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On the existence of weak solutions of stationary Boussinesq equation

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§1. Notations and results.

In this paper, we discuss the existence of weak solutions of equations which describe the motion of fluid of natural convection (Boussinesq approximation) in a bounded domain Ω in R^n, $2 \leq n$. We consider the following system of differential equations which is called stationary Boussinesq equation:

$$(1\text{-}1)\quad \begin{cases} (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p + \nu\,\Delta u + \beta\, g\,\theta\,, & \\ \operatorname{div} u = 0\,, & \text{in } \Omega \\ (u\cdot\nabla)\theta = \kappa\,\Delta\theta\,, & \end{cases}$$

where $u\cdot\nabla = \sum_j u_j \frac{\partial}{\partial x_j}$. Here u is the fluid velosity, p is the pressure , θ is the temperature, g is the gravitational vector function, and ρ(density), ν(kinematic viscosity), β(coefficient of volume expansion), κ(thermal conductivity) are positive constants. We study this system of equations with mixed boundary condition for θ.

In the previous paper[8], we treated this problem only for the case $n = 3$. By using the Galerkin method, we can show the existence of weak solution, for any integer n greater than or equal to 2. Some uniqueness result is also obtained.

Let $\partial\Omega$ (the boundary of Ω) be devided into two parts Γ_1, Γ_2 such that

$$\partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \phi.$$

The boundary conditions are as follows.

$$(1\text{-}2) \quad \begin{cases} u = 0, \ \theta = \xi , & \text{on } \Gamma_1, \\ u = 0, \ \dfrac{\partial\theta}{\partial n} = 0 , & \text{on } \Gamma_2, \end{cases}$$

where ξ is a given function on Γ_1, n is the outer normal vector to $\partial\Omega$. If we can find a function θ_0 defined on Ω, of class $C^2(\Omega)\cap C^1(\bar{\Omega})$, satisfying $\theta_0 = \xi$ on Γ_1 and $\frac{\partial}{\partial n}\theta_0 = 0$ on Γ_2, then we can transform the equations (1-1),(1-2) for u and $\tilde{\theta} = \theta - \theta_0$ and we obtain the following:

$$(1\text{-}3) \quad \begin{cases} (u\cdot\nabla)u = -\dfrac{1}{\rho}\nabla p + \nu\Delta u + \beta g\tilde{\theta} + \beta g\theta_0 , & \text{in } \Omega , \\ \operatorname{div} u = 0 , & \text{in } \Omega , \\ (u\cdot\nabla)\tilde{\theta} = \kappa\Delta\tilde{\theta} - (u\cdot\nabla)\theta_0 + \kappa\,\Delta\theta_0 , & \text{in } \Omega , \\ u = 0, \ \tilde{\theta} = 0 , & \text{on } \Gamma_1, \\ u = 0, \ \dfrac{\partial\tilde{\theta}}{\partial n} = 0 , & \text{on } \Gamma_2. \end{cases}$$

For the domain Ω, we assume:

<u>Condition(H)</u>

Ω is a bounded domain in R^n with C^2 boundary. The boundary $\partial\Omega$ of Ω is devided as follows:

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 , \ \Gamma_1 \cap \Gamma_2 = \phi , \ \text{measure of } \Gamma_1 \neq 0,$$

and the intersection $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is a n-1 dimensional C^1 manifold.

In order to state the definition of weak solution and our result, we introduce some

<u>Function spaces</u> :

D_σ = {vector function $\varphi \in C^\infty(\Omega)$ | supp $\varphi \subset \Omega$, div $\varphi=0$ in Ω}

H = completion of D_σ under the $L^2(\Omega)$-norm

V = completion of D_σ under the $H^1(\Omega)$-norm

$\tilde{V}$ = completion of D_σ under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$.

D_0 = { scalar function $\varphi \in C^\infty(\bar{\Omega})$ |

$\varphi \equiv 0$ in a neighborhood of Γ_1} ,

W = completion of D_0 under the $H^1(\Omega)$-norm .

$\tilde{W}$ = completion of D_0 under the norm $\|u\|_{H^1(\Omega)} + \|u\|_{L^n(\Omega)}$.

Consider L^2 inner product of the first equation of (1-3) with v in $\tilde{V}$, and the third equation of (1-3) with τ in $\tilde{W}$. Then we obtain:

<u>Auxiliary problem</u>: Find $u \in V$ and $\tilde{\theta} \in W$ satisfying

$$(1\text{-}4)\quad \begin{cases} \nu(\nabla u,\nabla v) + B(u,u,v) - (\beta g\tilde{\theta},v) - (\beta g\theta_0,v) = 0 , \\ \qquad\qquad\qquad\qquad \text{for all } v \text{ in } \tilde{V} , \\ \kappa(\nabla\tilde{\theta},\nabla\tau) + b(u,\tilde{\theta},\tau) + b(u,\theta_0,\tau) + \kappa(\nabla\theta_0,\nabla\tau) = 0, \\ \qquad\qquad\qquad\qquad \text{for all } \tau \text{ in } \tilde{W} , \end{cases}$$

where

$$B(u,v,w) = ((u\cdot\nabla)v,w) = \int_\Omega \sum_{i,j=1}^{n} u_j(x)\,\frac{\partial v_i(x)}{\partial x_j}\,w_i(x)\,dx,$$

and

$$b(u,\theta,\tau) = ((u\cdot\nabla)\theta,\tau) = \int_\Omega \sum_{j=1}^{n} u_j(x)\,\frac{\partial\theta(x)}{\partial x_j}\,\tau(x)\,dx .$$

Now, we define the weak solution of (1-1),(1-2).

<u>Definition 1</u>. The pair of functions $\{u,\theta\}$ is called a weak solution of (1-1),(1-2), if there exists a function θ_0 in $C^1(\bar{\Omega})$

such that $u \in V$, $\theta - \theta_0 \in W$, $\theta_0 = \xi$ on Γ_1, $\frac{\partial}{\partial n}\theta_0 = 0$ on Γ_2, and, $\{u, \tilde{\theta}\}$ ($\tilde{\theta} = \theta - \theta_0$) satisfies (1-4).

Now, we state our results.

<u>Theorem 1</u>

Let Ω be a bounded domain in R^n with C^2 boundary satisfying the condition (H). If the function $g(x)$ is in $L^\infty(\Omega)$ and ξ is of class $C^1(\overline{\Gamma_1})$, then there exists a weak solution of (1-1),(1-2).

<u>Remark 1</u>

Generaly, $\tilde{V} \subset V \cap L^n(\Omega)$ and $\tilde{W} \subset W \cap L^n(\Omega)$. For $2 \le n \le 4$, $\tilde{V} = V$ and $\tilde{W} = W$ (c.f.Masuda[7], Giga[3]). Therefore our theorem contains the result of [8].

Let $g_\infty = \|g\|_{L^\infty(\Omega)}$, and c, c_1, c_2 be constants in Lemma 3 (§2). As for the uniqueness, we have:

<u>Theorem 2</u>

The weak solution $\{u, \theta\}$ of (1-1),(1-2) satisfying

(i) $u \in L^n(\Omega)$, $\theta \in L^n(\Omega)$,

(ii) $c\|u\|_n + \frac{\beta g_\infty c c_1 c_2}{\kappa}\|\theta\|_n < \nu$, when $n \ge 3$,

((ii)' $c\|u\|_p + \frac{\beta g_\infty c c_1 c_2}{\kappa}\|\theta\|_p < \nu$, for some $p > 2$, when $n = 2$)

is , if it exists, unique.

<u>Remark 2</u>

The condition (i) is automatically satisfied when $2 \le n \le 4$.

<u>Remark 3</u>

If we set

$$Re = \frac{c}{\nu}\|u\|_n \qquad \text{(Reynolds number)},$$

$$Ra = \frac{\beta g_\infty c c_1 c_2}{\nu\kappa} \|\theta\|_n \quad \text{(Rayleigh number)},$$

then the condition (ii) reads as

$$Re + Ra < 1.$$

See also Joseph[5].

§2. Some lemmas.

In this section, we prepare some lemmas.

Lemma 1

$\tilde{V}$ and $\tilde{W}$ are separable Banach spaces.

Proof. A subset of separable metric space is separable(e.g. Brezis[2]). If we show $V \cap L^n(\Omega)$ is separable, Lemma is proved. We can identify $V \cap L^n(\Omega)$ as a subset

$$F = \{(v, \frac{\partial}{\partial x_1} v, \ldots, \frac{\partial}{\partial x_n} v);\ v \in V \cap L^n(\Omega)\}$$

of $L^n(\Omega) \times L^2(\Omega) \times \cdots \times L^2(\Omega)$. Since the latter space is separable, the set F is also separable and Lemma 1 is proved.

Lemma 2 (Sobolev)

Sobolev space $H^1(\Omega)$ is continuously imbedded in $L^q(\Omega)$, where $q = \frac{2n}{n-2}$ for $n \geq 3$, and $+\infty > q \geq 1$ for $n = 2$.

For the proof , see Adams[1].

Lemma 3 (Poincaré)

There exist constants c_1, c_2, c depending on Ω and n such that

(i) $\|u\| \le c_1\|\nabla u\|$ for $^\forall u \in V$,

(ii) $\|u\|_q \le c\|\nabla u\|$ for $^\forall u \in V$, $q = \frac{2n}{n-2}$ $(n \ge 3)$,

$q = 4$ $(n=2)$,

(iii) $\|\theta\| \le c_2\|\nabla\theta\|$ for $^\forall\theta \in W$.

These constants are used in the statement of Theorem 2. For the proof of (i),(iii), see Morimoto[8]. (ii) follows from (i) and Lemma 2.

By Hölder's inequality and Lemmas 2,3, we have:

<u>Lemma 4</u>

Let $n \ge 3$. There exists a constant c_B depending on Ω and n such that

$$|B(u,v,w)| \le c_B\|\nabla u\|\|\nabla v\|\|w\|_n \quad \text{for } ^\forall u \in V, {}^\forall v \in H^1(\Omega), {}^\forall w \in L^n(\Omega),$$

$$|b(u,\theta,\tau)| \le c_B\|\nabla u\|\|\nabla\theta\|\|\tau\|_n \quad \text{for } ^\forall u \in V, {}^\forall\theta \in H^1(\Omega), {}^\forall\tau \in L^n(\Omega),$$

hold.

Using the integration by parts, we obtain:

<u>Lemma 5</u>

(i) $B(u,v,w) = -B(u,w,v)$ for $^\forall u \in V$, $^\forall v,w \in H^1 \cap L^n$ holds. In particular,

$B(u,v,v) = 0$ for $^\forall u \in V$, $^\forall v \in H^1 \cap L^n$.

(ii) $b(u,\theta,\tau) = -b(u,\tau,\theta)$ for $^\forall u \in V$, $^\forall \theta,\tau \in H^1 \cap L^n$, holds. In particular,

$b(u,\theta,\theta) = 0$ for $^\forall u \in V$, $^\forall \theta \in H^1 \cap L^n$.

Lemma 6 (Whitney)

Let Ω be a bounded domain in R^n with C^2 boundary $\partial\Omega$. If ξ is a C^1 function defined on $\partial\Omega$, then for any positive number ε and any $p \geq 1$, there exists an extension θ_0 of ξ such that

$$\theta_0 \in C^1(R^n),$$

$$\theta_0 = \xi \ , \quad \frac{\partial\theta_0}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$\|\theta_0\|_p < \varepsilon.$$

Proof. It is well known as Whitney's extension theorem(see Malgrange[6]). In the case $n = 3$, we can prove it directly(Morimoto[8]), and it is easy to extend to the general case.

§3. Proof of Theorem 1.

Under our assumptions on $\partial\Omega$ and ξ, we have an extension θ_0 of ξ (Lemma 6), and we study the equation (1-4). Using the Galerkin method, we construct approximate solutions of (1-4). Let $\{\varphi_j\}$ be a sequence of functions in D_σ, linearly independent and total in $\hat{V}$. We can assume $(\nabla\varphi_j, \nabla\varphi_k) = \delta_{jk}$ without loss of generality. Let $\{\psi_j\}$ be a sequence of functions in D_0, linearly independent and total in $\hat{W}$. We can assume $(\nabla\psi_j, \nabla\psi_k) = \delta_{jk}$. Since $\hat{V}$ (resp. $\hat{W}$) is separable and D_σ (resp. D_0) is dense there, we can find these functions. We put

$$u^{(m)} = \sum_{j=1}^{m} \xi_j \varphi_j \quad , \ \theta^{(m)} = \sum_{j=1}^{m} \xi_{m+j} \psi_j \ ,$$

and we consider the following system of equations:

$$\text{(3-1)}\quad \nu(\nabla u^{(m)},\nabla\varphi_j) + ((u^{(m)}\cdot\nabla)u^{(m)},\varphi_j) - (\beta g\theta^{(m)},\varphi_j) - (\beta g\theta_0,\varphi_j) = 0, \qquad 1 \le j \le m.$$

$$\text{(3-2)}\quad \kappa(\nabla\theta^{(m)},\nabla\psi_j) + ((u^{(m)}\cdot\nabla)\theta^{(m)},\psi_j) + ((u^{(m)}\cdot\nabla)\theta_0,\psi_j) + \kappa(\nabla\theta_0,\nabla\psi_j) = 0, \qquad 1 \le j \le m.$$

Substituting $u^{(m)}$, $\theta^{(m)}$ into these equations, we obtain:

$$\text{(3-3)}\quad \xi_j + \frac{1}{\nu}\sum_{k,\ell}\xi_k\xi_\ell((\varphi_k\cdot\nabla)\varphi_\ell,\varphi_j) - \frac{1}{\nu}\sum_k \xi_{m+k}(\beta g\psi_k,\varphi_j) - \frac{1}{\nu}(\beta g\theta_0,\varphi_j) = 0 , \qquad 1 \le j \le m,$$

$$\text{(3-4)}\quad \xi_{m+j} + \frac{1}{\kappa}\sum_{k,\ell}\xi_k\xi_{m+k}((\varphi_k\cdot\nabla)\psi_\ell,\psi_j) + \frac{1}{\kappa}\sum_k \xi_k((\varphi_k\cdot\nabla)\theta_0,\psi_j) + (\nabla\theta_0,\nabla\psi_j) = 0, \qquad 1 \le j \le m.$$

The left hand side of (3-3),(3-4) determines a polynomial which we denote by

$$\xi_j - P_j(\xi_1,\xi_2,\cdots,\xi_{2m}) , \quad 1 \le j \le 2m .$$

P_j is a polynomial in $\xi = (\xi_1,\cdots,\xi_{2m})$ of degree 2. Let P be a mapping from R^{2m} to R^{2m} defined by $P(\xi) = (P_1(\xi),\cdots,P_{2m}(\xi))$. Then the fixed point ξ of P, if it exists, is a solution of (3-3),(3-4). We show the existence of a fixed point of P. Let $\xi = \xi(\lambda)$ be any solution of $\xi = \lambda P(\xi)$, $0 \le \lambda \le 1$. First we treat the case $n \ge 3$.

$$\begin{aligned}
\sum_{j=1}^m |\xi_j|^2 &= \|\nabla u^{(m)}\|^2 = \lambda\sum_{j=1}^m P_j(\xi)\xi_j \\
&= -\frac{\lambda}{\nu}\sum_{j,k,\ell}\xi_j\xi_k\xi_\ell((\varphi_k\cdot\nabla)\varphi_\ell,\varphi_j) + \frac{\lambda\beta}{\nu}\sum_{j,k}\xi_{m+k}\xi_j(g\psi_k,\varphi_j) \\
&\qquad + \frac{\lambda\beta}{\nu}\sum_j \xi_j(g\theta_0,\varphi_j) \\
&= -\frac{\lambda}{\nu}((u^{(m)}\cdot\nabla)u^{(m)},u^{(m)}) + \frac{\lambda\beta}{\nu}\{(g\theta^{(m)},u^{(m)}) + (g\theta_0,u^{(m)})\} \\
&\le \frac{\lambda\beta g_\infty}{\nu}(\|\theta^{(m)}\| + \|\theta_0\|)\|u^{(m)}\| \\
&\le \frac{\lambda\beta g_\infty c_1}{\nu}(c_2\|\nabla\theta^{(m)}\| + \|\theta_0\|)\|\nabla u^{(m)}\|,
\end{aligned}$$

where we have used Lemmas 4,5. Thereby,

(3-5) $\quad \|\nabla u^{(m)}\| \le \frac{\lambda\beta g_\infty c_1}{\nu} \; (c_2\|\nabla\theta^{(m)}\| + \|\theta_0\|).$

Similarly,

$$\sum_{j=1}^{m} |\xi_{m+j}|^2 = \|\nabla\theta^{(m)}\|^2 = \lambda \sum_{j=1}^{m} P_{m+j}(\xi)\xi_{m+j}$$

$$= -\frac{\lambda}{\kappa} \sum_{j,k,\ell} \xi_k \xi_{m+\ell} \xi_{m+j}((\varphi_k\cdot\nabla)\psi_\ell,\ \psi_j)$$

$$+ \frac{\lambda}{\kappa} \sum_{j,k} \xi_k \xi_{m+j}((\varphi_k\cdot\nabla)\psi_j, \theta_0) - \lambda\sum_j \xi_{m+j}(\nabla\theta_0, \nabla\psi_j)$$

$$= -\frac{\lambda}{\kappa}\{((u^{(m)}\cdot\nabla)\theta^{(m)}, \theta^{(m)}) - ((u^{(m)}\cdot\nabla)\theta^{(m)}, \theta_0)\} - \lambda(\nabla\theta_0, \nabla\theta^{(m)})$$

$$\le \frac{\lambda}{\kappa}\|u^{(m)}\|_{2n/(n-2)}\|\nabla\theta^{(m)}\|\|\theta_0\|_n + \lambda\|\nabla\theta^{(m)}\|\|\nabla\theta_0\|$$

(by Hölder's inequality)

$$\le \frac{\lambda c}{\kappa}\|\nabla u^{(m)}\|\|\nabla\theta^{(m)}\|\|\theta_0\|_n + \lambda\|\nabla\theta_0\|\|\nabla\theta^{(m)}\|$$

(by Lemma 3).

For $n = 2$, we have

$$\|\nabla\theta^{(m)}\|^2 \le \frac{\lambda c}{\kappa}\|\nabla u^{(m)}\|\|\nabla\theta^{(m)}\|\|\theta_0\|_4 + \lambda\|\nabla\theta_0\|\|\nabla\theta^{(m)}\|.$$

Thereby,

(3-6) $\quad \|\nabla\theta^{(m)}\| \le \frac{\lambda c}{\kappa}\|\theta_0\|_p \|\nabla u^{(m)}\| + \lambda\|\nabla\theta_0\|.$

where $p = n$ when $n \ge 3$, and $p = 4$ when $n = 2$. Substituting (3-6) into (3-5), we obtain:

$$\left(1 - \frac{c c_1 c_2 \beta g_\infty \lambda^2}{\kappa\nu}\|\theta_0\|_p\right)\|\nabla u^{(m)}\| \le \frac{\lambda c_1\beta g_\infty}{\nu}(c_2\lambda\|\nabla\theta_0\| + \|\theta_0\|).$$

According to Lemma 6, we can choose θ_0 such that

(3-7) $\quad 1 - \frac{c c_1 c_2 \beta g_\infty}{\kappa\nu}\|\theta_0\|_p > \frac{1}{2}$

holds. Then, we have

(3-8) $\quad \|\nabla u^{(m)}\| \le \frac{2\lambda c_1\beta g_\infty}{\nu}(c_2\lambda\|\nabla\theta_0\| + \|\theta_0\|)$

$$\le \frac{2c_1\beta g_\infty}{\nu}(c_2\|\nabla\theta_0\| + \|\theta_0\|) \equiv \rho_1.$$

Similarly, using (3-7), we have:

(3-9) $\quad \|\nabla\theta^{(m)}\| \le 2\|\nabla\theta_0\| + \frac{1}{c_2}\|\theta_0\| \equiv \rho_2$.

Note that ρ_1 and ρ_2 are constants independent of λ and m.

Thereby the solution ξ of $\xi = \lambda P(\xi)$ satisfies:

$$\sum_{j=1}^{2m} |\xi_j|^2 \le \rho_1^2 + \rho_2^2 \equiv \rho^2 \text{ , for } 0 \le {}^\forall\lambda \le 1.$$

Leray-Schauder's Theorem[4] tells us the existence of a fixed point of the mapping P: $\xi = P(\xi)$, such that $|\xi| \le \rho$. Thus we have obtained the solutions $u^{(m)}, \theta^{(m)}$ of (3-1),(3-2).

Moreover, they satisfy the estimates:

$$\|\nabla u^{(m)}\| \le \rho_1 \quad , \quad \|\nabla\theta^{(m)}\| \le \rho_2 \ .$$

Since V (resp.W) is compactly imbedded in H_σ (resp. L^2), we can choose subsequences of $\{u^{(m)}\},\{\theta^{(m)}\}$ which we denote by the same symbols, and elements $u \in V$, $\theta \in W$ such that the following convergences hold:

(3-10) $u^{(m)} \longrightarrow u$ weakly in V, strongly in H_σ

(3-11) $\theta^{(m)} \longrightarrow \theta$ weakly in W, strongly in $L^2(\Omega)$.

For these convergent sequences, the following lemma holds:

<u>Lemma 7</u>

$$B(u^{(m)},u^{(m)},v) \longrightarrow B(u,u,v) \text{ , for } {}^\forall v \in D_\sigma$$

$$b(u^{(m)},\theta^{(m)},\tau) \longrightarrow b(u,\theta,\tau) \text{ , for } {}^\forall\tau \in D_0.$$

The proof is found in [9] and omitted. Using this lemma for (3-1),(3-2), we find

(3-12) $\quad \nu(\nabla u,\nabla v)+B(u,u,v)-(\beta g\theta,v)-(\beta g\theta_0,v) = 0$,

(3-13) $\quad \kappa(\nabla\theta,\nabla\tau)+b(u,\theta,\tau)+b(u,\theta_0,\tau)+\kappa(\nabla\theta_0,\nabla\tau) = 0$,

hold for $v = \varphi_j$, $\tau = \psi_j$, ${}^\forall j$. By Lemma 4, we see the linear functional

$$v \longrightarrow B(u,u,v) \text{ (resp. } \tau \longrightarrow b(u,\theta,\tau) \text{)}$$

is continuous in L^n . Thereby the linear functional

$$v \longrightarrow \text{the left hand side of (3-12)}$$

(resp. $\tau \longrightarrow$ the left hand side of (3-13))
is continuous in $V \cap L^n$ (resp. $W \cap L^n$). Since $\{\varphi_j\}$ (resp. $\{\psi_j\}$) is total in $\hat{V}$ (resp. $\hat{W}$) , (3-12)(resp.(3-13)) holds for any v in $\hat{V}$ (resp. $\hat{W}$). Thereby $\{u,\theta\}$ is a required weak solution.

§4. Proof of Theorem 2.

Let $\{u_i,\theta_i\}$, $i=1,2$, be weak solutions of (1-1),(1-2) satisfying (i),(ii). For $i = 1,2$, there is a function $\theta_0^{(i)}$ satisfying the condition in Definition 1. Then u_i and $\theta_i - \theta_0^{(i)}$ satisfy (1-4). Since $\theta_0^{(1)} - \theta_0^{(2)}$ is 0 on Γ_1 , it belongs to W. Thereby, $\theta_1 - \theta_2$ is also in W. Put $u = u_1 - u_2$ $\theta = \theta_1 - \theta_2$. Then, they satisfy the following:

$$(4\text{-}1)\quad \begin{cases} \nu(\nabla u,\nabla v) + B(u,u_1,v) + B(u_2,u,v) - (\beta g u,v) = 0, & \forall v \in \hat{V}, \\ \kappa(\nabla\theta,\nabla\tau) + b(u,\theta_1,\tau) + b(u_2,\theta,\tau) = 0, & \forall \tau \in \hat{W}. \end{cases}$$

Here we have used Lemma 5. From the condition (i), we see

$$u \in \hat{V},\ \theta \in \hat{W}.$$

Therefore, we can take $v = u, \tau = \theta$, and we have

$$(4\text{-}2)\quad \begin{cases} \nu\|\nabla u\|^2 + B(u,u_1,u) - \beta g(\theta,u) = 0, \\ \kappa\|\nabla\theta\|^2 + b(u,\theta_1,\theta) = 0. \end{cases}$$

Let $n \geq 3$. Making use of the Hölder's inequality and Lemma 5 to estimate (4-2), we have

$$\nu\|\nabla u\|^2 \leq \|u\|_{2n/(n-2)}\ \|\nabla u\|\ \|u_1\|_n + \beta g_\infty\|\theta\|\|u\| ,$$
$$\kappa\|\nabla\theta\|^2 \leq \|u\|_{2n/(n-2)}\ \|\nabla\theta\|\ \|\theta_1\|_n .$$

By Lemma 3, we estimate the right hand side of the above equations, and we obtain:

$$\nu\|\nabla u\| \le c\|u_1\|_n \|\nabla u\| + \beta g_\infty c_1 c_2 \|\nabla\theta\| ,$$

$$\kappa\|\nabla\theta\| \le c\|\theta_1\|_n \|\nabla u\| .$$

Thereby,

$$\nu\|\nabla u\| \le (c\|u_1\|_n + \frac{\beta g_\infty c c_1 c_2}{\kappa} \|\theta_1\|_n)\|\nabla u\|$$

holds. Since u_1 , θ_1 satisfy the condition (ii):

$$c\|u_1\|_n + \frac{\beta g_\infty c c_1 c_2}{\kappa} \|\theta_1\|_n \le \nu,$$

therefore $\|\nabla u\| = \|\nabla\theta\| = 0$. Since $u = 0$ on $\partial\Omega$ and $\theta = 0$ on Γ_1, we see $u = 0$, $\theta = 0$ in Ω. Thereby $u_1 = u_2$, $\theta_1 = \theta_2$ in Ω .

When $n = 2$, we have

$$\nu\|\nabla u\|^2 \le \|u\|_{p'}\|\nabla u\| \|u_1\|_p + \beta g_\infty \|\theta\|\|u\| ,$$

$$\kappa\|\nabla\theta\|^2 \le \|u\|_{p'}\|\nabla\theta\| \|\theta_1\|_p .$$

where $1/p + 1/p' = 1/2$. We discuss in a similar way to the case $n \ge 3$, and we have $u = 0$, $\theta = 0$. Theorem is proved.

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