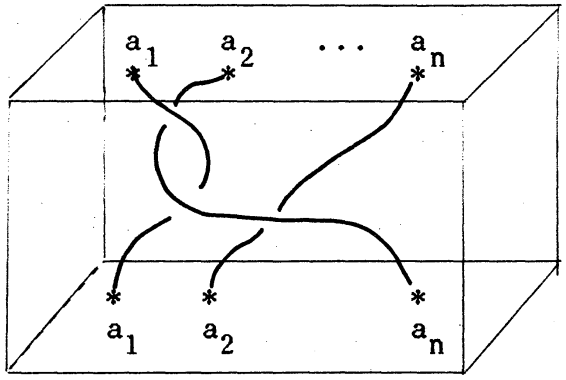
INTRODUCTION. The purpose of this note is to present a brief introduction on recent advances concerning Artin's braid groups principally from a point of view of complex analytic geometry. In Section 1, we recall the definition and basic properties of the braid groups. In Section 2, we study the operation of the braid groups on free groups. As an application we focus on a recent work of E. Brieskorn on the action of the braid groups on basis of quadratic forms. In Section 3, we give an alternative definition of the braid group as the fundamental group of the configuration space. Related topics such as the $K(\pi,1)$ properties, the lower central series etc. are reported. Section 4 is devoted to introduce several approaches to construct linear representations of the braid groups containing a method to investigate integrable connections on the configuration space. We shall also explain an algebraic method using Iwahori's Hecke algebra and discuss their relations. In Section 5, we review recent works of V. Jones and several authors on new invariants of links. A complete account on these subjects is found in :

"Artin's braid groups" Proceeding of the conference held at Santa Cruz, to appear as a volume of Cotemporary Math.

1. DEFINITION OF THE BRAID GROUP. Let us first recall the geometric definition proposed by E. Artin in 1925 [A]. We fix distinct n points $a_1, \ldots, a_n$ in the Euclidean plane $\mathbb{R}^2$. We consider disjoint arcs $\mathscr{C}_i = (f_i(t), t)$ in $\mathbb{R}^2 \times I$, $1 \le i \le n$, such that $f_i(1) = a_{\sigma(i)}$ with some element σ of the symmetric group $\mathfrak{S}_n$. Their union $\mathscr{C} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_n$ is called a *geometric braid* (see Fig. 1). The arc $\mathscr{C}_i$ is called the i-th braid.

Figure 1



Now we define an equivalence relation between geometric braids. Let $\mathscr{C}$ and $\mathscr{C}'$ be geometric braids. We shall say that $\mathscr{C}$ and $\mathscr{C}'$ are equivalent if there exists an isotopic deformation F_t of $\mathbb{R}^2 \times I$ which is the identity on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ and for each $t \in [0,1]$, the image $F_t(\mathscr{C})$ satisfies the properties :

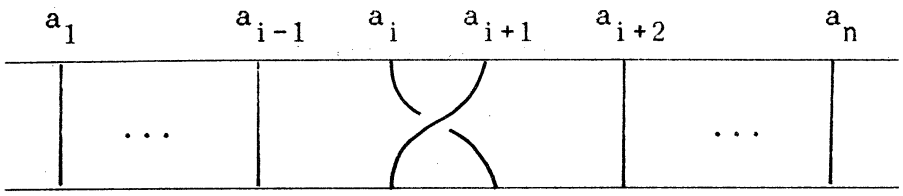
(i) $F_t(\mathscr{C})$ is a geometric braid.

(ii) $F_0(\mathscr{C}) = \mathscr{C}$ and $F_1(\mathscr{C}) = \mathscr{C}'$.

We denote by B_n the above equivalence classes of the geometric braids. The set B_n has a structure of a group by means of the composition of geometric braids. We call B_n the *Artin's braid*

group with n *strings*. Let σ_i be the element of B_n represented by the following geometric braid (see Fig. 2).

Figure 2



It was shown by E. Artin that B_n is generated by σ_i , $1 \le i \le n-1$, with relations :

(1.1) $$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i-j|>1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Since each geometric braid determines a permutation of the set $\{a_1, \ldots, a_n\}$ by associating the end points to its initial points we have a surjective homomorphism $\pi : B_n \longrightarrow \mathfrak{S}_n$. The kernel of π is called the *pure braid group* with n strings which is denoted by P_n .

The presentation 1.1 leads us to the following generalization of the Artin's braid group (see [Br1]). Let $M = (m_{ij}) \in M(m, \mathbb{N})$ be a Coxeter matrix (see [Bou]). We associate to M a group A_M with generators $g_1, \ldots, g_m$ and relations :

(1.2) $$\underbrace{g_i g_j g_i \ldots}_{m_{ij} \text{ times}} = \underbrace{g_j g_i g_j \ldots}_{m_{ij} \text{ times}}$$

We call A_M the *Artin group* associated with M. We observe that Artin's braid group B_n can be considered as the Artin group associated with the Coxeter matrix of type A_n .

Several group theoretic problems such as the word problem, the conjugacy problem and the determination of the center for the Artin's braid group are solved in [A] and [G], where the center in an infinite cyclic group generated by

$$\Delta^2 = (\sigma_1 \ \cdots \ \sigma_{n-1})^n$$

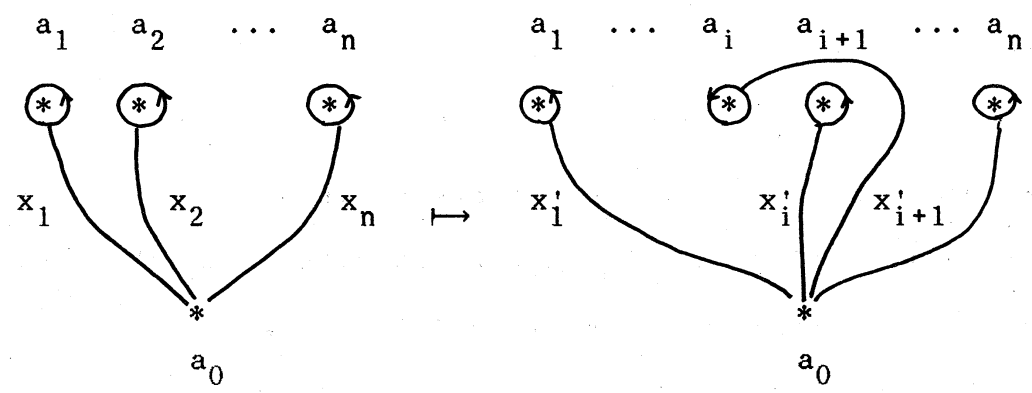
Here Δ is called a fundamental element which represents a rotation of 180^0 of the Euclidean plane. The same problems for Artin groups of finite type are solved in [BS] and [D].

2. OPERATION OF BRAID GROUPS ON FREE GROUPS. Let F_n denote the free group generated by n letters $x_1, \ldots, x_n$. We define a homomorphism $\rho : B_n \longrightarrow \mathrm{Aut}(F_n)$ by

$$\rho(\sigma_i)(x_j) = \begin{cases} x_j \ , & j \neq i \ , \ i+1 \\ x_{i+1} x_i x_{i+1}^{-1} \ , & j = i+1 \\ x_{i+1} \ , & j = i \end{cases}$$

This operation is visualized in the following way by identifying F_n with the fundamental group $\pi_1(\mathbb{C} - \{a_1, \ldots, a_n\}, a_0)$ where the letters x_i are realized by the paths turning around a_i once positively without intersections except a_0.

Figure 3



It is known that the above homomorphism ρ is injective. The image $\rho(B_n)$ is characterized in the following way. An automorphism α of F_n is an element of $\rho(B_n)$ if and only if the following two conditions are satisfied :

(i) $\alpha(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n$

(ii) α leaves invariant the set $\{\bar{x}_1, \ldots, \bar{x}_n\}$, where $\bar{x}_i$ denotes the image of x_i in the abelianization $F_n^{ab} = F_n / [F_n, F_n]$.

Let D' denote the commutator subgroup $[F_n, F_n]$ and let D'' denote the second commutator $[D', D']$. We see that D'/ D'' is a free module over $\mathbb{Z}[F_n^{ab}]$. Here $\mathbb{Z}[F_n^{ab}]$ stands for the group algebra of F_n^{ab}. Identifying this algebra with $\mathbb{Z}[t, t^{-1}]$ the above operation ρ induces a linear representation ψ : $B_n \longrightarrow GL_n(\mathbb{Z}[t, t^{-1}])$ defeined by

$$\psi(\sigma_i) = \begin{array}{c} \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & \cdots & & \\ & & & t & 1-t & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix} \end{array} \; i$$

This representation is called the *Burau represetation*. It is known by Magnus and Peluso (see [B]) that ψ is injective if $n = 3$. It is an open question to determine wheather ψ is faithful or not in the case $n \geq 4$.

An application (d'après Brieskorn [Br2] *et al) : operation of B_n on basis of quadratic forms.* Let us consider a proper flat family $\pi : X \longrightarrow \Delta$ of complex varieties possibly with boundary over a disc Δ (with some additional condition on the boundary). Let $\{a_1, \ldots, a_n\} \subset \Delta$ be the set of critical values of π and we assume that the fibre X_{a_i} over a_i has a single ordinary double point. By fixing a base point $a_0 \in \Delta - \{a_1, \ldots a_n\}$ and choosing paths x_i as before, we obtain generators of the free group $\pi_1(\Delta - \{a_1, \ldots, a_n\}, a_0)$. On the other hand inside a fibre X_t, $t \in \Delta$, for t near to a_i there exists a middle homology class e_i, which vanishes to the singular point of X_{a_i}. By transporting e_i along the path x_i to a_0, we obtain a cycle in X_{a_0} denoted by $x_i^* e_i$. Let us now consider a lattice defined as a formal sum

$$L = \bigoplus_{i=1}^{n} \mathbb{Z}\, x_i^* e_i$$

with the natural intersection form $\langle\ ,\ \rangle : L \times L \longrightarrow \mathbb{Z}$. We obtain the monodromy representation

$$\theta : \pi_1(\Delta - \{a_1, \ldots, a_n\}, a_0) \longrightarrow \mathrm{Aut}(L)$$

By the Picard-Lefschetz theorem the monodromy θ is expressed as

$$\theta(x_i)(u) = u \pm \frac{2\,\langle u, x_i^* e_i\rangle}{\langle x_i^* e_i, x_i^* e_i\rangle}$$

for $u \in L$. We put $B = \langle x_i^* e_i, x_j^* e_j\rangle_{i,j=1,\ldots,n} \in M(n, \mathbb{Z})$.

Now we consider the operation on an element b of B_n on $\pi_1(\Delta - \{a_1, \ldots, a_n\}, a_0)$ via ρ to get another basis x_1', ... x_n' (see Fig. 3). Thus we obtain the matrix $B' = \langle (x_i')^* e_i, (x_j')^* e_j\rangle$

By means of this process we have an operation of B_n on $M(n, \mathbb{Z})$, which can be written down explicitly (see [Br2]). We should remark that this is not a linear representation.

For example in the case $n=3$ the space of the solutions of

$$\det \begin{bmatrix} 2 & x & z \\ x & 2 & y \\ z & y & 2 \end{bmatrix} = \ell \quad , \quad x, y, z \in \mathbb{Z}$$

admits an action of B_3. The orbits of this action were studied by Markov, Mordel, Brieskorn (see [Br2]), Naruki [N] and Saito [S2].

3. CONFIGURATION SPACE. Let H_{ij}, $1 \le i < j \le n$, denote the hyperplanes in $\mathbb{C}^n$ defined by $H_{ij} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \; ; \; z_i = z_j\}$. The symmetric group $\mathfrak{S}_n$ acts freely on the complementary space $\mathbb{C}^n - \bigcup_{1 \le i < j \le n} H_{ij}$ by the permutation of the coordinates. The *configuration space* of distinct n points in $\mathbb{C}$ is by definition

$$\mathcal{C}_n(\mathbb{C}) = (\mathbb{C}^n - \bigcup H_{ij}) / \mathfrak{S}_n \quad .$$

Let $p : \mathbb{C}^n \longrightarrow \mathbb{C}^n / \mathfrak{S}_n$ denote the natural projection. We may identify $\mathbb{C}^n / \mathfrak{S}_n$ with $\mathbb{C}^n$ via elementary symmetric polynomials σ_i, $1 \le i \le n$. The image of the union $\bigcup H_{ij}$ by the projection p is called the *discriminant* which is a complex hypersurface

$$\mathcal{D}_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{C}^n \; ; \; \Delta(\sigma_1, \ldots, \sigma_n) = 0\}$$

where $\Delta(\sigma_1, \ldots, \sigma_n) = \prod_{i \ne j} (z_i - z_j)$ is the discriminant polynomial. Then p is an unramified covering outside $\mathcal{D}_n$ and $\mathcal{C}_n(\mathbb{C})$ is also written as $\mathbb{C}^n - \mathcal{D}_n$. The braid group B_n is isomorphic to the fundamental group $\pi_1(\mathbb{C}^n - \mathcal{D}_n, *)$. It is

known by Fox and Neuwirth (see [Br1]) that

$$\pi_i(\mathbb{C}^n - \mathcal{D}_n, *) = 0 \quad \text{for} \quad i \geq 2 .$$

This leads us to the isomorphisms :

$$H^*(B_n) \cong H^*(\mathbb{C}^n - \mathcal{D}_n)$$
$$H^*(P_n) \cong H^*(\mathbb{C}^n - \bigcup H_{ij})$$

An analogous statement for an arbitrary Artin group of finite type was proved by Deligne [D].

The structure of $H^*(\mathbb{C}^n - \bigcup H_{ij})$ was studied by Arnold [Ar] (see [Br1] and [OS] for a more extensive treatment). It is known that $H^*(\mathbb{C}^n - \bigcup H_{ij})$ is generated by $\omega_{ij} = \dfrac{d(z_i - z_j)}{z_i - z_j}$, $1 \leq i < j \leq n$, with fundamental relations :

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0 \quad , \quad 1 \leq i < j < k \leq n .$$

The Poincaré polynomial $\sum_{p \geq 0} \operatorname{rank} H^p(\mathbb{C}^n - \bigcup H_{ij}).t^p$ is given as

$$(1+t)(1+2t) \ldots (1+(n-1)t) .$$

Let us remark here that the numbers $\{1, 2, \ldots, n-1\}$ appearing in the above formula are the Coxeter exponents of type A_n (see [Br1]). Let us look at several properties relevant to these numbers. Let $\mathcal{D}er_{\mathbb{C}^n}(\log \mathcal{D}_n)_0$ denote the set of germs of the holomorphic vector fields at 0 tangent to $\mathcal{D}_n$. Then $\mathcal{D}er_{\mathbb{C}^n}(\log \mathcal{D}_n)_0$ is a free $\mathcal{O}_{\mathbb{C}^n,0}$ -module generated by homogeneous vector fields X_i , $1 \leq i \leq n-1$, such that $\deg X_i = i$ (see [S]). This relation between the Poincaré polynomial and the logarithmic vector fields was investigated by Terao [Te] in a more general situation. Another property which we want to describe here is a relation with the lower central series

of P_n . Let $\Gamma_1 \supset \Gamma_2 \supset \dots \supset \Gamma_j \supset \dots$ be the lower central series of P_n defined inductively by $\Gamma_1 = P_n$ and $\Gamma_{j+1} = [\Gamma_1, \Gamma_j]$. We put $\varphi_j = \mathrm{rank}\,(\Gamma_j / \Gamma_{j+1})$. Then we have

$$\prod_{j=1}^{n-1} (1 - jt) = \prod_{p=1}^{\infty} (1 - t)^{\varphi_p} \quad \text{in } \mathbb{Z}[[t]]$$

(see [K] and [KO]).

4. LINEAR REPRESENTATIONS. By the Riemann-Hilbert correspondence, we know philosophically that the study of finite dimensional linear representations of B_n is "equivalent" to the study of systems of linear differential equations with regular singularities along the discriminant $\mathcal{D}_n$. But at this stage we do not have a sufficient understanding of either of them. We list up several approaches in this direction.

(i) hypergeometric differential equations : Appell, Picard, Terada [Te] and Deligne-Mostow [DM]. Linear representations of the braid groups appear as the monodromy of the integrals of Pochhammer type.

(ii) total differential equations with logarithmic singularities : By means of solutions of classical Yang-Baxter equations we have a method to construct an integrable connection over the configuration space associated with any finite dimensional irreducible representation of a complex simple Lie algebra (see Kohno's note in this volume). Another approach using the integrals of difference products is due to Aomoto [Ao].

(iii) Gauss-Manin connections : K. Saito, Anbai ... . The reader may refer to Anbai's note in this volume.

(iv) conformal field theory on $\mathbb{P}^1$ *:* Belavin-Polyakov-Zamolodchikov [BPZ], Knizhnik-Zamolodchikov [KZ] and Tsuchiya-Kanie [TK] ... . Linear repreresentations of B_n appear as the monodromy of n-point functions. These functions satisfy the total differential equations mentioned in (ii).

(v) Iwahori's Hecke algebra representations (von Neumann algebra representations) : Jones [Jo], Ocneanu [O] .... . These representations were used effectively to define new invariants of links (see the next Section).

(vi) algebra with two parameters : Birman-Wenzl [BW], J. Murakami [M1] ... . This new algebra was introduced motivated by Kauffman polynomial. It is an extension of Iwahori's Hecke algebra.

Let us recall here the definition of Iwahori's Hecke algebra H(n,q) of the symmetric group. Let us define H(n,q) as the algebra over $\mathbb{Z}[q, q^{-1}]$ generated by $1, T_1, \ldots, T_{n-1}$ with relations :

$$T_iT_j = T_jT_i \quad \text{if} \quad |i-j|>1$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$

$$(T_i+1)(T_i-q) = 0 \quad , \quad i = 1, 2, \ldots, n-1$$

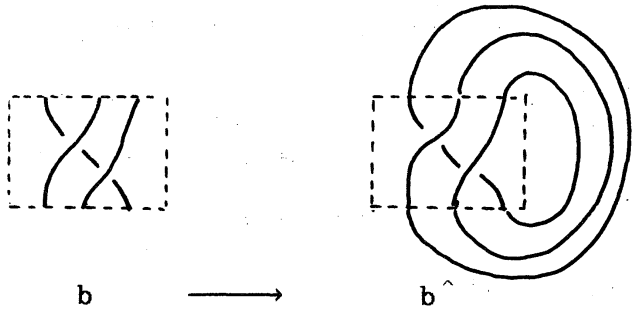
We observe that H(n,q) is a free $\mathbb{Z}[q, q^{-1}]$ - module of rank n !

and that $H(n,1)$ is identified with the group algebra of the symmetric group $\mathfrak{S}_n$. We have a natural homomorphism $\pi : \mathbb{Z}[B_n] \longrightarrow H(n,q)$ defined by $\pi(\sigma_i) = T_i$. A linear representation of B_n factoring through π is called a *Hecke algebra representation.*

The above approches (i) - (vi) are closely intricated. Hecke algebra representaions appear as the monodromy of the connection associated with the vector representation of $sl(m,\mathbb{C})$ (see [K2]). In [TK] it was shown that as the monodromy of n-point function in two dimensional conformal field theory with gauge symmetry of type $A_1^{(1)}$ we obtain unitarizable representations of B_n factoring through the Jones algebra. The study of the monodromy of n-point functions with other types of symmetry is in progress. The monodromy of the connection associated with the vector representation of the other types of non-exceptional simple Lie algebras is related with solutions of quantum Yang-Baxter equation obtained by Jimbo [J1] and [J2] (see [K2]). They commute with the diagonal action of the q-analogue of the corresponding Lie algebra and factor through the algebra with two parameters due to Birman-Wenzl.

5. APPLICATION TO LINK POLYNOMIALS. Given a geometric braid b we obtain an oriented link $b^{\wedge}$ by means of the closing process illustrated in Fig.4.

Figure 4



Let $\mathcal{L}$ denote the set of the isotopy classes of oriented links. We ontroduce an equivalence relation in $\coprod_{n=1}^{\infty} B_n$ generated by the following Markov equivalence :

(MI) for $b, c \in B_n$, $b \sim c$ if b and c are conjugate in B_n .

(MII) for $b \in B_n$, $b \sim b\,\sigma_n^{\pm 1}$

It is known by Alexander and Yamada that the natural map

$$\coprod_{n=1}^{\infty} B_n / \sim \;\longrightarrow\; \mathcal{L}$$

obtained by the closing of braids is surjective. Yamada's argument gives us a stronger statement. Namely he asserts that the minimal number of strings of the geometric braid to obtain an oriented link L is equal to the minimal number of the Seifert circles of L (see [Y] for a precise statement). Moreover we know by Markov (see [B] for a proof) that the above natural map is bijective.

Let us now proceed to review known link polynomials. The *Alexander polynomial* $\Delta_L(t)$ may be defined in the following way from a point of view of the braid group. Let $\psi : B_n \longrightarrow$

$GL_n(\mathbb{Z}[t, t^{-1}])$ be the Burau representation. Then the Alexander polynomial of $L = b^{\wedge}$, $b \in B_n$, is given by

$$\Delta_L(t) = (1+t+ \dots +t^{n-1})^{-1} \det((\psi(b) - 1) \quad .$$

After the definition of the *Jones polynomial* $V_L(t)$ it was generalized to the *two variable polynomial* $X_L(q,\lambda)$ independently by Freyd-Yetter, Lickorish-Millet, Ocneanu and Hoste. Let us recall the definition. We know by Ocneanu [O] that there exists a linear map $tr : H(\infty,q) \longrightarrow \mathbb{Q}(\sqrt{q}, z)$ uniquely defined by

(i) $tr(ab) = tr(ba)$

(ii) $tr(1) = 1$

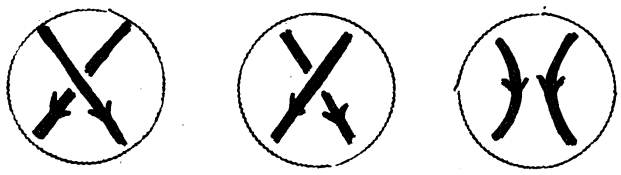
(iii) $tr(xT_n) = z\, tr(x)$ for $x \in H(q,n)$

Here we denote by $H(\infty,q)$ the inductive limit of $H(n,q)$ with respect to the natural inclusion $H(n,q) \subset H(n+1,q)$. Putting $\lambda = (qz)^{-1}(1-q+z)$, we define $X_L(q,\lambda)$ by

$$X_L(q,\lambda) = \left(- \frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1} (\sqrt{\lambda})^e \, tr(\pi(\alpha))$$

where $\alpha \in B_n$ is any braid with $\alpha^{\wedge} = L$, e denotes the exponent sum of α as a word on σ_i's and π is a natural homomorphism to $H(n,q)$ defined in the previous section. By means of Markov's theorem and the properties of tr we verify that $X_L(q,\lambda)$ is an invariant of the isotopy type of L . We put $t = \sqrt{\lambda q}$, $x = \sqrt{q} - (\sqrt{q})^{-1}$ and $P_L(t,x) = X_L(q,\lambda)$. Then $P_L(t,x)$ satisfies the following "skein rule". If L_+ , L_- and L_0 are links which have projections identical, except in one crossing where they are as in Fig.5 .

Figure 5



L_- L_+ L_0

Then we have $t^{-1}P_{L_+} - t\,P_{L_-} = x\,P_{L_0}$. The two variable polynomial P_L is characterized by the above skein rule together with the property that $P_L = 1$ for a trivial knot L .

Both the Alexander polynomial and the Jones polynomial are obtained as the following specializations.

$$\Delta_L(t) = X_L(t,\ t^{-1}) = P_L(1,\ \sqrt{t} - 1/\sqrt{t})$$
$$V_L(t) = X_L(t,t) = P_L(it^{-1},\ -i(\sqrt{t} - 1/\sqrt{t}))$$

They also satisfy the skein rules :

$$\Delta_{L_+} - \Delta_{L_-} + (\sqrt{t} - 1/\sqrt{t})\Delta_{L_0} = 0 \qquad \text{(Conway)}$$
$$t\,V_{L_+} - t^{-1}V_{L_-} + (\sqrt{t} - 1/\sqrt{t})\,V_{L_0} = 0$$

Several results have been obtained for the meaning of special values of $V_L(t)$. The reader may refer to [M2] and [Jo] for works in this direction.

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