数理解析研究所講究録
689巻 1989年 38-56

JACOBI FORMS AND ODA'S LIFTING

Takashi SUGANO 菅野孝史 (三重大・教育)

0. Introduction

In the space of Siegel cusp forms of genus two, a special subspace, which is constructed by H. Maass and so is called the Maass space, is defined. Many authors studied this subspace in connection with the Saito-Kurokawa lifting. The coincidence between the Maass space and the image of the lifting was proved by using Jacobi forms (cf. Zagier [12]). In [02] and [03], Kohnen and Skoruppa showed that $\rho_1 \iota_1 f$ is a constant multiple of f for any Hecke eigen cusp form f, where ι_1 [resp. ρ_1] is the Saito-Kurokawa lifting [resp the adjoint mapping of ι_1]. Moreover the constant appearing above is described by a special value of the L-function of f.

In our previous note [11], we constructed a lifting ι_0 of Jacobi cusp forms to holomorphic cusp forms on $SO(2,m+2)$ ($m\geq 1$). This was a natural generalization of the original Maass' lifting ($m=1$) and the hermitian modular case of genus two ($m=2$) by Kojima [04] and Gritsenko [01]. We also regard our construction as a Jacobi form version of Oda's lifting [06]. The purpose of this paper is to generalize Kohnen and Skoruppa's result, namely

<u>Theorem</u> Let S be a even integral positive definite symmetric matrix of rank m and f be a Jacobi cusp form of weight

k ($k > 2m+4$) with respect to Γ_S . We assume that $\mathbb{Z}_p^m$ is a maximal lattice with respect to S for any prime p . If f is a simultaneous eigen function of $\mathcal{H}'_{S,p}$ ($^\forall p$) , then the identity

$$\rho_0 \, \iota_0 \, f = C_{S.k} \, L(f;(m+2)/2) \, f$$

holds. Here ρ_0 is the adjoint operator of ι_0 , $L(f;s)$ is the L-function associated with f and $C_{S,k}$ is an explicitly determined constant.

The proof is based on the Fourier expansion of $\iota_0 f$ and $\rho_0 F$, which is essentially given in Oda [06]. Therefore our method is similar as Oda's treatment in [07] and Kohnen's first manner in [02] (see also Oda [08]).

The author sincerely express his hearty thanks to Professor T. Oda for drawing his attention to this subject and for constant encouragement.

1. Notation

S : even integral positive definite symmetric matrix of rank m ,

$Q_0 = -S$, $L_0 = \mathbb{Z}^m \subset V_0 = L_0 \otimes \mathbb{Q} = \mathbb{Q}^m$, $L_0^* = Q_0^{-1} L_0$,

$Q_1 = \begin{bmatrix} & & 1 \\ & Q_0 & \\ 1 & & \end{bmatrix}$, $L_1 = \begin{bmatrix} \mathbb{Z} \\ L_0 \\ \mathbb{Z} \end{bmatrix} = \mathbb{Z}^{m+2} \subset V_1 = \mathbb{Q}^{m+2}$, $L_1^* = Q_1^{-1} L_1$,

$Q = \begin{bmatrix} & & 1 \\ & Q_1 & \\ 1 & & \end{bmatrix}$, $L = \begin{bmatrix} \mathbb{Z} \\ L_1 \\ \mathbb{Z} \end{bmatrix} = \mathbb{Z}^{m+4} \subset V = \mathbb{Q}^{m+4}$, $L^* = Q^{-1} L$,

$G = O(Q) / \mathbb{Q}$, $\mathbb{G} = G_{\mathbb{R}}^0$: the identity component of $G_{\mathbb{R}}$,

$G_1 = O(Q_1) / \mathbb{Q}$, $\mathbb{G}_1 = G_{1,\mathbb{R}}^0$: the identity component of $G_{1,\mathbb{R}}$,

$\Gamma = \mathbb{G} \cap SL(m+4,\mathbb{Z})$, $\Gamma_1 = \mathbb{G}_1 \cap SL(m+2,\mathbb{Z})$,

$\Gamma^* = \{ \gamma \in \Gamma \; ; \; (\gamma-1)L^* \subset L \}$,

$G' = SL(2) / \mathbb{Q}$, $\mathbb{G}' = SL(2,\mathbb{R})$, $\Gamma' = SL(2,\mathbb{Z})$,

$$\mathcal{D} = \{ \mathfrak{Z} \in V_1 \otimes_{\mathbb{R}} \mathbb{C} \ ; \ Q_1[\mathrm{Im}\ \mathfrak{Z}] > 0 \}^0 ,$$

$$\mathfrak{Z}_0 = \sqrt{-1}\begin{bmatrix}1\\0\\1\end{bmatrix} \in \mathcal{D} , \quad \mathfrak{Z}^{\sim} = \begin{bmatrix}-Q_1[\mathfrak{Z}]/2\\ \mathfrak{Z}\\ 1\end{bmatrix} \quad (\mathfrak{Z} \in \mathcal{D}) ,$$

$$g\cdot\mathfrak{Z}^{\sim} = (g\langle\mathfrak{Z}\rangle)^{\sim}\ J(g,\mathfrak{Z}) ,$$

$$\mathbb{K} = \{ g \in \mathbb{G} \ ; \ g\langle\mathfrak{Z}_0\rangle = \mathfrak{Z}_0 \} \quad (\text{a maximal compact subgroup of } \mathbb{G}),$$

$$F|[g]_k(\mathfrak{Z}) = J(g,\mathfrak{Z})^{-k}\ F(g\langle\mathfrak{Z}\rangle) \quad (g \in \mathbb{G} , \mathfrak{Z} \in \mathcal{D}) ,$$

$$G_S = H_S\cdot G' = \{[\xi,\eta,\zeta]g \ ; \ \xi,\eta \in V_0 , \ \zeta \in \mathbb{Q} , \ g=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \in G'\} / \mathbb{Q} ,$$

$$[\xi,\eta,\zeta]\ g = \begin{bmatrix} 1 & 0 & {}^t\eta S & S(\xi,\eta)-\zeta & S[\eta]/2 \\ 0 & 1 & {}^t\xi S & S[\xi]/2 & \zeta \\ & & 1_m & \xi & \eta \\ & & & 1 & 0 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} a & -b & & & \\ -c & d & & & \\ & & 1_m & & \\ & & & a & b \\ & & & c & d \end{bmatrix} ,$$

$$[\xi,\eta,\zeta]\ [\xi',\eta',\zeta'] = [\xi+\xi',\eta+\eta',\zeta+\zeta'+S(\xi,\eta')] ,$$

$$g^{-1}\ [\xi,\eta,\zeta]\ g = [\xi',\eta',\zeta'] ,$$

$$(\xi',\eta') = (\xi,\eta)\ g \quad , \quad \zeta' = \zeta + S(\xi',\eta')/2 - S(\xi,\eta)/2 ,$$

$$Z = \{[0,0,\zeta]\} = \text{the center of } G_S , \quad \mathbb{G}_S = G_{S,\mathbb{R}} ,$$

$$\Gamma_S = G_S \cap \Gamma = \{[\xi,\eta,\zeta]g \ ; \ \xi,\eta \in L_0 , \ \zeta \in \mathbb{Z} , \ g \in \Gamma'\} ,$$

$$\mathcal{D}_S = \mathfrak{H} \times \mathbb{C}^m , \quad Z_0 = (\sqrt{-1},0) \in \mathcal{D}_S ,$$

$$\underline{g}\langle Z\rangle = (g\langle z\rangle, wj(g,z)^{-1}+\xi g\langle z\rangle+\eta) \quad (\underline{g} = [\xi,\eta,\zeta]g) ,$$

$$J_{S,k}(\underline{g},Z) = j(g,z)^k\ e[-\zeta+\{\tfrac{1}{2}S[w]c-S(\xi,w)\}j(g,z)^{-1}-g\langle z\rangle\tfrac{1}{2}S[\xi]],$$

$$f|[\underline{g}]_{S,k}(Z) = J_{S,k}(\underline{g},Z)^{-1}\ f(\underline{g}\langle Z\rangle) \quad (\underline{g} \in \mathbb{G}_S , Z \in \mathcal{D}_S) ,$$

$$e[z] = \exp(2\pi\sqrt{-1}\ z) \quad (z \in \mathbb{C}) .$$

2. Jacobi cusp forms

For a positive integer k we denote by $\mathfrak{S}_{S,k}(\Gamma_S)$ the space of Jacobi cusp forms of weight k and with respect to Γ_S, namely the set of holomorphic functions f on $\mathcal{D}_S$ satisfying

(i) $f|[\gamma]_{S,k} = f$ for $^\forall\ \gamma \in \Gamma_S$ and

(ii) $f|[\underline{g}]_{S,k}(Z_0)$ is bounded as a function of $\underline{g} \in \mathbb{G}_S$.

Each form f in $\mathfrak{S}_{S,k}(\Gamma_S)$ is expanded as

(1) $$f(z,w) = \sum_{\substack{a\in\mathbb{Z},\alpha\in L_0^* \\ 2a-S[\alpha]>0}} a_f(a,\alpha)\, e[az - S(\alpha,w)] \quad .$$

We introduce a inner product in $\mathfrak{S}_{S,k}(\Gamma_S)$, which is called the Petersson inner product :

(2) $$\langle f_1,f_2\rangle_{S,k} = \int_{\Gamma_S\backslash \mathfrak{D}_S} f_1(Z)\, \overline{f_2(Z)}\, y^k \exp(-2\pi y S[\xi])\, dZ \ ,$$

where $Z = (z,w)$, $z = x+\sqrt{-1}y$, $w = \xi z+\eta$, and $dZ = y^{-2}\, dx\, dy\, d\xi\, d\eta$ (dx , dy [resp. $d\xi$, $d\eta$] are the Lebesgue measures on $\mathbb{R}$ [resp. $V_{0,\mathbb{R}}$]).

3. Cusp forms on $\mathfrak{D}$

We denote by $S_k(\Gamma^*)$ the space of holomorphic functions F satisfying the following (i) and (ii) ,

(i) $F|[\gamma]_k = F \quad \forall\ \gamma \in \Gamma^*$,

(ii) $F|[g]_k(\mathfrak{Z}_0)$ is bounded as a function of $g \in \mathbb{G}$.

The Fourier expansion of F is given by

(3) $$F\left(\begin{bmatrix}\tau\\ w\\ z\end{bmatrix}\right) = \sum_{\mu\in L_1^*,\ \sqrt{-1}\mu\in\mathfrak{D}} a_F(\mu)\, e[Q_1(\mu,\begin{bmatrix}\tau\\ w\\ z\end{bmatrix})]$$
$$= \sum_{\substack{a,b\in\mathbb{N},\ \alpha\in L_0^* \\ 2ab-S[\alpha]>0}} a_F(a,\alpha,b)\, e[az - S(\alpha,w) + b\tau] \ .$$

The Petersson inner product in $S_k(\Gamma^*)$ is defined by

(4) $$\langle F_1,F_2\rangle_k = \int_{\Gamma^*\backslash \mathfrak{D}} F_1(\mathfrak{Z})\, \overline{F_2(\mathfrak{Z})}\, (\tfrac{1}{2}Q_1[\mathrm{Im}\ \mathfrak{Z}])^k\, d\mathfrak{Z} \ ,$$

where $d\mathfrak{Z} = (\frac{1}{2}Q_1[\mathrm{Im}\ \mathfrak{Z}])^{-(m+2)}\, dx_0dy_0\cdots dx_{m+1}dy_{m+1}$, $\mathfrak{Z} = (z_j)_{0\le j\le m+1}$ and $z_j = x_j + \sqrt{-1}y_j$. We often view $F = F^{dm}$ as a left Γ^*-invariant function F^{gr} on $\mathbb{G}$, through the identification $F^{gr}(g)$

$$= F^{dm}(g\langle \mathfrak{Z}_0\rangle)\ J(g,\mathfrak{Z}_0)^{-k}\ .$$

4. Lifting of Maass' type

Let f be in $\mathfrak{S}_{S,k}(\Gamma_S)$. We define a holomorphic function $\iota_0 f$ on $\mathfrak{D}$ by

$$(5)\qquad \iota_0 f\left(\begin{bmatrix}\tau\\ w\\ z\end{bmatrix}\right) = \sum_{\substack{a,b\in\mathbb{N},\alpha\in L_0^*\\ 2ab-S[\alpha]>0}}\ \sum_{\substack{r\in\mathbb{N},r^{-1}\alpha\in L_0^*\\ r\mid a,b}} r^{k-1}\ a_f(r^{-2}ab,r^{-1}\alpha) \times e[az-S(\alpha,w)+b\tau]\ .$$

It has been proved that $\iota_0 f$ belongs to $S_k(\Gamma^*)$. This lifting is a natural generalization of Siegel modular case of genus 2 (m=1) by Zagier [12] and hermitian modular case of genus 2 (m=2) by Gritsenk [01] and Kojima [04]. The proof is based on taking a suitable generator system of Γ^* and the symmetry of τ and z in the definition (5) (for the proof see [11]) . We denote by ρ_0 its adjoint mapping from $S_k(\Gamma^*)$ into $\mathfrak{S}_{S,k}(\Gamma_S)$.

$$(6)\qquad \langle \iota_0 f,F\rangle_k = \langle f,\rho_0 F\rangle_{S,k} \quad \text{for } \forall\ f\in\mathfrak{S}_{S,k}(\Gamma_S)\ ,\ \forall\ F\in S_k(\Gamma^*)\ .$$

Since ι_0 is injective from our definition, we know that ρ_0 is surjective.

5. Theta functions and the definition of Oda's lifting

For $\alpha\in L_0^*/L_0$ and $Z\in\mathfrak{D}_S$, put

$$(7)\qquad \theta_\alpha(z,w) = \sum_{\ell\in L_0^*} e[\tfrac{1}{2}zS[\ell+\alpha] - S(\ell+\alpha,w)]\ .$$

We define a Schwartz-Bruhat function $f_{z,k}$ on V by

$$f_{z,k}(v) = Q(\mathfrak{Z}_0^{\sim}, v)^k \, e[\tfrac{1}{2} Q_z[v]] \ ,$$

where $Q_z = xQ + \sqrt{-1}yR$ and $R = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & S & \\ & & & 1 \\ & & & & 1 \end{bmatrix}$.

We introduce a kind of theta function on $\mathfrak{D}_S \times \mathbb{G}$ which plays a role of the kernel function of Oda's lifting.

(8) $$\Theta_k(Z,g) = \sum_{\mu \in L^*/L} \theta_k(z,g;\mu) \, \theta_{\pi(\mu)}(Z) \ ,$$

where $\theta_k(z,g;\mu) = \sum_{\ell \in L} y^{(m+2)/2} f_{z,k}(g^{-1}(\ell+\mu))$ and $\pi : L^* \longrightarrow L_0^*$ is the natural projection. From the transformation formula of $\theta_\alpha(z,w)$ and $\theta_k(z,g;\mu)$ under the action of Γ' (cf. Shintani [09]), we easily obtain that

(9) $$\Theta_k(\gamma\langle Z\rangle, \gamma g \kappa) = J_{S,k}(\gamma, Z) \, J(\kappa, \mathfrak{Z}_0)^k \, \Theta_k(Z,g) \qquad \forall \ \gamma \in \Gamma_S, \ \gamma \in \Gamma, \ \kappa \in \mathbb{K} \ .$$

For an $x \in V_1$ and a $y \in V_0$ we set

$$n(x) = \begin{bmatrix} 1 & -{}^t x Q_1 & -Q_1[x]/2 \\ & 1_{m+2} & x \\ & & 1 \end{bmatrix} \quad \text{and} \quad n'(y) = \begin{bmatrix} 1 & -{}^t y Q_0 & -Q_0[y]/2 \\ & 1_m & y \\ & & 1 \end{bmatrix} .$$

For $g_0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R})$ ($\det g_0 = N$), we put

$$g_0^{\sim} = \begin{bmatrix} a & b & & & \\ c & d & & & \\ & & 1_m & & \\ & & & a/N & -b/N \\ & & & -c/N & d/N \end{bmatrix} .$$

It is easy to see $\mathbb{K} = \mathbb{K}_0 \cdot \mathbb{K}_1$ (semi-direct product) , where $\mathbb{K}_0 = \{ \kappa_\theta^{\sim} \ ; \ 0 \le \theta < 2\pi, \ \kappa_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \}$. and $\mathbb{K}_1 = \{ \kappa \in \mathbb{K} \ ; \ J(\kappa, \mathfrak{Z}_0) = 1 \}$. We normalize a Haar measure $\omega_0(g)$ on $\mathbb{G}$ by

(10) $$\int_{\mathbb{G}} \Phi(g) \, \omega_0(g)$$
$$= \int_{V_1} \int_{V_0} \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_{\mathbb{K}_1} \Phi(n(x) \, n'(y) \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^{\sim} \kappa_\theta^{\sim} \, \kappa_1)$$
$$\pi^{-1} \, a^{-(m+3)} \, b^{-(m+1)} \, dx \, dy \, da \, db \, d\theta \, d\kappa_1 \ ,$$

where dx, dy, da, db, $d\theta$ are the Lebesgue measures and $\int_{\mathbb{K}_1} d\kappa_1 = 1$. Hence the identity $\int_{\mathbb{G}} \Phi(g) \, \omega_0(g) = \int_{\mathfrak{D}} \Phi(g) \, d(g\langle \mathfrak{Z}_0 \rangle)$ holds for any

integrable right $\mathbb{K}$-invariant function Φ on $\mathbb{G}$.

Now let us introduce a mapping ι from $\mathfrak{S}_{S,k}(\Gamma_S)$ into $S_k(\Gamma^*)$ and its adjoint mapping ρ .

(11) $$\iota f(g) = \int_{\Gamma_S \backslash \mathfrak{D}_S} f(Z)\ \overline{\theta_k(Z,g)}\ y^k \exp(-2\pi y S[\xi])\ dZ \quad ,$$

(12) $$\rho F(Z) = \int_{\Gamma^* \backslash \mathbb{G}} F(g)\ \theta_k(Z,g)\ \omega_0(g) \quad ,$$

(13) $$\langle \iota f, F\rangle_k = \langle f, \rho F\rangle_{S,k} \qquad \forall\ f \in \mathfrak{S}_{S,k}(\Gamma_S)\ ,\ \forall\ F \in S_k(\Gamma^*)\ .$$

The convergence and well-definedness are assured by T. Oda [05] for $k > 2m + 4$. Our aim in this note is to prove that $\rho\iota f$ is a constant multiple for any Hecke eigen function f in $\mathfrak{S}_{S,k}(\Gamma_S)$ and to determine the constant appearing there.

<u>Remark</u> Any Jacobi form f in $\mathfrak{S}_{S,k}(\Gamma_S)$ is written as $f(z,w) = \sum_{\alpha \in L_0^*/L_0}{}^{*} f_\alpha(z)\ \theta_\alpha(z,w)$. Then $(f_\alpha(z))$ becomes a vector valued holomorphic cusp form of weight $k - m/2$ with respect to $\Gamma' = SL(2,\mathbb{Z})$. Moreover each component $f_\alpha(z)$ is a usual cusp form with respect to some congruence subgroup.

6. Fourier coefficients of Oda's lifting

First we note that Oda's lifting ι coincides with Maass' lifting ι_0 up to constant multiple.

(A) $$\iota f(g)\ J(g,\mathfrak{I}_0)^k = C(\iota)\ \iota_0 f(g\langle\mathfrak{I}_0\rangle) \quad ,$$

where $C(\iota) = (-\sqrt{-1})^k\ 2^{k-m/2}\ (\det S)^{-1/2}$.

Secondly we shall describe Fourier coefficients of ρF . We needs some notation for this purpose. Take any $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix} \in L_1^*$ and

assume $Q_1[\eta] > 0$.

Put $\Delta_\eta = \frac{1}{2} Q_1[\eta] = a - \frac{1}{2} S[\alpha] > 0$,

$g_\eta \in G_1 \ : \ g_\eta \mathfrak{X}_0 = \sqrt{-1}\ \Delta_\eta^{-1/2}\ \eta$,

$\underline{S}_\eta = \begin{bmatrix} S & S\alpha \\ {}^t\alpha S & 2a \end{bmatrix}$, $\underline{Q}_\eta = \begin{bmatrix} & & 1 \\ & & \\ 1 & & -\underline{S}_\eta \end{bmatrix}$,

$\underline{G}_\eta = O(\underline{Q}_\eta)$, $\underline{\mathbb{G}}_\eta = \underline{G}^0_{\eta,\mathbb{R}}$: the identity component of $\underline{G}_{\eta,\mathbb{R}}$,

$\underline{\mathfrak{D}}_\eta = \{ X = (x,r) \ ; \ x \in \mathbb{R}^{m+1} \ , \ r > 0 \}$ ($X_0 = (0,1) \in \underline{\mathfrak{D}}_\eta$) ,

$g \cdot X^\sim = (g\{X\})^\sim \ j(g,X)$ ($X^\sim = \begin{bmatrix} r+\underline{S}_\eta[x]/2 \\ x \\ 1 \end{bmatrix}$, $g \in \underline{\mathbb{G}}_\eta$) ,

$\underline{\mathbb{K}}_\eta$ = the stabilizer subgroup of $X_0 \simeq SO(m+2)$.

It is clear that $\underline{G}_\eta$ is identified to $G_{\eta^\sim} = \{ g \in G \ ; \ g\,\eta^\sim = \eta^\sim \}$ ($\eta^\sim = \begin{bmatrix} 0 \\ \eta \\ 0 \end{bmatrix}$) . We identify $\underline{V}_\eta = \mathbb{Q}^{m+1}$ with the orthogonal complement of η in V_1 . Let us define a Haar measure on $\underline{\mathbb{G}}_\eta$ by

(14) $$\int_{\underline{\mathbb{G}}_\eta} \Phi(g)\ \underline{\omega}_\eta(g)$$

$$= \int_{\underline{V}_{\eta,\mathbb{R}}} \int_0^\infty \int_{\underline{\mathbb{K}}_\eta} \Phi(\underline{n}(x) \begin{bmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{bmatrix} \kappa)\ a^{-(m+2)}\ dx\ da\ d\kappa \ ,$$

where dx and da are Lebesgue measures and $\int_{\underline{\mathbb{K}}_\eta} d\kappa = 1$. By using the above notation, we can write down Fourier coefficients of ρF .

(B) $$\rho F(Z) = C(\rho) \sum_{\substack{a\in\mathbb{N},\alpha\in L_0^*/L_0 \\ 2\Delta_\eta = 2a - S[\alpha] > 0}} \Delta_\eta^{(k-m-1)/2} \int_{\Gamma_\eta^* \backslash \underline{\mathbb{G}}_\eta} F(h\ g_\eta)\ \underline{\omega}_\eta(h)$$

$$\times \sum_{n\in\mathbb{N}} \theta_{n\alpha}(Z)\ e[n^2 \Delta_\eta z]\ n^{k-(m+2)} \ ,$$

where $\Gamma_\eta^* = \underline{\mathbb{G}}_\eta \cap \Gamma^*$ and $C(\rho) = \sqrt{-1}^k\ 2^{k-m/2}\ (\det S)^{-1/2} = \overline{C(\iota)}$.

Theorem **(A)** and **(B)** , which are essentially given in [06], are proved by chasing the Oda's arguments carefully (we need to rewrite all his discussion in terms of Jacobi forms and to determine some constants explicitly).

7. Hecke theory for Jacobi forms

First we introduce the notion of Hecke algebra action on $\mathfrak{S}_{S,k}(\Gamma_S)$ after T.Shintani. For any prime p, we denote by $\mathcal{H}_{S,p}$ the set of functions ϕ on $G_{S,p}$ satisfying

(15-i) $\phi([0,0,\xi]\kappa_1 g\kappa_2) = \chi_p(\xi)\,\phi(g)$ for any $\xi \in \mathbb{Q}_p$, $\kappa_1,\kappa_2 \in K'_{S,p}$

and

(15-ii) the support of ϕ is compact modulo Z_p,

where $K'_{S,p} = G'_{S,p} \cap SL(m+4,\mathbb{Z}_p)$ and χ_p is the p-component of the character χ of $\mathbb{Q}\backslash\mathbb{Q}_A$ ($\chi_\infty(x) = e[x]$). Clearly $\mathcal{H}_{S,p}$ forms a $\mathbb{C}$-algebra with the convolution product :

(16) $$\phi_1 * \phi_2(g) = \int_{Z_p\backslash G_{S,p}} \phi_1(gg'^{-1})\,\phi_2(g')\,dg' ,$$

where we normalize the measure dg' as $\mathrm{vol}(Z_p\backslash Z_p K_{S,p}) = 1$. Hereafter we assume that $L_{0,p} = L_0 \otimes \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-integral lattice with respect to S for any prime p ; namely, if M is a lattice in $V_{0,p}$ containing $L_{0,p}$ and satisfies $\frac{1}{2}S[x] \in \mathbb{Z}_p$ for any $x \in M$, then $M = L_{0,p}$. We define two elements $\phi_{1,p}$ and $\phi_{0,p}$ in $\mathcal{H}_{S,p}$ as follows :

(17) $$\begin{aligned} &\phi_{1,p}\left(\begin{bmatrix} p & \\ & p^{-1}\end{bmatrix}\right) = 1 , \quad \mathrm{supp}\,\phi_{1,p} = Z_p K_{S,p}\begin{bmatrix} p & \\ & p^{-1}\end{bmatrix}K_{S,p} , \\ &\phi_{0,p}([0,y,0]) = p^{-\partial_p} \quad (\forall y \in L'_{0,p}), \\ &\mathrm{supp}\,\phi_{1,p} = Z_p K_{S,p}\{[0,y,0]; y\in L'_{0,p}\}K_{S,p} , \end{aligned}$$

where we put $L'_{0,p} = \{x \in L^*_{0,p} ; \frac{1}{2}S[x] \in p^{-1}\mathbb{Z}_p\}$ and ∂_p stands for the dimension of $L'_{0,p}/L_{0,p}$ over $\mathbb{Z}_p/p\mathbb{Z}_p$. We note $0 \le \partial_p \le 2$. The next relations are easily checked.

$$\phi_{1,p} * \phi_{0,p} = \phi_{0,p} * \phi_{1,p} = \phi_{1,p} ;$$

(18) $\phi_{0,p} = 1$ if $\partial_p = 0$,

$\phi_{0,p}^{\ 2} = 1$ if $\partial_p = 1$,

$\phi_{0,p}^{\ 2} = (1 - p^{-1})\phi_{0,p} + p^{-1}$ if $\partial_p = 2$.

We denote by $\mathcal{H}'_{S,p}$ the commutative subalgebra of the Hecke algebra $\mathcal{H}_{S,p}$ generated by $\phi_{0,p}$ and $\phi_{1,p}$. For any character λ of $\mathcal{H}'_{S,p}$, local L-function $L_p(\lambda;s)$ is defined as follows :

(19) $$L_p(\lambda;s) = B_{S,p}(p^{-s}) \left\{ \begin{matrix} (1 - \chi_S(p)p^{-s})^{-1} & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{matrix} \right\}$$

$$\times\{1 - (\lambda_1 p^{-(1+m/2)} - p^{\partial_p - n_0/2} + p^{-1+n_0/2})p^{-s} + \lambda_0^{-1}p^{-2s}\}^{-1}.$$

Here ν_p = the Witt index of S over $\mathbb{Q}_p$, $n_0 = m - 2\nu_p$,

$\chi_S(p) = \left(\frac{D_0(S)}{p}\right)$, $D(S) = (-1)^{m(m-1)/2}(\det S)$,

$D_0(S)$ = the discriminant of $\mathbb{Q}(D(S)^{1/2})$,

$\lambda_j = \lambda(\phi_{j,p})$ $(j=0,1)$,

$B_{S,p}(T) = 1$ when $(n_0,\partial_p) = (0,0), (1,0), (2,0), (2,1)$,

$= (1 + p^{1/2}T)$ when $(n_0,\partial_p) = (1,1)$,

$= (1 - p^{1/2}T)$ when $(n_0,\partial_p) = (3,1)$,

$= (1 + p^{1/2}T)(1 - p^{1/2}T)$ when $(n_0,\partial_p) = (3,2)$,

$= (1 + pT)(1 + T)$ when $(n_0,\partial_p) = (2,2)$,

$= (1 - pT)(1 - T)$ when $(n_0,\partial_p) = (4,2)$,

$B_S(s) = \prod_p B_{S,p}(p^{-s})$, $L(\chi_S;s) = \prod_p (1 - \chi_S(p)p^{-s})^{-1}$.

We let $\mathcal{H}'_{S,p}$ act on $\mathfrak{S}_{S,k}(\Gamma_S)$ naturally. As usual, $\mathfrak{S}_{S,k}(\Gamma_S)$ has an basis consisting of common eigen function of all $\mathcal{H}'_{S,p}$ $(\forall p)$. Let f be such a simultaneous eigen function . When $f*\phi_{j,p} = \lambda(\phi_{j,p})f$, we define the global L-function $L(f;s)$ as the product $\prod_p L_p(\lambda;s)$ of local ones. We put

$$(20)\qquad \xi(f;s) = 2^{-s}\,\pi^{-3s/2}\,(\det S)^{s/2}\left\{\begin{matrix}\Gamma(\frac{s+1}{2}) & m\equiv 0 \pmod 4\\ \Gamma(\frac{s}{2}) & m\equiv 2 \pmod 4\end{matrix}\right\}$$

$$\times\,\Gamma(s+k-\tfrac{m+2}{2})\,L(f;s) \qquad \text{if } m:\text{even},$$

$$= (2\pi)^{-s}\,(2^{-1}\det S)^{s/2}\,\Gamma(s+k-\tfrac{m+2}{2})\,L(f;s) \text{ if } m:\text{odd}.$$

On the analytic continuation and the functional equation of $L(f;s)$ the following theorem holds.

(C) When $k > (m+1)/2$, $\xi(f;s)$ is continued to whole s-plane as a meromorphic function with possible poles at $s = 0, 1$. It satisfies the functional equation $\xi(f;s) = \varepsilon_S\,\xi(f;1-s)$, where ε_S is -1 if $m \equiv 1$ or $3 \pmod 8$ and $+1$ otherwise. Moreover except the case $m \equiv 6 \pmod 8$, $\xi(f;s)$ is entire.

<u>Remark</u> The definition of the Hecke algebra $\mathcal{H}_{S,p}$ is given by T. Shintani in more general context (see for example A. Murase [05]) It is not difficult to see that $\mathcal{H}'_{S,p}$ coincides with $\mathcal{H}_{S,p}$ if $\partial_p = 0$ or 1. On the other hand, when $\partial_p = 2$, $\mathcal{H}'_{S,p}$ is a proper subalgebra of $\mathcal{H}_{S,p}$; in particular $\mathcal{H}'_{S,p}$ is not commutative.

An element $\eta = \begin{bmatrix} a \\ \alpha \\ 1\end{bmatrix}$ in L_1^* is said to be reduced if $\underline{L} = \begin{bmatrix} L_0 \\ \mathbb{Z}\end{bmatrix}$ is a maximal lattice with respect to $\underline{S}_\eta$. We note that there exists some reduced η such that $a_f(a,\alpha) \neq 0$. For such η we have

$$\textbf{(D)}\qquad \sum_{n=1}^{\infty} a_f(n^2a, n\alpha)\, n^{-(s+k-(m+2)/2)}$$

$$= L(f;s)\, B_{\underline{S}_\eta}(s+1/2)\left\{\begin{matrix}\zeta(2s)^{-1} & m:\text{even}\\ L(\chi_{\underline{S}_\eta}, s+1/2)^{-1} & m:\text{odd}\end{matrix}\right\} a_f(a,\alpha).$$

Rewriting this by adelic language, we get

$$\textbf{(D')}\qquad \int_{\mathbb{Q}_A^\times} (\iota_0 f)^{gr}_\eta\left(\begin{bmatrix} t & & \\ & 1_{m+2} & \\ & & t^{-1}\end{bmatrix} g_\eta\right) |t|_A^{s-(m+2)/2}\, d^\times t$$

$$= a_f(a,\alpha)\ \zeta(s-m/2)\ L(f;s)\ B_{\underline{S}_\eta}(s+1/2)^{-1}$$

$$\times \frac{\Gamma(s+k-(m+2)/2)}{(4\pi\sqrt{\Delta_\eta})^{s+k-(m+2)/2}} \left\{ \begin{array}{ll} \zeta(2s)^{-1} & m:\text{even} \\ L(\chi_{\underline{S}_\eta};s+1/2)^{-1} & m:\text{odd} \end{array} \right\} .$$

where for a cusp form F on G_A we have put

(21) $$F_\eta(g) = \int_{V_{1,\mathbb{Q}}\backslash V_{1,A}} F(n(x)g)\ \chi(-Q_1(\eta,x))\ dx .$$

Theorem **(D)** is shown by the standard discussion, namely the the study of the action of $\mathcal{H}'_{S,p}$ on Fourier coefficients of f. The left hand side of this theorem is represented as a convolution product of f and a kind of Eisenstein series on $G_{\underline{S}_\eta}$ (this representation was given by T.Shintani in more general context). Therefore **(C)** is reduced to the analytic property of this Eisenstein series. By calculating all Fourier coefficients of it, we can verify **(C)**. For more precise treatment, refer our forthcoming paper.

Moreover by comparing the actions of Hecke operators of G and G_S, we can show that

(E) If f is a common eigen function of $\mathcal{H}'_{S,p}$ for any p, then $\rho_\ell f$ is also eigen function with same eigenvalues.

8. Eisenstein series on $O(1,q+1)$

Let T be an even integral positive definite symmetric matrix of rank q. Put $T^\sim = \begin{bmatrix} & & 1 \\ & -T & \\ 1 & & \end{bmatrix}$, $H = O(T)$ and $H^\sim = O(T^\sim)$. We put $L = \mathbb{Z}^q$, $L^* = T^{-1}L$, $L^\sim = \mathbb{Z}^{q+1}$ and $L^{\sim *} = T^{\sim -1}L^\sim$. For each prime p, we assume that $L_{0,p} = L_0 \otimes \mathbb{Z}_p$ is a maximal lattice with respect to T and set

$$U_p = \{ g \in H_p \ ; \ g\, L_p = L_p \} ,$$
$$U_p^{\sim} = \{ g \in H_p^{\sim} \ ; \ g\, L_p^{\sim} = L_p^{\sim} \} ,$$
$$U_p^{*} = \{ g \in U_p \ ; \ (g-1)\, L_p^{*} \subset L_p \} ,$$
$$U_p^{\sim *} = \{ g \in U_p^{\sim} \ ; \ (g-1)\, L_p^{\sim *} \subset L_p^{\sim} \} .$$

$H_\infty^{\sim}$ acts on $D = \{ X = (x,r) \ ; \ x \in \mathbb{R}^q , \ r > 0 \}$ through

$$g \cdot X^{\sim} = (g\{X\})^{\sim} \ j(g,X) ,$$

where $X^{\sim} = \begin{bmatrix} r+T[x]/2 \\ x \\ 1 \end{bmatrix}$, $X = (x,r) \in D$. We denote by $U_\infty^{\sim} = U_\infty^{\sim *}$ the stabilizer subgroup of $X_0 = (0,1)$ and put $U_\infty = U_\infty^{*} = H_\infty$. As is well-known, the Iwasawa decomposition and the class number one property holds :

(22) $\quad H_A^{\sim} = P_A^{\sim}\, U_A^{\sim *}$ ($P^{\sim}$ denotes the upper triangular subgroup) ,

$$= H_{\mathbb{Q}}^{\sim}\, H_\infty^{\sim}\, U_{A,f}^{\sim *} .$$

Hence we can write g as

$$g = n(x) \begin{bmatrix} \alpha(g) & & \\ & \beta(g) & \\ & & \alpha(g)^{-1} \end{bmatrix} k(g) \quad , \quad x \in V_A = \mathbb{Q}_A{}^q \ , \ \alpha(g) \in \mathbb{Q}_A^{\times} ,$$

$$\beta(g) \in H_A^{\sim} \quad \text{and} \quad k(g) \in U_A^{\sim} .$$

We consider a space $\mathcal{V}_H$ of automorphic forms on H and a linear operator $M(s;*)$ acting on it.

(23) $\quad \mathcal{V}_H = \{ f\colon H_{\mathbb{Q}} \backslash H_A \,/\, U_A^{*} \longrightarrow \mathbb{C} \}$ (finite dimensional) .

(24) $\quad M(s;f)(h) = \int_{V_A} f(h\ \beta(n^{\vee}(x)))\ |\alpha(n^{\vee}(h))|_A^{s+q/2}\, dx$,

where $n^{\vee}(x) = \begin{bmatrix} 1 & & \\ x & 1_q & \\ T[x]/2 & {}^t xT & 1 \end{bmatrix}$ and $\int_{V_A/V_{\mathbb{Q}}} dx = 1$.

Now we introduce Eisenstein series attached to f .

(25) $\quad E(g,s;f) = \sum_{\gamma \in P_{\mathbb{Q}}^{\sim} \backslash H_{\mathbb{Q}}^{\sim}} f(\beta(\gamma g))\ |\alpha(\gamma g)|_A^{s+q/2} \quad (g \in H_A^{\sim})$.

By virtue of Langlands' general theory on Eisenstein series, the analytic properties of $E(g,s;f)$ is reduced to that of its constant

term $E_0(g,s;f) = \int_{V_{\mathbb{Q}}\backslash V_A} E(n(x)g,s;f)\,dx$. Using the Bruhat decomposition

(26) $$H_{\tilde{\mathbb{Q}}} = P_{\tilde{\mathbb{Q}}} \sqcup P_{\tilde{\mathbb{Q}}}\, w\, N_{\tilde{\mathbb{Q}}} \quad , \quad w = \begin{bmatrix} & & 1 \\ & 1_q & \\ 1 & & \end{bmatrix} \in H_{\tilde{\mathbb{Q}}} \cap U_A^* ,$$

we obtain

(27) $$E_0\left(\begin{bmatrix} a & & \\ & b & \\ & & a^{-1} \end{bmatrix}, s; f\right) = f(b)\,|a|_A^{s+q/2} + M(s;f)(b)\,|a|_A^{-s+q/2} .$$

We consider a suitable subalgebra $\mathcal{H}'_p$ of the Hecke algebra associated with the pair (H_p, U_p^*) . For each $\mathcal{H}'_p$ eigen function f in $\mathcal{V}_H$, we can define a L-function $L(f;s)$. Then $M(s;f)$ is described by this L-function.

(F) If $f \in \mathcal{V}_H$ is a simultaneous eigen function of $\mathcal{H}'_p$ (for any prime p) , then

$$M(s;f) = \frac{L(f;s)}{L(f;s+1)} \frac{(2\pi)^{q/2}}{(\det T)^{1/2}} \frac{\Gamma(s)}{\Gamma(s+q/2)} \left\{ \begin{matrix} 1 & q:\text{even} \\ \dfrac{\zeta(2s)}{\zeta(2s+1)} & q:\text{odd} \end{matrix} \right\} \times f .$$

For $f = \mathbf{1}$ (the constant function on H_A with value 1), our L-function is written by usual Riemann zeta functions.

(28) $$L(\mathbf{1};s) = \prod_{j=1}^{q-1} \zeta(s+j-q/2) \times B_T(s) \times \left\{ \begin{matrix} 1 & q:\text{odd} \\ L(\chi_T,s) & q:\text{even} \end{matrix} \right\} ,$$

(29) $$M(s;\mathbf{1}) = \frac{(2\pi)^{q/2}}{(\det T)^{1/2}} \frac{\Gamma(s)}{\Gamma(s+q/2)} \frac{\zeta(s+1-q/2)}{\zeta(s+q/2)} \frac{B_T(s)}{B_T(s+1)} \times \left\{ \begin{matrix} \zeta(2s)\ \zeta(2s+1)^{-1} & q:\text{odd} \\ L(\chi_T;s)\ L(\chi_T;s+1)^{-1} & q:\text{even} \end{matrix} \right\} \times \mathbf{1} .$$

Therefore we have obtained that

(30) $$\begin{aligned} \operatorname*{Res}_{s=q/2} E(g,s;\mathbf{1}) &= \operatorname*{Res}_{s=q/2} M(s;\mathbf{1}) \\ &= \frac{(2\pi)^{q/2}}{(\det T)^{1/2}} \frac{\Gamma(q/2)}{\Gamma(q)} \frac{\operatorname{Res}_{s=1} \zeta(s)}{\zeta(q)} \frac{B_T(q/2)}{B_T(q/2+1)} \\ &\quad \times \left\{ \begin{matrix} \zeta(q)\ \zeta(q+1)^{-1} & q:\text{odd} \\ L(\chi_T;q/2)\ L(\chi_T;q/2+1)^{-1} & q:\text{even} \end{matrix} \right\} . \end{aligned}$$

For any reduced η with $\Delta_\eta > 0$, $\underline{S}_\eta$ fulfills the assumption

on T. So we can apply the above results to the case $T = \underline{S}_\eta$ ($q = m+1$). Define a Haar measure dh on $H_A^\sim$ as follows :

(31) $$\int_{G_A} \Phi(h)\, dh$$
$$= \int_{\underline{V}_A} \int_{\mathbb{Q}_A^\times} \int_{H_A} \int_{U_A^\sim} \Phi(n(x)\begin{bmatrix} a & b \\ & a^{-1}\end{bmatrix} u)\ |a|_A^{-(m+2)}\, dx\, da\, db\, du\ ,$$

where $\int_{U_v^*} db_v = 1$ and $\int_{U_v^{\sim *}} du_v = 1$. Then it is easily seen that

(32) $$\int_{H_{\mathbb{Q}}^\sim \backslash H_A^\sim} F(h\, g_\eta)\, dh = \int_{\Gamma_\eta^* \backslash G_\eta} F(h\, g_\eta)\, \underline{\omega}_\eta(h) \qquad (F \in S_k(\Gamma^*)).$$

On account of the argument in the study of L-functions associated with F (cf. [10]), we obtain

(G) $$\int_{H_{\mathbb{Q}}^\sim \backslash H_A^\sim} (\iota f)^{gr}(h\, g_\eta)\, E(h, s-1/2; \mathbf{1})\, dh$$
$$= \mathrm{vol}(H_{\mathbb{Q}} \backslash H_A) \int_{\mathbb{Q}_A^\times} (\iota f)_\eta^{gr}(\begin{bmatrix} t & \\ & t^{-1}\end{bmatrix} g_\eta)\ |t|_A^{s-(m+2)/2}\, d^\times t\ .$$

<u>Remark</u> Since $E(g,s;\mathbf{1})$ is continued meromorphically to whole s-plane and satisfies the functional equation $E(g,s;\mathbf{1}) = M(s;\mathbf{1})E(g,-s;\mathbf{1})$, the meromorphic continuation and the functional equation of $\xi(f;s)$ are also proved by **(G)**.

9. Siegel formula

From the celebrated Siegel's theorem on quadratic forms, we can calculate the volume of $H_{\mathbb{Q}} \backslash H_A$.

(H) $$\mathrm{vol}(H_{\mathbb{Q}} \backslash H_A) = 2^{1-q}\, \gamma_q^{-1} \prod_{j=1}^{[(q-1)/2]} \zeta(2j)\, B_T(\tfrac{q}{2})\, (\det T)^{(q-1)/2}$$
$$\times \begin{cases} 1 & q:\text{odd} \\ L(\chi_T; q/2) & q:\text{even} \end{cases} ,$$

where $\gamma_q = \pi^{q(q+1)/4} \prod_{j=1}^{q} \Gamma(j/2)^{-1}$, $H = O(T)$ and $\mathrm{vol}(U_v^*) = 1$.

10. Main result

Theorem Assume $k > 2m + 4$ and let $f \in \mathfrak{S}_{S,k}(\Gamma_S)$ be a simultaneous eigen function of $\mathcal{H}'_{S,p}$ for all p. Then

$$\rho_0 \iota_0 f = C_{S,k}\, L(f;(m+2)/2)\, f ,$$

where $C_{S,k} = (4\pi)^{-k}\, \Gamma(k)\, C_S$,

$$C_S = (\det S)^{(m+1)/2} \prod_{j=1}^{[(m+1)/2]} B_{2j}$$

$$\times \begin{cases} 2^{-m/2}\, \pi^{-1-m/2} & \text{if } m \text{ is even} \\ 2^{-(m+1)/2}\, \Gamma(\frac{m+3}{2})^{-1} & \text{if } m \text{ is odd} \end{cases} ,$$

B_{2j} is the $2j$-th Bernoulli number.

Consequently, we obtain the connection between the norm of f and that of $\iota_0 f$.

Corollary Notation and assumption being as above,

$$\langle \iota_0 f, \iota_0 f \rangle_k = C_{S,k}\, L(f;(m+2)/2)\, \langle f, f \rangle_{S,k} .$$

Remark 1 In particular, when $m = 1$ and $S = 2$, $C_{S,k} = (4\pi)^{-k}\, \Gamma(k)\, 6^{-1}$. This coincides with the Kohnen-Skoruppa's results ([03]). Their method, which is based on the study of Eisenstein series on $Sp(2,\mathbb{R})$, is entirely different from ours. Probably our inner product formula is obtained also by calculating the constant term of Eisenstein series on $O(2,m+2)$. However, our theorem is not reduced to this formula, because the multiplicity one property is not known for Jacobi forms.

Remark 2 Taking account of the connection between Jacobi cusp forms of weight k and elliptic cusp forms of weight $k-m/2$, it is easily checked from Rankin's evaluation that the Dirichlet series

in (**D**) converges absolutely at $s = (m+2)/2$ for even m. As is shown in [07] ($m=1$) by using the Shimura-Shintani correspondence, it is probably valid for odd m. If so, the non-vanishingness of the special value of L-function appearing in the above theorem is also verified directly.

11. Proof of Theorem

Take any reduced $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix} \in L_1^*$ with $\Delta_\eta > 0$ and put $T = \underline{S}_\eta$. We calculate the residue at $s = (m+2)/2$ of

$$(33) \qquad \int_{H_{\tilde{\mathbb{Q}}}\backslash H_{\tilde{A}}} (\iota_0 f)^{gr}(h\, g_\eta)\, E(h, s-1/2; \mathbf{1})\, dh \quad .$$

First, from (30) and (**G**) we have

$$(34) \qquad \operatorname*{Res}_{s=(m+1)/2} M(s;\mathbf{1}) \times \int_{H_{\tilde{\mathbb{Q}}}\backslash H_{\tilde{A}}} (\iota_0 f)^{gr}(h\, g_\eta)\, dh$$

$$= \mathrm{vol}(H_{\mathbb{Q}}\backslash H_A) \times \operatorname*{Res}_{s=(m+2)/2} \int_{\mathbb{Q}_A^\times} (\iota_0 f)^{gr}_\eta\left(\begin{bmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{bmatrix} g_\eta\right) |t|_A^{s-(m+2)/2}\, d^\times t$$

Secondly, accounting (32) we get

$$(35) \qquad \int_{\Gamma_\eta^*\backslash \mathbb{G}_\eta} (\iota_0 f)^{gr}(h\, g_\eta)\, \underline{\omega}_\eta(h)$$

$$= \frac{\mathrm{vol}(H_{\mathbb{Q}}\backslash H_A)}{\operatorname*{Res}_{s=(m+1)/2} M(s;\mathbf{1})} \operatorname*{Res}_{s=(m+2)/2} \int_{\mathbb{Q}_A^\times} (\iota_0 f)^{gr}_\eta\left(\begin{bmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{bmatrix} g_\eta\right) |t|_A^{s-(m+2)/2}\, d^\times t$$

$$= (\det T)^{m/2}\, 2^{-m}\, \gamma_{m+1}^{-1}\, B_T\left(\tfrac{m+1}{2}\right) \prod_{j=1}^{[m/2]} \zeta(2j) \left\{ \frac{1}{L(\chi_T;(m+1)/2)} \right\}$$

$$\times \frac{(\det T)^{1/2}}{(2\pi)^{(m+1)/2}} \frac{\Gamma(m+1)}{\Gamma((m+1)/2)} \frac{\zeta(m+1)}{\operatorname*{Res}_{s=1} \zeta(s)} \frac{B_T((m+3)/2)}{B_T((m+1)/2)}$$

$$\times \left\{ \frac{\zeta(m+2)\, \zeta(m+1)^{-1}}{L(\chi_T;(m+3)/2)\, L(\chi_T;(m+1)/2)^{-1}} \right\}$$

$$\times\, a_f(a,\alpha) \operatorname*{Res}_{s=1} \zeta(s)\, L(f;(m+2)/2)\, B_T\left(\tfrac{m+3}{2}\right)^{-1} \Delta_\eta^{-k/2}\, (4\pi)^{-k}\, \Gamma(k)$$

$$\times \left\{ \begin{matrix} \zeta(m+2)^{-1} \\ L(\chi_T;(m+3)/2)^{-1} \end{matrix} \right\}$$

(we used (**H**), (30), (**G**))

$$= (4\pi)^{-k}\ \Gamma(k)\ 2^{-m}\ \pi^{-(m+1)/2}\ \gamma_{m+1}^{-1}\ \Delta_\eta^{(m+1-k)/2}\ \frac{\Gamma(m+1)}{\Gamma((m+1)/2)}$$

$$\times \prod_{j=1}^{[(m+1)/2]} \zeta(2j)\ \ L(f;(m+2)/2)\ \ a_f(a,\alpha)$$

(note $\det T = 2\Delta_\eta\ (\det S)$) .

Since $\rho_0 F = \overline{C(\iota)}^{-1}\ \rho F$ for any F in $S_k(\Gamma^*)$ and $C(\rho) = \overline{C(\iota)}$, we obtain for any reduced $\eta = \begin{bmatrix} a \\ \alpha \\ 1 \end{bmatrix}$

(36) $$a_{\rho_0 \iota_0 f}(a,\alpha) = C_{S,k}\ L(f;(m+2)/2)\ a_f(a,\alpha)$$

(we used (**B**) and (35)).

Put $f' = \rho_0 \iota_0 f - C_{S,k}\ L(f;(m+2)/2)\ f$. It belongs to the same eigen space as f and $a_{f'}(a,\alpha) = 0$ for any reduce η . So f' must be 0 ((**E**)) and our theorem is now proved.

References

[01] Gritsenko, V. : Maass space for SU(2,2) , Hecke algebra and zeta function ; preprint.

[02] Kohnen, W. : On the Petersson norm of a Siegel-Hecke eigen form of degree two in the Maass space ; J. reine angew. Math. 357 (1985), 96-100.

[03] Kohnen, W. and Skoruppa, N.-P. : A certain Dirichlet series attached to Siegel modular forms of degree two ; preprint.

[04] Kojima, H. : An arithmetic of hermitian modular forms of degree two ; Invent. Math. 69 (1982), 217-227.

[05] Murase, A. : L-functions attached to Jacobi forms of degree n ,

Part I. The basic identity ; preprint.

[06] Oda, T. : On modular forms associated with indefinite quadratic forms of signature (2,n-2) ; Math. Ann. 231 (1977), 97-144.

[07] Oda, T. : On the poles of Andrianov L-functions ; Math. Ann. 256 (1981), 323-340.

[08] Oda, T. : Periods of Hilbert modular surfaces ; Progress in Math. 19, Birkhauser, 1982.

[09] Shintani, T. : On construction of holomorphic cusp forms of half integral weight ; Nagoya Math. J. 58 (1975), 83-126.

[10] Sugano, T. : On Dirichlet series attached to holomorphic cusp forms on SO(2,q) ; Advanced studies in Pure Math. vol. 7, Kinokuniya, 1985, 333-362.

[11] Sugano, T. : A note on Maass' type lifting of Jacobi forms ; preprint.

[12] Zagier, D. : Sur la conjecture de Saito-Kurokawa (d'apres H.Maass) ; Sem. Delange-Pisot-Poitou, 1979/80, Progress in Math. 12, Birkhauser, 1980, 371-394.