数理解析研究所講究録
699巻 1989年 55-73

The initial-boundary value problem for a nonlinear degenerate parabolic equation

福田　勇（国士館大学）　Isamu Fukuda

堤　正義（早稲田大学）　Masayoshi Tsutsumi

1. Introduction and main results.

Let $a < b$ and $\lambda > 0$. We consider nonnegative solutions of the initial-boundary value problem

$$\begin{cases} u_t = uu_{xx} - \lambda|u_x|^2 & (a < x < b,\ t > 0) \quad (1.1) \\ u(a,t) = u(b,t) = 0 & (t > 0) \quad (1.2) \\ u(x,0) = u_0(x) & (a < x < b) \quad (1.3) \end{cases}$$

where initial data u_0 satisfy

(H.1) $\quad u_0 \in W^{1,\infty}(a,b)$ and $u_0(x) \geqq 0 \quad (a \leqq x \leqq b)$

In order to construct a solution to the problem (1.1)-(1.3), it might be natural to employ the well-known viscosity method: Let $\varepsilon > 0$ and let $u_\varepsilon(x,t)$ be an unique classical solution of the initial-boundary value problem for the uniformly parabolic equation:

$$\begin{cases} u_{\varepsilon t} = (u_\varepsilon + \varepsilon)u_{\varepsilon xx} - \lambda|u^2_{\varepsilon x}| & (a < x < b,\ t > 0) \quad (1.1)_\varepsilon \\ u_\varepsilon(a,t) = u_\varepsilon(b,t) = 0 & (t > 0) \quad (1.2)_\varepsilon \\ u_\varepsilon(x,0) = u_0(x) & (a < x < b) \quad (1.3)_\varepsilon \end{cases}$$

We call u the viscosity solution of the problem (1.1)-(1.3) if $u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t)$.

Let us consider solutions with compact support and define the interface $\zeta_\pm(t)$ by

$$\zeta_\pm(t)=\pm\sup\{\ \pm x:\ u(x,t) > 0\ \} \qquad \text{for } t > 0$$

Differentiating $u(\zeta_\pm(t),t)=0$ with respect to t and using eq. (1.1), we easily see that the interface $\zeta_\pm(t)$ satisfies formally

$$\frac{d\zeta_\pm(t)}{dt} = \lambda u_x(\zeta_\pm(t),t)\ , \tag{1.4}$$

provided $u_x(\zeta_\pm(t),t)\neq 0$. Thus we might expect that the support of solutions shrinks if $u_x(\zeta_\pm(t),t)\neq 0$. Indeed, for $\lambda > \frac{1}{2}$ we have a special weak solution of the form

$$u(x,t)=(T_0-t)^{\frac{1}{2\lambda-1}}[C_0^2-\frac{1}{2(2\lambda-1)}x^2(T_0-t)^{\frac{2\lambda}{2\lambda-1}}]_+ \tag{1.5}$$

where T_0 and C_0 are positive constants such that

$$(-\sqrt{2(2\lambda-1)}\,C_0T_0^{\frac{\lambda}{2\lambda-1}},\ \sqrt{2(2\lambda-1)}\,C_0T_0^{\frac{\lambda}{2\lambda-1}})\subset[a,b]$$

and $[\cdot]_+ = \max(\cdot,0)$.

Apparently its support shrinks to one point. But this conjecture is not true for viscosity solutions. In [1], Bertch, Dal Passo and Ughi show that every viscosity solution of the Cauchy problem for (1.1) has a property that

$$\text{supp } u(t)=\text{supp } u_0 \qquad \text{for } t > 0\ . \tag{1.6}$$

It is a striking result. If $\lambda < 0$, equation (1.1) is called the pressure equation, related to the porous medium equation and the support of solutions spreads out as time goes , as is suggested by the interface equation(1.4).

Another curious property of eq.(1.1) is the nonuniqueness phenomenon which was discovered by Dal Passo and Luckhaus [2] ($\lambda = 0$), Ughi [5] ($\lambda = 0$) and Bertch, Dal Passo and Ughi [1] ($\lambda \geqq 0$). The existence of our special weak solution u also suggests the nonuniqueness phenomenon.

We now define weak solutions of the problem (1.1)-(1.3) as follows:

Definition 1. A nonnegaive function $u \in L^\infty([0,\infty):W^{1,\infty}[a,b])$ is called a weak solution of (1.1)-(1.3) if for any $T > 0$

$$u_t \in L^2([a,b]\times[0,T])$$

and for all $t \geqq 0$

$$\int_a^b u(x,t)\psi(x,t)\,dx = \int_a^b u_0(x)\psi(x,0)\,dx$$

$$+\int_0^t\int_a^b \{u(x,s)\psi_t(x,s) - u(x,s)u_x(x,s)\psi_x(x,s) - (\lambda+1)|u_x(x,s)|2\psi(x,s)\}\,dxdt$$

for any function $\psi \in C^{2,1}([a,b]\times[0,\infty))$ with compact support in (a,b).

Note that $u \in L^\infty([0,\infty):W^{1,\infty}([a,b]))$ with $u_t \in L^2([a,b]\times[0,T])$ for any $T > 0$ implies that u is continuous in x and t.

In this paper we establish the global existence of (weak) solutions of (1.1)-(1.3) and investigate the uniqueness of solutions. We propose a new uniqueness class of solutions which is different from [1], [2] and [5].

As to the existence theorem, we have

Theorem 1. Let u_0 satisfy (H1). Then the problem (1.1)-(1.3) has at least one weak solution.

Theorem 2. Let $\lambda > \frac{1}{2}$. Assume that u_0 satisfies (H1) and

$$\text{(H2)} \quad \lim_{x \downarrow a} \frac{u_0(x)}{(x-a)^2} < \infty \quad \text{and} \quad \lim_{x \uparrow b} \frac{u_0(x)}{(b-x)^2} < \infty .$$

Then u satisfies

$$|u_{xx}(x,t)| \leq \frac{1}{t} \tag{1.8}$$

and, in particular, $u \in L^\infty([\delta,\infty):W^{2,\infty}([a,b]))$ as well as $u_t \in L^\infty([\delta,\infty):L^\infty([a,b]))$ for any $\delta > 0$. Moreover, if we assume that u_0 is semiconcave, that is,

$$u_{0xx} \leq C \quad \text{in} \quad \mathcal{D}'$$

for some constant C, then u is also semiconcave almost everywhere, that is,

$$u_{xx}(x,t) \leq C \quad \text{for a.e. } (x,t) \in [a,b] \times (0,\infty)$$

where C is also a positive constant.

Remark 1. In theorem 2 the hypotheses (H1) can be weakened as follows:

$(H1)_w$ $u_0 \in L^\infty([a,b])$, $u_0(x) \geqq 0$ a.e. .

Collorary 1. Under the assumption $(H1)_w$ and (H2), the problem (1.1)-(1.3) has at least one weak solution which has properties in Theorem 2.

Concerning the uniqueness and continuous-dependence-on-data of solutions, we have

Theorem 3. Let u and v be two weak solutions coresponding to the initial data u_0 and v_0 , respectively. Assume that u and v are semiconcave almost everywhere. Then the inequality

$$\int_a^b |u(x,t)-v(x,t)|dx \leqq e^{ct}\int_a^b |u_0(x)-v_0(x)|dx$$

holds valid for any $t > 0$ and a positive constant c.

Corollary 2. Let u_0 satisfy $(H1)_w$,(H2) and be semiconcave. Then the problem (1.1)-(1.3) has an unique weak solution u which is also semiconcave and depends on initial data continuously in $L^1(a,b)$.

Remark 2. Our special solution (1.5) is not semiconcave. Uniqueness theorem does not hold valid for the problem (1.1)-(1.3) with initial data

$$u_0(x)=(T_0)^{\frac{1}{2\lambda-1}}[C_0^2-\frac{1}{2(2\lambda-1)}\ x^2(T_0)^{\frac{2\lambda}{2\lambda-1}}]_+ \ ,$$

which does not satisfy (H2).

2. Proof of Theorem 1.

Before proving Theorem 1, we shall obtain a priori estimates of u_ε.

Lemma 1. Let u_0 satisfy (H1). Then

$$\|u_\varepsilon\|_{L^\infty([0,\infty):W^{1,\infty}([a,b]))} \leqq C \tag{2.1}$$

and

$$\int_0^\infty\int_a^b (u_\varepsilon(x,t)+\varepsilon)|u_{\varepsilon x}(x,t)|^{p-1}u_{\varepsilon xx}^2(x,t)\ dx \leqq C \tag{2.2}$$

for any $p \geqq 1$, where and in the sequel C denotes various positive constants independent of ε.

Proof. The maximum principle gives

$$0 \leqq u_\varepsilon(x,t) \leqq \max_{a\leqq x\leqq b} u_0(x)\ . \tag{2.3}$$

Multiplying $(1.1)_\varepsilon$ by $\frac{1}{p}\ (|u_{\varepsilon x}(x,t)|^{p-1}u_{\varepsilon x})_x$ and integrating by parts on [a, b], we have

$$\frac{1}{p(p+1)}\frac{d}{dt}\int_a^b |u_{\varepsilon x}|^{p+1}dx + \int_a^b (u_\varepsilon+\varepsilon)|u_{\varepsilon x}|^{p-1}u_{\varepsilon xx}^2\ dx$$

$$+\frac{\lambda}{p+1}|u_{\varepsilon x}(a,t)|^p u_{\varepsilon x}(a,t) - \frac{\lambda}{p+1}|u_{\varepsilon x}(b,t)|^p u_{\varepsilon x}(b,t) = 0 \quad (2.4)$$

Here and from now on we abbreviate x and t variables in the integrand. Since u_ε is nonnegative, we easily see that

$$u_{\varepsilon x}(a,t) \geqq 0 \quad \text{and} \quad u_{\varepsilon x}(b,t) \leqq 0 \ .$$

Hence integrating (2.4) from 0 to t, we obtain that, any $p \geqq 1$

$$\frac{1}{p(p+1)}\int_a^b |u_{\varepsilon x}|^{p+1}dx + \int_0^t\int_a^b (u_\varepsilon+\varepsilon)|u_{\varepsilon x}|^{p-1}u_{\varepsilon xx}^2\ dxdt$$

$$\leqq \frac{1}{p(p+1)}\int_a^b |u_{0x}|^{p+1}\ dx$$

form which it follows that

$$\|u_{\varepsilon x}(t)\|_{L^{p+1}(a,b)} \leqq \|u_{0x}\|_{L^{p+1}(a,b)} \quad \text{for any } t > 0 \quad (2.5)$$

and

$$\int_0^\infty\int_a^b (u_\varepsilon+\varepsilon)|u_{\varepsilon x}|^{p-1}u_{\varepsilon xx}^2\ dxdt \leqq C\|u_{0x}\|_{L^{p+1}(a,b)}^{p+1} . \quad (2.6)$$

From (2.5) we easily have (2.1).

Lemma 2. Let $0 < \varepsilon < \varepsilon_0$ where ε_0 is a fixed number. For any $T > 0$,

$$\|u_{\varepsilon t}\|_{L^2([a,b]\times[0,T])} \leqq C \tag{2.7}$$

where C is a positive constant independent of ε.

Proof. Using $(1.1)_\varepsilon$ and integrating by parts, we get

$$\int_0^T\int_a^b u_{\varepsilon t}^2 \, dxdt = \int_0^T\int_a^b (u_\varepsilon+\varepsilon)^2 u_{\varepsilon xx}^2 \, dxdt - \frac{2}{3}\varepsilon\lambda\int_0^T \{u_x(b,t)^2 - u_x(a,t)^2\} dt$$

$$+ \left(\frac{2}{3}\lambda+\lambda^2\right)\int_0^T\int_a^b u_{\varepsilon x}^4 \, dxdt$$

$$\leqq (\|u_\varepsilon\|_{L^\infty([a,b]\times[0,T])} + \varepsilon_0)\int_0^T\int_a^b (u_\varepsilon+\varepsilon)u_{\varepsilon xx}^2 \, dxdt$$

$$+ \frac{4}{3}\varepsilon\lambda T\|u_{\varepsilon x}\|^3_{L^\infty([a,b]\times[0,\infty))} + \left(\frac{2}{3}\lambda+\lambda^2\right)(b-a)T\|u_{\varepsilon x}\|^4_{L^\infty([a,b]\times[0,\infty))}.$$

From (2.1) and (2.2) with $p = 1$, we can easily obtain (2.7).

Proof of Theorem 1. From (2.1), (2.3) and (2.7), we see that there exists a nonnegative function $u \in L^\infty([0,\infty):C([a,b])\cap W^{1,\infty}[a,b])$ with $u_t \in L^2([a,b]\times[0,T])$ (for any $T > 0$) and we can extract a subsequence of $\{u_\varepsilon\}$, which is denoted by $\{u_{\varepsilon_i}\}$, such that, as $\varepsilon_i \longrightarrow 0$,

$$u_{\varepsilon_i} \longrightarrow u \quad \text{strongly in } C([a,b]\times[0,T])$$

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{weakly star in } L^\infty([a,b]\times[0,\infty))$$

and

$$u_{\varepsilon_i t} \longrightarrow u_t \quad \text{weakly in } L^2([a,b]\times[0,T]) .$$

In order to show that u is a weak solution of (1.1)-(1.3), it suffices to show that, for any $T > 0$

$$|u_{\varepsilon_i x}|^2 \longrightarrow |u_x|^2 \quad \text{in } L^1([a,b]\times[0,T]) \quad ,$$

and this implies

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{strongly in } L^2([a,b]\times[0,T]) .$$

From (2.1) and (2.2), we have

$$\|(u_\varepsilon^2)_{xx}\|_{L^2([a,b]\times[0,T])} \leqq 2\|u_\varepsilon u_{\varepsilon xx}+u_{\varepsilon x}^2\|_{L^2([a,b]\times[0,T])} \leqq C.$$

We also have

$$\|(u_\varepsilon^2)_{xt}\|_{L^2(0,T:H^{-1}(a,b))} \leqq C\|(u_\varepsilon^2)_t\|_{L^2([a,b]\times[0,T])} \leqq C .$$

By virtue of Aubin's compactness theorem (see J.L.Lions [4]), we may assume that

$$(u_{\varepsilon_i}^2)_x = 2u_{\varepsilon_i}u_{\varepsilon_i x} \longrightarrow 2uu_x = (u^2)_x \quad \text{strongly in } L^2([a,b]\times[0,T]).$$

Hence we may also assume that

$$u_{\varepsilon_i}u_{\varepsilon_i x} \longrightarrow uu_x \quad \text{a.e. in } [a,b]\times[0,\infty)$$

from which it follows that

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{a.e. in } [a,b]\times[0,\infty) \tag{2.8}$$

since $\frac{\partial u}{\partial x} = 0$ a.e. in $E=\{x\in[a,b];u=o\}$ (see Kinderlehrer-Stampacchia [3], p53) and

$$u_{\varepsilon_i x} \longrightarrow u_x \quad \text{a.e.} \quad \text{in} \quad {}^cE=\{x\in[a,b];u>o\}.$$

In view of Lebesgue's bounded convergence theorem we can easily obtain

$$\lim_{m\to\infty}\int_0^T\int_a^b |u_{mx}^2|\,dxdt = \int_0^T\int_a^b |u_x^2|\,dxdt. \qquad (2.9)$$

On the other hand, from (2.1) we may assume that u_{mx} converges to u_x weakly in $L^2([a,b]\times[0,T])$. Hence

$$u_{mx} \longrightarrow u_x \quad \text{strongly in} \quad L^2([a,b]\times[0,T]) \qquad (2.10)$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2.

Lemma 3. Let u_0 satisfy $(H1)_w$ and (H2). Then, for any $t > 0$

$$|u_\varepsilon(a,t)| \leqq \sqrt{\varepsilon}\, C \qquad (3.1)$$

and

$$|u_\varepsilon(b,t)| \leqq \sqrt{\varepsilon}\, C \qquad (3.2)$$

Proof. We only show that (3.1) hold valid. From (H2) we see that for some $\delta > 0$ and $C_1 > 0$

$$0 \leqq u_0(x) \leqq C_1\{(x-a)^2+\sqrt{\varepsilon}\,(x-a)\} \quad \text{for any } x\in(a,a+\delta) \tag{3.3}$$

Let $T > 0$ be fixed. For any $(x,t)\in[a,a+\delta]\times[0,T]$ set

$$\overline{u}(x,t) = A\{(x-a)^2+\sqrt{\varepsilon}\,(x-a)\} \tag{3.4}$$

where A is chosen so large that

$$A \geqq C_1 \tag{3.5}$$

and

$$A(\delta^2+\sqrt{\varepsilon}\,\delta) \geqq \max_{\substack{a\leqq x\leqq a+\delta \\ 0\leqq t\leqq T}} u_\varepsilon(x,t) \ . \tag{3.6}$$

Note that $u\in C^{2,1}((a,b)\times[0,T])$. Direct calculation gives

$$\begin{aligned} &\overline{u}_t-(\overline{u}+\varepsilon)\overline{u}_{xx}+\lambda(\overline{u}_x)^2 \\ &= 2(2\lambda-1)A^2(x-a)^2+ 2(2\lambda-1)A^2\sqrt{\varepsilon}\,(x-a) + 2\varepsilon A(\lambda A-2) \\ &\geqq 0 \quad \text{in} \quad (a,a+\delta)\times(0,T) \end{aligned} \tag{3.7}$$

provided that $\lambda \geqq \frac{1}{2}$ and that A is so large that

$$\lambda A-2 > 0 \ .$$

By virtue of (3.3)-(3.7) we apply the maximum principle to obtain

$$0 \leqq u_\varepsilon(x,t) \leqq \overline{u}(x,t) \quad \text{in} \quad [a,a+\delta]\times[0,T] \ .$$

Hence

$$0 \leqq u_\varepsilon(a,t) = \lim_{h\downarrow 0}\frac{u_\varepsilon(a+h,t)-u_\varepsilon(a,t)}{h} \leqq \lim_{h\downarrow 0}\frac{u(a+h)}{h} = A\sqrt{\varepsilon} \ .$$

Thus we have (3.1).

Lemma 4. Under the same assumption

$$|u_{\varepsilon xx}| \leqq \frac{C}{t} \quad \text{for ant } t > 0 . \tag{3.8}$$

Moreover, if $u_{0xx} \leqq C_2$ then

$$u_{\varepsilon xx} \leqq C_3 \tag{3.9}$$

where C_3 is a constant.

Proof. Putting $p=\dfrac{u_{\varepsilon t}}{u_\varepsilon + \varepsilon}$, we have

$$p_t=(u_\varepsilon+\varepsilon)p_{xx}+2(1-\lambda)u_{\varepsilon x}p_x+p^2 \qquad (x,t)\in(a,b)\times(0,\infty)$$

$$p(a,t)=p(b,t)=0 \qquad t\in(0,\infty)$$

$$p(x,0)=u_{0xx}-\frac{\lambda u_{0x}^2}{u_0+\varepsilon} \qquad x\in(a,b) .$$

The standard comparison theorem yields that

$$p \geqq -\frac{1}{t}$$

Using $(1.1)_\varepsilon$, we easily see that

$$u_{\varepsilon xx} \geqq -\frac{1}{t} \tag{3.10}$$

We put $q=u_{\varepsilon xx}$ to obtain that

$$q_t=(u_\varepsilon+\varepsilon)q_{xx}+2(1-\lambda)u_{\varepsilon x}q_x+(1-2\lambda)q^2 \tag{3.11}$$

As for the boundary conditions, we utilize $(1.1)_\varepsilon$ to get

$$q(a,t)=\frac{\lambda}{\varepsilon}|u_{\varepsilon x}(a,t)|^2 \quad , \quad q(b,t)=\frac{\lambda}{\varepsilon}|u_{\varepsilon x}(b,t)|^2 \tag{3.12}$$

for any $t>0$. In view of Lemma 3, we see that

$$0 \leqq q(a,t) \leqq \lambda C^2 \quad , \quad 0 \leqq q(b,t) \leqq \lambda C^2 \tag{3.13}$$

Hence the comparison theorem yields that, if $\lambda > \frac{1}{2}$

$$q(x,t)=u_{\varepsilon xx}(x,t) \leqq \frac{C}{(2\lambda-1)}$$

for some constant $C > 0$.

If $u_{0xx} \leqq C_2$, $\lambda \geqq \frac{1}{2}$ and (3.11)-(3.13) yield that

$$u_{\varepsilon xx}(x,t) \leqq C_3 \tag{3.14}$$

where $C_3 = \max(\lambda C^2, C_2)$ is independent of ε.

Proof of Theorem 2. Because of Lemma 4, we see that $\{u_{\varepsilon xx}\}$ is bounded in $L^\infty([a,b]\times[\delta,\infty))$ for every $\delta > 0$. Hence we can assume that

$$u_{\varepsilon_i xx} \longrightarrow u_{xx} \quad \text{weakly star in } L^\infty([a,b]\times[\delta,\infty))$$

and

$$|u_{xx}(x,t)| \leqq \frac{C}{t} \quad \text{for any } (x,t)\in[a,b]\times[\delta,\infty).$$

If $u_{0xx} \leqq C$, from (3.14) we have

$$u_{xx}(x,t) \leqq C \quad \text{for any } (x,t)\in[a,b]\times[0,\infty).$$

This completes the proof of Theorem 2.

4. Proof of Theorem 3.

Let u and v be two weak soluions of (1.1)-(1.3) with initial data u_0 and v_0, respectively. Let $T > 0$ be fixed and put $w(x,t) = u(x,t) - v(x,t)$ and $w_0(x) = u_0(x) - v_0(x)$. Then we have

$$\int_a^b w(x,T)\psi(x,T)\,dx = \int_a^b w_0(x)\psi(x,0)\,dx$$

$$+\int_0^T\int_a^b \{w\psi_t - (uu_x - vv_x)\psi_x - (\lambda+1)(|u_x|^2 - |v_x|^2)\psi\}\,dx\,dt \tag{4.1}$$

for any $\psi \in C^{2,1}([a,b]\times[0,\infty))$ with compact support in (a,b).

For each $n \in \mathbb{N}$ define

$$g_n(x) = \begin{cases} 1 & \text{if } \frac{1}{n} < s \\ ns & \text{if } |s| \leq \frac{1}{n} \\ -1 & \text{if } s < -\frac{1}{n} \end{cases}$$

and

$$\Psi = \{g_n((u^2 - v^2)\theta_k\theta_m \star \rho_\nu \star \sigma_\mu) \star \rho_\nu \star \sigma_\mu\}\theta_k\theta_m$$

where ρ_ν and σ_μ are the standard molifiers with respect to x and t, respectively; $\theta_k(\frac{x}{k})$ where $\theta \in C_0^\infty((a,b))$ with $0 \leqq \theta \leqq 1$ and $\theta(x)=1$ in a neighborhood of 0 (we may assume $0 \in (a,b)$) and

$\theta_m(t) \in C_0^\infty((0,\infty))$ such that $0 \leqq \theta_m \leqq 1$ and $\theta_m(t)$ tends to the indicator function of $[s_1, s_2]$ $(0 < s_1 < s_2)$ as $m \to \infty$. Then $\Psi \in C_0^\infty((a,b)\times(0,\infty))$ and $\Psi(x,t) \geqq 0$ for any $(x,t) \in (a,b)\times(0,\infty)$.

Substituting Ψ for a test function ψ in (3.1), we have

$$\int_0^T\int_a^b \{w\Psi_t - (uu_x - vv_x)\Psi_x - (\lambda+1)(|u_x|^2 - |v_x|^2)\Psi\}dxdt . \qquad (4.2)$$

From $w_t \in L^2([a,b]\times[0,T])$ for any $T > 0$ and $\Psi \in C_0^\infty((a,b)\times(0,\infty))$ we get

$$\int_0^T\int_a^b w\Psi_t dxdt = -\int_0^T\int_a^b w_t \Psi dxdt \quad .$$

Letting ν and μ tend to infinity, we can easily see that

$$I_1(k,m,n) - I_2(k,m,n) - I_3(k,m,n)$$

$$\equiv \int_0^T\int_a^b w_t \theta_k \theta_m g_n((u^2-v^2)\theta_k\theta_m)dxdt$$

$$-[-\int_0^T\int_a^b (uu_x - vv_x)\{g_n((u^2-v^2)\theta_k\theta_m)\}_x dxdt]$$

$$-[-(\lambda+1)\int_0^T\int_a^b (|u_x|^2 - |v_x|^2)\theta_k\theta_m g_n((u^2-v^2)\theta_k\theta_m dxdt] = 0 \quad (4.3)$$

As n tends to infinity, we find $I_1(k,m,n)$ tends to

$$\tilde{I}_1(k,m) = \int_0^T\int_a^b w_t\theta_k\theta_m \,\mathrm{sgn}((u-v)\theta_k\theta_m)\,dxdt \tag{4.4}$$

since $\mathrm{sgn}((u^2-v^2)\theta_k\theta_m) = \mathrm{sgn}((u-v)\theta_k\theta_m)$.

Moreover, $\theta_m(t)=0$ near 0 and T , then we have

$$\tilde{I}_1(k,m) = \int_0^T\int_a^b (|w\theta_k\theta_m|)_t\,dxdt - \int_0^T\int_a^b |w\theta_k|(\theta_m)_t\,dxdt$$

$$= -\int_0^T\int_a^b |w\theta_k|(\theta_m)_t\,dxdt \tag{4.5}$$

As for $I_2(k,m,n)$, using chain rule, we get

$$I_2(k,m,n) = -2\int_0^T\int_a^b (uu_x-vv_x)^2 g_n'((u^2-v^2)\theta_k\theta_m)\theta_k\theta_m\,dxdt$$

$$-\int_0^T\int_a^b (uu_x-vv_x)\, g_n'((u^2-v^2)\theta_k\theta_m)(u^2-v^2)\theta_m^2\theta_k(\theta_k)_x\,dxdt$$

$$-\int_0^T\int_a^b (uu_x-vv_x)\, g_n((u^2-v^2)\theta_k\theta_m)\,\theta_m^2\theta_k(\theta_k)_x\,dxdt\ .$$

Since the first term onthe right hand side is nonpositive and $|(\theta_k)_x| \leq \frac{C}{k}$, we have

$$I_2(k,m,n) \leq \frac{C}{k}(\|u\|^3_{L^\infty} + \|v\|^3_{L^\infty} + \|u\|_{L^\infty} + \|v\|_{L^\infty})(\|u_x\|_{L^2} + \|v_x\|_{L^2})$$

where $L^p = L^p([a,b]\times[0,T])$ $(p=2,\infty)$. Since $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$ and $\|v_x\|_{L^2}$ are bounded, we get

$$I_2(k,m,n) \leq \frac{C}{k} \tag{4.6}$$

where C depends on $\|u\|_{L^\infty}$, $\|v\|_{L^\infty}$, $\|u_x\|_{L^2}$ and $\|v_x\|_{L^2}$.

Since $\operatorname{sgn}((u^2-v^2)\theta_k\theta_m) = \operatorname{sgn}(w\theta_k\theta_m)$, letting $n \longrightarrow \infty$ we see that $I_3(k,m,n)$ tends to

$$\tilde{I}_3(k,m) = -(\lambda+1)\int_0^T\int_a^b (|u_x|^2 - |v_x|^2)\theta_k\theta_m \operatorname{sgn}(w\theta_k\theta_m)\,dxdt .$$

Recalling that u_{xx} and v_{xx} are semiconcave, we have

$$\tilde{I}_3(k,m) = -(\lambda+1)\int_0^T\int_a^b (|w\theta_k\theta_m|)_x(u_x+v_x)\,dxdt$$

$$-(\lambda+1)\int_0^T\int_a^b (u-v)(u_x-v_x)(\theta_k)_x\theta_m \operatorname{sgn}(w\theta_k\theta_m)\,dxdt$$

$$\leqq (\lambda+1)\int_0^T\int_a^b |w\theta_k\theta_m|(u_{xx}+v_{xx})dxdt$$

$$+ (\lambda+1)\int_0^T\int_a^b (|u|+|v|)(|u_x|+|v_x|)|(\theta_k)_x|dxdt$$

$$\leqq C\int_0^T\int_a^b |w\theta_k\theta_m|dxdt + \frac{C}{k} . \qquad (4.7)$$

Hence eq.(4.3) with (4.5),(4.6) and (4.7) implies that

$$\int_0^T\int_a^b |w\theta_k|(\theta_m)_t dxdt \leqq C\int_0^T\int_a^b |w\theta_k\theta_m|dxdt + \frac{C}{k} \qquad (4.8)$$

In (4.8) letting $k,m \longrightarrow \infty$, we find that

$$\int_a^b |w(x,s_2)|dx - \int_a^b |w(x,s_1)|dx \leqq C\int_{s_1}^{s_2}\int_a^b |w(x,s)|dxds$$

for any s_1 and s_2 ($0 < s_1 < s_2$).

As $s_2=t$ and s_1 tends to 0, we have

$$\int_a^b |w(x,t)|dx - \int_a^b |w_0(x)|dx \leqq C\int_0^t\int_a^b |w(x,s)|dxds$$

from which it follows that, for any $t \geqq 0$

$$\int_a^b |w(x,t)|dx \leq e^{ct}\int_a^b |w_0(x)|dx \quad . \qquad (4.9)$$

This completes the proof of Theorem 3. Corollary 2 is easily obtained from (4.9).

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