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A C*-ALGEBRA WITH THE REPRESENTATION EXTENSION PROPERTY

東北大　理　斎藤和之　（Kazuyuki Saitô）

Every C*-algebra A sits inside its regular completion $\hat{A}$ very nicely. $\hat{A}$ is a monotone complete C*-algebra, which is, in general, not a von Neumann algebra. $\hat{A}$ reflects closely the structure of A, for examples, any bounded derivation of A extends to a unique <u>inner</u> derivation of $\hat{A}$, any automorphism of A extends to a unique automorphism of $\hat{A}$, and if A is simple, any <u>outer</u> automorphism of A extends to a unique <u>outer</u> automorphism of $\hat{A}$.

The following question naturally arises: Let A be any C*-algebra and let π be any irreducible representation of A.

(*) Can we extend π to a *-representation $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)\hat{}$?

If A is abelian, every character of A can be extended to a character of $\hat{A}$ and so (*) has an affirmative answer.

Unfortunately, the answer to this question (*) is negative in general.

Definition. Let A be a C*-algebra. Then A is said to have the representation extension property (the (RE)-property) if, for every irreducible representation π of A, π can be extended to a *-homomorphism $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)\hat{}$. We say that A has the strong representation extension property (the

(SRE)-property) if, any quotient of A has the (RE)-property.

In [6], a characterization of C^*-algebras with the (SRE)-property is given in the following form:

Theorem. <u>Let</u> A <u>be</u> <u>a</u> <u>separable</u> C^*<u>-algebra</u>. <u>Then</u> A <u>has</u> <u>the</u> (SRE)-<u>property</u> <u>if</u>, <u>and</u> <u>only</u> <u>if</u>, A <u>is</u> <u>a</u> <u>restricted</u> <u>direct</u> <u>sum</u> <u>of</u> <u>a</u> <u>sequence</u> $\{B_n\}$, <u>consisting</u> <u>of</u> <u>infinite</u> <u>dimensional</u> <u>simple</u> C^*<u>-algebras</u> <u>or</u> <u>homogeneous</u> C^*<u>-algebras</u> <u>of</u> <u>finite</u> <u>orders</u> (including $\{0\}$).

In this note, we would like to give you an outline of its proof with negative examples to the question (*), which, we do hope, would be helpful to understand the proof of Theorem. We also would like to give an example to show that there is a C^*-algebra which has the (RE)-property, but has not the (SRE)-property.

Detailed proof will appear in the J. London Math. Soc. (see [6]).

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Note first that if A is abelian, every character of A can be extended to a character of $\hat{A}$ and so A has the (SRE)-property.

Example 1. Let A be the set of all continuous functions F on [0,1], taking values in the C*-algebra $M_2(\mathbb{C})$ of all 2 by 2 matrices over $\mathbb{C}$, such that $F(0)_{2j} = F(0)_{12} = 0$ for $i,j = 1,2$. A becomes a C*-algebra with respect to the pointwise operations and the norm $\|F\| = \mathrm{Sup}\{\|F(t)\| \mid t \in [0,1]\}$. Then A has not got the (RE)-property.

In fact, the one dimensional representation of A defined by $F \in A \longrightarrow F(0)_{11}$ cannot be extended to $\hat{A} = M_2(C[0,1]\hat{\,})$.

Example 2. Let H be an infinite dimensional separable Hilbert space and let $A = K(H)\otimes C[0,1]$, the C*-tensor product of the algebra of compact operators on H, by the C*-algebra $C[0,1]$. Then A has never got the (RE)-property.

Note, first, that $\hat{A} = L(H)\overline{\otimes}C[0,1]\hat{\,}$ and $\hat{A}$ is a properly infinite and σ-finite AW*-algebra. Let $\pi = i\otimes\pi_t$ be the irreducible representation of A on H induced by $t \in [0,1]$. Then $\pi(A) = K(H)$. If it were possible to extend π to a *-homomorphism $\hat{\pi}$ from $\hat{A}$ onto $K(H)\hat{\,} = L(H)$, then, since H is separable, this would imply that π is <u>normal</u> (see [1]). Let ξ be any unit vector in H, and let $\phi_t(a) = (\hat{\pi}(1\otimes a)\xi,\xi)$ for any $a \in C[0,1]\hat{\,}$. Then $\phi_t(1) = 1$, and so ϕ_t is a state on $C[0,1]\hat{\,}$, which is completely additive on projections. This were, however, a contradiction, because $C[0,1]\hat{\,}$ has no normal states.

Example 3. Let F be the Fermion algebra acting irreducibly on an infinite dimensional separable Hilbert space H.

Let $B = F + K(H)$ (note that $F \cap K(H) = \{0\}$). Let π be the irreducible representation of B which is canonically defined by $B/K(H)$ ($\cong F$). Then π cannot be extended to a *-homomorphism from $\hat{B}$ ($= L(H)$) onto (into) $\hat{F}$.

In fact, if it were possible, then, since $\hat{B}$ is properly infinite and $\hat{F}$ is σ-finite, π would be a direct summand. This is, however, a contradiction, because $\hat{B}$ is a factor and π is not faithful.

The fact that <u>the regular completion</u> $\hat{A}$ <u>of a separable</u> C*-<u>algebra</u> A <u>has no type II direct summand</u> is used to prove our Theorem. See for details ([3], Theorem 2).

Also we use the following Theorem proved by Feldman and Fell ([1]).

Lemma 1. <u>Let</u> A, B <u>be two</u> AW*-<u>algebras</u>. <u>Let</u> A <u>be properly infinite and</u> σ-<u>finite</u>. <u>Suppose that</u> B <u>is</u> σ-<u>finite</u>. <u>Then any</u> *-<u>homomorphism from</u> A <u>into</u> B <u>is compltely additive on projections</u>.

Some routine calculations in the regular completions of C*-algebras (see [5], [7], [10] and [11]) give us the following:

Lemma 2. <u>Let</u> $\{B_n\}$ <u>be any sequence of separable</u> C*-<u>algebras</u> <u>and let</u> $A = \sum_{n=1}^{\infty}{}' B_n$ <u>be the restricted direct sum of</u> $\{B_n\}$. <u>Then</u> $\hat{A} = \sum_{n=1}^{\infty} \hat{B}_n$.

To prove this, we only have to check the following two properties by the uniqueness of the regular completions.

Let $\tilde{A} = \sum_{n=1}^{\infty} \hat{B}_n$. Then $\tilde{A}$ has the following two properties:

1) $C^*(A,1)_h$ is order dense in $\tilde{A}_h$,

and

2) $C^*(A,1)_h$ σ-generates $\tilde{A}_h$,

where $C^*(A,1)$ is the C^*-algebra obtained from the adjunction of a unit to A if A is non-unital, otherwise, $C^*(A,1) = A$ and B_h is the self-adjoint part of a C^*-algebra B.

By an homogeneous C^*-algebra of order n (n < ∞), we mean a C^*-algebra A such that $A/J \cong M_n(\mathbb{C})$ for every primitive ideal J of A ([2], [8]).

By using the fact that the Borel envelop of a separable homogeneous C^*-algebra of order n is $*$-isomorphic to the n by n matrix algebra over the C^*-algebra $\mathcal{B}(PrimA)$ of all bounded complex Borel functions on the 2nd countable Hausdorff locally compact space PrimA, where PrimA is the primitive ideal space of A, we can show the following lemma (see [2], [4] and [5]).

Lemma 3. <u>Let</u> A <u>be any separable homogeneous C^*-algebra of finite order n, then</u> $\hat{A}$ <u>is an AW*-algebra of type</u> I_n, <u>whose centre is $*$-isomorphic to the regular completion</u> $C_b(PrimA)\hat{}$ <u>of the ideal centre</u> $C_b(PrimA)$ <u>of A, where</u> $C_b(PrimA)$ <u>means the C^*-algebra of all bounded complex continuous functions on</u> PrimA.

A careful examination of an irreducible representation of an homogeneous algebra of a finite order tells us that every separable homogeneous C*-algebra of a finite order has the (SRE)-property. By using this fact, together with Lemma 2 and Lemma 3, we can show the following proposition, which states the "if" part of Theorem is valid.

Proposition 1. <u>Let</u> A <u>be a restricted direct sum of a sequence</u> $\{B_n\}$ <u>of simple C*-algebras or homogeneous C*-algebras of finite orders, then</u> A <u>has the</u> (SRE)-<u>property</u>.

Next, we shall sketch a proof of "only if" part of Theorem. First of all, we shall start with the following proposition, which treats a C*-algebra whose regular completion is properly infinite.

Proposition 2. <u>Let</u> B <u>be a separable C*-algebra whose regular completion</u> $\hat{B}$ <u>is properly infinite</u>. <u>Suppose that</u> B <u>has the</u> (RE)-<u>property</u>. <u>Then</u> PrimB <u>is discrete, and so</u> B <u>is a restricted direct sum of an at most countable family of separable infinite dimensional simple C*-algebras</u>.

Let $J \in$ PrimB. Let $\hat{\pi}_J$ be a *-homomorphism from $\hat{B}$ onto $(B/J)^{\hat{}}$ which is an extension of π_J. Since $\hat{B}$ is σ-finite, properly infinite and $(B/J)^{\hat{}}$ is σ-finite, $\hat{\pi}_J$ is normal by Lemma 1, that is, there is a unique central projection e_J in $\hat{B}$ such that

$$J = \hat{B}e_J \cap B.$$

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Moreover, $\hat{B}(1 - e_J) \cong (B(1 - e_J))^\wedge \cong (B/J)^\wedge$, which implies

that $\hat{B}(1 - e_J)$ is a non-zero factor and so every $J \in \text{Prim}B$

is <u>maximal</u> in PrimB.

For any $J \in \text{Prim}B$, let $J^c = \hat{B}(1 - e_J) \cap B \,(\neq \{0\})$, which

is a closed two-sided ideal of B.

Let $K \subset \text{Prim}B$. Let $J_0 \in \overline{K}$, that is, $J_0 \supset \cap \{ J | J \in K \}$.

We shall show that there is $J \in K$ such that $J_0 + J \neq B$.

Suppose that $J_0 + J = B$ for all $J \in K$. Then, $J^c \subset J_0$

for all $J \in K$. In fact, for any fixed $J \in K$, if $y \in J^c$ (

$y \geq 0$), then one can find $x_0 \in J_0$ ($x_0 \geq 0$) and $x_1 \in J$ ($x_1 \geq 0$)

such that $y = x_0 + x_1$. Since $y \in J^c$, $e_J y = 0$ follows and so

$e_J x_0 e_J + x_1 = 0$, that is, $x_1 = 0$. This means that $y \in J_0$.

Hence $1 - e_{J_0} \leq e_J$ for all $J \in K$.

On the other hand, $J_0 \supset \cap \{J | J \in K\}$ tells us that $1 - e_{J_0} \leq$

$\text{Sup}\{1 - e_J | J \in K\}$ and hence $1 = e_{J_0}$. This is a contradiciton

and so, if $J_0 \in \overline{K}$, then there is $J \in K$ such that $J_0 + J \neq B$.

Since J and J_0 are maximal, this means that $J_0 = J \in K$,

that is, $\overline{K} = K$ for all $K \subset \text{Prim}B$. Hence PrimB is discrete.

This completes the proof.

Let A be a separable C^*-algebra and let

$$X_0 = \{J \in \text{Prim}A \mid (A/J)^\wedge \text{ is properly infinite}\},$$

$$Y = \{J \in \text{Prim}A \mid (A/J)^\wedge \text{ is finite}\},$$

and for each k ($k \geq 1$),

$$X_k = \{J \in \text{Prim}A \mid (A/J)^\wedge \cong M_k(\mathbb{C})\}.$$

Since $(A/J)^\wedge$ has no type II-direct summand for each

$J \epsilon$ PrimA,

$$Y = \bigcup_{k=1}^{\infty} X_k \quad (\text{some } X_k \text{ may be } \emptyset).$$

Suppose that A has the (SRE)-property. Then, by a rather lengthy speculation, we have the following:

Lemma 4. Keep the notations and assumptions above in mind. X_k (k $\geq$ 0) is open and closed in PrimA.

The following F.B. Wright's theorem is used to prove Lemma 4, but we shall omit the details (see [6] and [9]).

Lemma 5([9]). Let B be a finite AW*-algebra of type I and let X be the set of all maximal ideals of B. Then B/M is a finite AW*-factor of type I for every M except possibly for a closed nowhere dense set in X. The exceptional set is empty if, and only if, the number of homogeneous summands of B is finite. If this set is non-empty, then B/M is a von Neumann factor of type II_1 for every such M.

So one can find closed two-sided ideals B and C such that

$$A = B \oplus C$$

and PrimB $\approx X_0$, PrimC $\approx \bigcup_{k=1}^{\infty} X_k$.

By our assumption, B has the (RE)-property and so, by Proposition 2, PrimB is discrete. So either B ={0} or else

B is a restricted direct sum of at most countable, infinite dimensional simple C*-algebras.

Let f_k be the projection in the ideal centre $Z(M(C))$ of the multiplier algebra of C corresponding to the continuous characteristic function of X_k $(k \geq 1)$ on PrimC, then, since C is separable, it is easy to check that the mapping

$$x \in C \longrightarrow (xf_k) \in \sum_{k=1}^{\infty}{}' \; Cf_k$$

is a *-isomorphism between C and $\sum_{k=1}^{\infty}{}' \; Cf_k$. Since Cf_k is k-homogeneous, this completes the proof.

Next, we shall construct an example of a C*-algebra which has the (RE)-property but not the (SRE)-property.

Let A be the set of all continuous functions F on [0,1], taking values in $M_2(\mathbb{C})$ such that $F(t)_{12} = F(t)_{2j} = 0$ for i,j = 1,2 and for t with $0 \leq t \leq 1/2$. Then A becomes a C*-algebra with respect to the pointwise operations and the norm $\| F \| = \text{Sup}\{\|F(t)\| \mid t \in [0,1]\}$. Let

$$I_1 = \{F \in A \mid F(t) = 0 \text{ for } t \in [1/2,1]\}$$

and let

$$I_2 = \{F \in A \mid F(t) = 0 \text{ for } t \in [0,1/2]\}.$$

Then I_1 and I_2 are closed ideals of A such that

(1) $I_1 \cap I_2 = \{0\}$,

(2) $A/I_2 \cong C[0,1/2]$,

(3) $A/I_1 \cong B$

where B is the C*-algebra of all continuous $M_2(\mathbb{C})$-valued

functions G on [1/2,1] such that $G(1/2)_{12} = G(1/2)_{2j} = 0$

for i,j = 1,2,

(4) $I_1 \cong C_0[0,1/2]$

where $C_0[0,1/2]$ is the C*-algebra of all complex-valued

continuous functions on [0,1/2], vanishing at 1/2,

(5) $I_2 \cong B_0$

where B_0 is the C*-subalgebra of B, vanishing at 1/2, and

(6) $I_1 + I_2$ is essential in A.

Lemma 6. *For* *any* $J \in \text{PrimA}$, *one* *can* *find* *a* *unique* $t \in [0,1]$,
such *that*, *either* $t \in [0,1/2]$ *and* $F + J = F(t)_{11}1$ *for* *all*
$F \in A$ *or* *else*, $t \in (1/2,1]$ *and* $F + J = F(t)$ *for* *all* $F \in A$.

Since, by (1)-(6),

$$\hat{A} \cong \begin{pmatrix} C[0,1/2]^{\wedge} & 0 \\ & \\ 0 & 0 \end{pmatrix} \oplus M_2(C[1/2,1]^{\wedge}),$$

by making use of Lemma 6, it is easy to check that A has the
(RE)-property. On the other hand, since $A/I_1 \cong B$ and B has
not got the (RE)-property (see example 1), A has not got the
(SRE)-property.

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Mathematical Institute

Tôhoku University

Sendai 980

Japan