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Max-flow min-cut theorems on an infinite network

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Introduction

Duality relations between the max-flow problems and the min-cut problems seem to be one of the most important results in the theory of networks. On a finite network, the celebrated max-flow min-cut theorem due to Ford and Fulkerson [2] has been the unique result for this direction before the work of Strang [6]. On an infinite network, there are several kinds of flows and cuts and several max-flow min-cut theorems have been obtained by Nakamura and Yamasaki [3] and Yamasaki [7]. In this paper, we shall study a duality relation between a max-flow problem for an important class of flows and a maximization problem on a class of min-cut problems. We use the notion of extremal width of the network which was introduced by Duffin [1]. For a set of exceptional cuts in the sense of extremal width, we can find a so-called penalty function by the same method as in Ohtsuka [5]. Our main theorem is proved by using the penalty method in the theory of mathematical programming.

More precisely, let X and Y be countable sets of nodes and arcs respectively and K be the node-arc incidence function. We assume that the graph G = {X, Y, K} is connected and locally finite and has no self-loop. For a strictly positive function r on Y, we call the pair N = {G, r} an infinite network. Denote by L(X) and

$L(Y)$ the sets of all real functions on X and Y respectively, by $L^+(Y)$ the set of all nonnegative functions on Y and by $L_0(Y)$ the set of $w \in L(Y)$ such that the support $\{y \in Y;\ w(y) \neq 0\}$ is a finite set. Let p be a number such that $p > 1$ and $H_p(w)$ be the energy of $w \in L(Y)$ of order p, i.e.,

$$H_p(w) = \sum_{y \in Y} r(y)|w(y)|^p.$$

The set $L_p(Y;\ r)$ of all $w \in L(Y)$ with finite energy of order p is a reflexive Banach space with the norm $[H_p(w)]^{1/p}$.

For notation and terminology, we mainly follow [3] and [4].

Let A and B be mutually disjoint nonempty finite subsets of X. We say that $w \in L(Y)$ is a flow from A to B if the following conditions are fulfilled:

(1.1) $\quad \sum_{y \in Y} K(x,\ y)w(y) = 0 \quad$ for all $x \in X - A - B$,

(1.2) $\quad -\sum_{x \in A} \sum_{y \in Y} K(x,\ y)w(y) = \sum_{x \in B} \sum_{y \in Y} K(x,\ y)w(y)$.

Condition (1.1) implies that the Kirchhoff's first law is valid at each nodes in $X - A - B$. Denote by $F(A,\ B)$ the set of all flows from A to B. The strength $I(w)$ of $w \in F(A,\ B)$ is defined by the common value in (1.2).

Let $F_p(A,\ B)$ be the closure of $F_0(A,\ B) = F(A,\ B) \cap L_0(Y)$ in the Banach space $L_p(Y;\ r)$.

First we state the max-flow problem in a general form.

Given a capacity function $W \in L^+(Y)$ and a nonempty subset $\mathcal{F}$ of flows, the max-flow problem related to W and $\mathcal{F}$ is formulated as follows:

($M\mathcal{F}$) $\quad$ Find $M(W;\ \mathcal{F}) = \sup\{I(w);\ w \in \mathcal{F}$ and $|w(y)| \leq W(y)$ on $Y\}$.

To state min-cut problems, we recall some notation. For mutually disjoint subsets X_1 and X_2 of X, denote by $X_1\ \theta\ X_2$ the set

of all arcs which connects directly X_1 and X_2. We say that a subset Q of Y is a cut (or cut-set) if $Q = X' \theta (X - X')$ for some nonempty set X'. Note that the pair $\{X', X - X'\}$ is uniquely determined by Q.

We say that Q is a cut between A and B if there exists a subset X' of X such that $Q = X' \theta (X - X')$, $X' \supset A$ and $X - X' \supset B$. For simplicity, we put $X' = Q(A)$ and $X - X' = Q(B)$. Denote by $Q_{A,B}$ the set of all cuts between A and B and by $Q^{(f)}_{A,B}$ the set of all $Q \in Q_{A,B}$ such that Q is a finite subset of Y.

Given $W \in L^+(Y)$ and a subset $\mathcal{C}$ of cuts, the min-cut problem related to W and $\mathcal{C}$ is formulated as follows:

$(M^*\mathcal{C})$ Find $M^*(W; \mathcal{C}) = \inf\{\sum_{y \in Q} W(y);\ Q \in \mathcal{C}\}$.

Here we use the convention that the infimum on the empty set is equal to ∞.

The aim of this paper is to obtain duality relations between problems $(M\mathcal{F})$ and $(M^*\mathcal{C})$ for several choices of $\mathcal{F}$, $\mathcal{C}$ and W.

§ 1. Known results

The characteristic function $u_Q \in L(Y)$ of $Q \in Q_{A,B}$ is defined by $u_Q(x) = 1$ on $Q(A)$ and $u_Q(x) = 0$ on $Q(B)$. Let us define a quantity $J(w; Q)$ for a cut $Q \in Q_{A,B}$ and $w \in L(Y)$ by

$$J(w; Q) = \sum_{y \in Y} w(y) \sum_{x \in X} K(x, y) u_Q(x)$$

if the sum is well-defined. Notice that $J(w; Q)$ is well-defined if Q is a finite cut or if $w \in L_0(Y)$.

We proved in [7]

Lemma 1.1. Let $w \in F(A, B)$ and $Q \in Q_{A,B}$. Then the equality $I(w) = -J(w; Q)$ holds if any one of the following conditions is

fulfilled:

(1.1) $w \in F_0(A, B)$ and $Q \in Q_{A,B}$.

(1.2) $w \in F_p(A, B)$ and $Q \in Q^{(f)}_{A,B}$.

Corollary. The following inequalities hold:

(1.3) $M(W; F_0(A, B)) \leq M^*(W; Q_{A,B})$.

(1.4) $M(W; F_p(A, B)) \leq M^*(W; Q^{(f)}_{A,B})$.

The following max-flow min-cut theorem was proved in [7] by using the standard labelling method.

Theorem 1.1. $M(W; F_0(A, B)) = M^*(W; Q_{A,B})$.

Corollary. $M^*(W; Q_{A,B}) \leq M(W; F_p(A, B)) \leq M^*(W; Q^{(f)}_{A,B})$.

The inequalities in the above corollary can be replaced by equalities if W satisfies the following condition

(∞) $M^*(W; Q_{F,\infty}) = 0$ for every nonempty finite set F of X.

We proved in [4]

Theorem 1.2. If W satisfies condition (∞), then

$$M(W; F(A, B)) = M^*(W; Q_{A,B}) = M^*(W; Q^{(f)}_{A,B}).$$

§ 2. Extremal value $\mu_p(\Lambda)$

For later use, we introduce an extremal value $\mu_p(\Lambda)$ as a scale of exceptional sets of cuts.

For a set Λ of cuts, we define the value $\mu_p(\Lambda)$ of Λ (of order p) as the inverse of the value of the following convex programming problem:

(CP) Minimize $H_p(W)$

subject to $W \in L^+(Y)$ and $\sum_{y \in Q} W(y) \geq 1$ for all $Q \in \Lambda$.

Denoting by $E(\Lambda)$ the set of feasible solutions of (CP), we have

$$\mu_p(\Lambda)^{-1} = \inf\{H_p(W);\ W \in E(\Lambda)\}.$$

By the standard method as in [5], we have

Lemma 2.1. If $\Lambda_1 \subset \Lambda_2$, then $\mu_p(\Lambda_1) \geq \mu_p(\Lambda_2)$.

Lemma 2.2. $\sum_{n=1}^{\infty} \mu_p(\Lambda_n)^{-1} \geq \mu_p(\cup_{n=1}^{\infty} \Lambda_n)^{-1}$.

We say that a set Λ of cuts is exceptional (with respect to μ_p) if $\mu_p(\Lambda) = \infty$. By the above lemmas, we see that any subset of an exceptional set is exceptional and that the countable union of exceptional sets is also exceptional.

By the same reasoning as in [3], we can prove

Lemma 2.3. A set Λ of cuts is exceptional if and only if there exists a penalty function W for Λ, i.e., W is a nonnegative function on Y such that $H_p(W) < \infty$ and $\sum_{y \in Q} W(y) = \infty$ for all $Q \in \Lambda$.

Corollary. Let Λ be an exceptional set of cuts. If $\sum_{y \in Y} r(y) < \infty$, then Λ does not contain a finite cut.

Lemma 2.4. Let Λ be a set of cuts such that every $Q \in \Lambda$ is an infinite subset of Y. If $\sum_{y \in Y} r(y) < \infty$, then Λ is an exceptional set.

Lemma 2.5. Let Λ be a set of cuts and assume that a sequence $\{W_n\}$ of nonnegative functions converges to 0 in $L_p(Y; r)$, i.e., $H_p(W_n) \to 0$ as $n \to \infty$. Then there exist a subsequence $\{n\}$ and an exceptional subset Λ' of Λ such that $\lim_{n\to\infty} \sum_{y \in Q} W_n = 0$ for every $Q \in \Lambda - \Lambda'$.

§ 3. Main results

Denote by $Q_{A,B}^{(\infty)}$ be the totality of exceptional subsets of $Q_{A,B}$ and consider the following maximin problem:

(MM) Find $M^{\#}(W; Q_{A,B}) = \sup\{M^*(W; Q_{A,B} - \Lambda); \Lambda \in Q_{A,B}^{(\infty)}\}$.

Note that $M^*(W; Q_{A,B} - \Lambda)$ is the value of a min-cut problem (M^*C)

with $C = Q_{A,B} - \Lambda$.

We shall prove the following duality theorem:

Theorem 3.1. If $W \in L^+(Y)$ and $H_p(W) < \infty$, then the following equality holds: $M(W; F_p(A, B)) = M^{\#}(W; Q_{A,B})$.

Proof. Let w be a feasible solution for our max-flow problem, i.e., $w \in F_p(A, B)$ and $|w(y)| \leq W(y)$ on Y. Then there exists a sequence $\{w_n\}$ in $F_0(A, B)$ such that $H_p(w - w_n) \to 0$ as $n \to \infty$ by our definition. For any $Q \in Q_{A,B}$, we have $I(w_n) = -J(w_n; Q)$ by Lemma 1.1, so that

$$|I(w_n)| \leq \sum_{y\in Y} |w_n(y)||\sum_{x\in X} K(x, y)u_Q(x)| = \sum_{y\in Q} |w_n(y)|.$$

Put $W_n(y) = |w_n(y) - w(y)|$. Then $H_p(W_n) \to 0$ as $n \to \infty$. On account of Lemma 2.5, we can find a subsequence $\{n\}$ (without changing notation) and an exceptional subset Λ_0 of $Q_{A,B}$ such that $\sum_{y\in Q} W_n(y) \to 0$ as $n \to \infty$ for every $Q \in Q_{A,B} - \Lambda_0$. Note that $I(w_n) \to I(w)$ as $n \to \infty$. From the relation

$$\sum_{y\in Q} |w_n(y)| - \sum_{y\in Q} |w(y)| \leq \sum_{y\in Q} W_n(y),$$

it follows that

$$|I(w)| \leq \limsup_{n\to\infty} \sum_{y\in Q}|w_n(y)| \leq \sum_{y\in Q} |w(y)| \leq \sum_{y\in Q} W(y)$$

for every $Q \in Q_{A,B} - \Lambda_0$, and hence

$$I(w) \leq M^*(W; Q_{A,B} - \Lambda_0) \leq M^{\#}(W; Q_{A,B}).$$

Therefore $M(W; F_p(A, B)) \leq M^{\#}(W; Q_{A,B})$. To prove the converse inequality, let t be any number such that $t < M^{\#}(W; Q_{A,B})$. There is an exceptional subset Λ_1 of $Q_{A,B}$ such that $M^*(W; Q_{A,B} - \Lambda_1) > t$. By Lemma 2.3, we can find a penalty function W' for Λ_1, i.e., $W' \in L^+(Y)$ such that $H_p(W') < \infty$ and $\sum_{y\in Q} W'(y) = \infty$ for all $Q \in \Lambda_1$. For any $\varepsilon > 0$, we see easily that

$$M^*(W + \varepsilon W'; Q_{A,B}) \geq M^*(W; Q_{A,B} - \Lambda_1) > t,$$

which is the so-called penalty method. By the elementary max-flow min-cut theorem (cf. Theorem 1.1), we have

$$M(W + \varepsilon W'; F_0(A, B)) = M^*(W + \varepsilon W'; Q_{A,B}).$$

Since $M(W + \varepsilon W'; F_0(A, B)) > t$, there exists $w_\varepsilon \in F_0(A, B)$ such that $I(w_\varepsilon) > t$ and $|w_\varepsilon(y)| \leq W(y) + \varepsilon W'(y)$ on Y. Noting that $H_p(w_\varepsilon) \leq 2^p[H_p(W) + \varepsilon^p H_p(W')]$ and taking $\varepsilon = 1/n$ for $n = 1, 2, \cdots$, we can find a weakly convergent subsequence of $\{w_\varepsilon\}$. Denote it by $\{w_n\}$ and let w be the limit. Since $F_p(A, B)$ is convex and strongly closed, it is weakly closed. Therefore $w \in F_p(A, B)$. Since $w_n(y) \to w(y)$ as $n \to \infty$ for each $y \in Y$, we have

$$|w(y)| \leq W(y) \text{ on } Y \text{ and } I(w) \geq t,$$

so that $M(W; F_p(A, B)) \geq t$. Thus $M^\#(W; Q_{A,B}) \leq M(W; F_p(A, B))$. This completes the proof.

We have

Theorem 3.2. Under the same assumption as in Theorem 3.1, both problems $(MF_p(A, b))$ and (MM) have optimal solutions.

By Theorem 3.1 and Lemma 2.4, we obtain a max-flow min-cut theorem:

Theorem 3.3. Assume that $\sum_{y \in Y} r(y) < \infty$. If $W \in L^+(Y)$ and $H_p(W) < \infty$, then $M(W; F_p(A, B)) = M^*(W; Q^{(f)}_{A,B})$ holds.

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