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Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori

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ABSTRACT

We prove that every expansive homeomorphism with the pseudo-orbit tracing property of the n-torus is topologically conjugate to a hyperbolic toral automorphism.

The strongest useful equivalence for study of the orbit structure of homeomorphisms will be topological conjugacy. Our investigation will be within the context of the conjugacy problem for homeomorphisms with expansiveness and the pseudo-orbit tracing property (abbrev. POTP). The author proved in [11] that every compact surface which admits such homeomorphisms is the 2-torus and moreover that such a homeomorphism of the 2-torus is topologically conjugate to a hyperbolic toral automorphism. Thus it seems that the orbit structure of homeomorphisms of the n-torus will be determined under the assumption of expansiveness and POTP. And so it will be natural to ask whether every homeomorphism with expansiveness

and POTP of the n-torus is topologically conjugate to a hyperbolic toral automorphism. An answer of this problem is given as follows.

<u>Theorem</u>. Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a homeomorphism of the n-torus. If f is expansive and has POTP, then f is topologically conjugate to a hyperbolic toral automorphism.

Let (X, d) be a metric space and $f : X \to X$ be a (self-) homeomorphism. We say that f is <u>expansive</u> if there is $c > 0$ (called an <u>expansive constant</u>) such that if $x, y \in X$ and $x \neq y$ then $d(f^n(x), f^n(y)) > c$ for some $n \in \mathbb{Z}$. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ of X is a δ-<u>pseudo-orbit</u> of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A point $x \in X$ ε-<u>traces</u> a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of X if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We say that f has <u>POTP</u> if for $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f is ε-traced by some point of X. Note that if X is compact, then expansiveness and POTP are independent of the metrics compatible with original topology, and preserved under topological conjugacy. For materials on topological dymanics on compact manifolds, the reader may refer to A. Morimoto [14].

For global analysis of our homeomorphisms, we prepare the notion of generalized foliations on topological manifolds.

Let M be a connected topological manifold without boundary and $\mathcal{F}$ be a family of subsets of M. We say that $\mathcal{F}$ is a <u>generalized foliation</u> on M if the following hold ;

(1) $\mathcal{F}$ is a decomposition of M,

(2) each $L \in \mathcal{F}$ (called a <u>leaf</u>) is arcwise connected,

(3) if $x \in M$ then there exist non-trivial connected subsets D_x, K_x with $D_x \cap K_x = \{x\}$, a connected open neighborhood N_x of x in M and a homeomorphism $\psi_x : D_x \times K_x \to N_x$ (called a <u>local coordinate</u>) such that

(a) $\psi_x(x, x) = x$,

(b) $\psi_x(y, x) = y$ $(y \in D_x)$ and $\psi_x(x, z) = z$ $(z \in K_x)$,

(c) for any $L \in \mathcal{F}$ there is at most countable set $B \subset K_x$ such that $N_x \cap L = \psi_x(D_x \times B)$.

Let $\mathcal{F}$ be a generalized foliation on M. If $x \in L$ for some $L \in \mathcal{F}$, then we see by (b) and (c) that $D_x \subset L$.

For fixed $L \in \mathcal{F}$ let Q_L be a family of subsets of L such that for any $D \in Q_L$ there is an open subset O of M such that D is a connected component in $O \cap L$. Then the topology generated by Q_L is called a <u>leaf topology</u> of L.

If D_x is as in (3), then D_x is open in L (with respect to Q_L). Hence D is open in L if and only if to $x \in L$, $D \cap D_x$ is open in D_x. Note that the leaf topology has countable base.

If $f : M \to M$ is a homeomorphism such that $f(\mathcal{F}) = \mathcal{F}$, then it is easily checked that $f : L \to L$ is a homeomorphism (with respect to Q_L).

Let $\mathcal{F}$ and $\mathcal{F}'$ be generalized foliations on M. We say that $\mathcal{F}$ is <u>transverse</u> to $\mathcal{F}'$ if to $x \in M$ there exist non-trivial connected subsets D_x, D_x' with $D_x \cap D_x' = \{x\}$, a connected open neighborhood N_x of x in M (such a

neighborhood N_x is called a <u>coordinate domain</u>) and a homeomorphism $\varphi_x : D_x \times D_x' \to N_x$ (in particular called <u>canonical coordinate</u>) such that

(a)' $\varphi_x(x, x) = x$,

(b)' $\varphi_x(y, x) = y$ $(y \in D_x)$ and $\varphi_x(x, z) = z$ $(z \in D_x')$,

(c)' for any $L \in \mathcal{F}$ there is at most countable set $B' \subset D_x'$ such that $N_x \cap L = \varphi_x(D_x \times B')$,

(d)' for any $L' \in \mathcal{F}'$ there is at most countable set $B \subset D_x$ such that $N_x \cap L' = \varphi_x(B \times D_x')$.

We denote by $L(x)$ and $L'(x)$ the leaves of transverse generalized foliations $\mathcal{F}$ and $\mathcal{F}'$ through x respectively. Let N be a coordinate domain and write $D(x)$ and $D'(x)$ the connected components of x in $N \cap L(x)$ and $N \cap L'(x)$ respectively. For $x, y \in N$ it is not difficult to see that $D(x) \cap D'(y)$ is a single point. And so we can define a map

$$\gamma_N : N \times N \longrightarrow N$$

by $(x, y) \longmapsto D(x) \cap D'(y)$, and have then γ_N is continuous and

$$\gamma_N(x, x) = x \cdot , \quad \gamma_N(x, \gamma_N(y, z)) = \gamma_N(x, z) \quad ,$$

$$\gamma_N(\gamma_N(x, y), z) = \gamma_N(x, z) \quad .$$

If N and $\tilde{N}$ are coordinate domains and if U is a coordinate domain such that $U \subset N \cap \tilde{N}$, then the following is checked from the definition

$$\gamma_U = \gamma_N|_{U\times U} = \gamma_{\tilde{N}}|_{U\times U}$$

Let f be a homeomorphism of a metric space (X, d). For $x \in X$, define the <u>stable set</u> $W^s(x)$ and the <u>unstable set</u> $W^u(x)$ by

$$W^s(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to \infty\},$$

$$W^u(x) = \{y \in X : d(f^n(x), f^n(y)) \to 0 \quad \text{as} \quad n \to -\infty\}$$

and put

$$\mathcal{F}_f^\sigma = \{W^\sigma(x) : x \in X\} \qquad (\sigma = s, u).$$

Then $\mathcal{F}_f^\sigma$ is a decomposition of X and $f(\mathcal{F}_f^\sigma) = \mathcal{F}_f^\sigma$.

For the proof of Theorem we need the following two propositions.

<u>Proposition A</u>. Let M be a closed topological manifold and $f : M \to M$ be a homeomorphism. If f is expansive and has POTP, then $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$ are transverse generalized foliations on M.

When $\mathcal{F}$ is a generalized foliation on M and there exists a generalized foliation transverse to $\mathcal{F}$, the orientability for $\mathcal{F}$ will be defined.

<u>Proposition B</u>. Let $f : M \to M$ be as in Proposition A. If the generalized foliation $\mathcal{F}_f^u$ is orientable, then there

exists $\ell \in \mathbb{N}$ such that for any $m \geq \ell$ all the fixed points of f have the same fixed point index 1 or -1.

If we established Proposition B, then our theorem will be obtained by using skilfully the techniques of J. Franks [8], M. Brin and A. Manning [4] and the author [11].

For the details of this paper, the author hope to appear elsewhere.

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