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Stability in L^r for the Navier-Stokes Flow in a n-dimensional Bounded Domain

By

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Introduction

The purpose of this paper is to investigate the stability for an incompressible fluid motion in a bounded domain in $\mathbb{R}^n$.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with smooth boundary $\partial\Omega$. The motion of the fluid occupying Ω is governed by the Navier-Stokes equations:

$$\left.\begin{aligned} & -\Delta w + w\cdot\nabla w + \nabla q = f, \quad \nabla\cdot w = 0 \quad \text{in} \quad \Omega, \\ & w\big|_{\partial\Omega} = 0, \end{aligned}\right\} \text{(S)}$$

where $w = w(x) = (w^1(x), \cdots, w^n(x))$ and $q = q(x)$ denote the velocity and the pressure of the fluid, respectively, and $f = f(x) = (f^1(x), \cdots, f^n(x))$ denotes the external force. If $w(x)$ and $f(x)$ are perturbed by $a(x)$ and $g(x, t)$, respectively, then the perturbed flow $v(x, t)$ is governed by the following time-dependent Navier-Stokes equations:

$$\left.\begin{aligned} & \partial_t v - \Delta v + v\cdot\nabla v + \nabla\pi = f + g \quad \text{in} \quad Q := \Omega \times (0,\infty), \\ & \nabla\cdot v = 0 \quad \text{in} \quad Q, \\ & v\big|_{\partial\Omega} = 0, \\ & v\big|_{t=0} = w + a. \end{aligned}\right\} \text{(N. S)}$$

There are many papers concerning the stability problem for the solutions of the Navier-Stokes equations. See, e.g., Ladyzenskaya [10], Heywood [6] [7], Masuda [11] and Sattinger [12]. These results, however, are obtained in L^2-setting or require some regularity assumptions on the perturbed flow at the initial time. Making use of the method developed by Giga & Miyakawa [5], *we consider the perturbed flow in* L^r *and take such assumptions away.*

To state our results, we need some preliminaries. For $m \in \mathbb{R}$ and $r > 1$, $W^{m,r}(\Omega)$ denotes the Sobolev space of order m, so that $W^{0,r}(\Omega) = L^r(\Omega)$. We set $\mathbb{W}^{m,r}(\Omega) = W^{m,r}(\Omega) \otimes \mathbb{C}^n$, $\mathbb{L}^r(\Omega) = L^r(\Omega) \otimes \mathbb{C}^n$. For $k \in \mathbb{N} \cup \{0\}$, a Banach space X and an interval $I \subset \mathbb{R}$, $C^k(I; X)$ denotes the space of continuously differentiable functions from I into X. For $0 < \mu < 1$, $C^\mu(I; X)$ denotes the space of functions in $C^0(I; X)$ satisfying the Hölder condition with exponent μ on compact subintervals of I. We set $BC(I; X) = C^0(I; X) \cap L^\infty(I; X)$. $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ denotes the set of all C^∞-vector fields φ with compact support in Ω such that $\nabla \cdot \varphi = 0$. For $r > 1$, X_r denotes the completion of $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ with respect to the $\mathbb{L}^r(\Omega)$-norm $\|\cdot\|_r$. Then by Fujiwara & Morimoto [1], we have the following decomposition:

$$\mathbb{L}^r(\Omega) = X_r \oplus G_r \quad (\text{direct sum}),$$

where $G_r = \{\nabla\pi;\ \pi \in W^{1,r}(\Omega)\}$. Let P_r be the projection operator from $\mathbb{L}^r(\Omega)$ onto X_r associated with this decomposition. We define the Stokes operator A_r by $A_r = -P_r\Delta$ with domain $D(A_r) = X_r \cap \{u \in \mathbb{W}^{2,r}(\Omega);\ u|_{\partial\Omega} = 0\}$. Applying P_r to both sides of (S) and (N.S), we have the equations in X_r:

$$A_r w + P_r w\cdot\nabla w = P_r f. \tag{S)'}$$

$$\left.\begin{aligned} &\frac{dv}{dt} + A_r v + P_r v\cdot\nabla v = P_r(f + g), \quad t > 0, \\ &v(0) = a + w. \end{aligned}\right\} \text{(N.S)'}$$

Our main results now read:

Theorem 1. *Let* $r > \max(n/3, 1)$ *and* $f \in \mathbb{L}^r(\Omega)$. *Then there is a positive number* $\lambda = \lambda(r)$ *such that* (S)' *has a unique solution* w *in* $D(A_r)$ *if* $\|P_r f\|_r \le \lambda$.

Theorem 2. *Let* $r > \max(n/3, 1)$ *and* $0 < \mu < 1$. *Let* σ *satisfy* $\sigma = n/2r - 1/2$ *for* $n/3 < r < n/2$, $\sigma = 1/2 + \varepsilon$ *for* $r \ge n/2$, *where* $0 < \varepsilon < \min(1/2, n/2r)$. *Let* γ *and* δ *satisfy* $n/2r - 1/2 \le \gamma < 1$, $\delta \ge 0$ *and* $-\gamma < \delta < \min((1 - |\gamma|)/2, 1 - \sigma)$. *Let* $\lambda(r)$ *be the number given by Theorem 1. Then, there are positive numbers* $\lambda' \le \lambda(r)$ *and* $\eta = \eta(r, n, \gamma, \delta)$ *such that for any* $(a, f, g) \in D(A_r^\gamma) \times \mathbb{L}^r(\Omega) \times C^\mu((0,\infty); \mathbb{L}^r(\Omega))$ *with* $\|P_r f\|_r \le \lambda'$, $\|A_r^\gamma a\|_r + \sup_{t>0} t^{1-\gamma-\delta}\|A_r^{-\delta} P_r g\|_r \le \eta$, (N. S)' *has a unique solution* v *satisfying:*

(1) $v \in C^0([0,\infty); D(A_r^\gamma)) \cap C^1((0,\infty); X_r)$;

(2) $v(t) - w \in D(A_r)$ *for* $t > 0$, $A_r(v - w) \in C^0((0,\infty); X_r)$, *where* w *is the unique solution given by Theorem 1;*

(3) $\|A_r^\alpha(v(t) - w)\|_r = O(t^{\gamma-\alpha})$ *as* $t \longrightarrow \infty$ *for* $\gamma \le \alpha < 1 - \delta$.

In section 1, we shall prove Theorem 1. In the special case $n \le 4$, every weak solution w of (S) in $W_0^{1,2}(\Omega)$ belongs to $W^{2,r}(\Omega)$ if $f \in \mathbb{L}^r(\Omega)$. See Temam [14, p. 172, Remark 1.4] and Gerhardt [2]. Little has been known, however, about the existence of strong solution of (S) in the case $n \ge 5$. Using the properties of the fractional powers of the Stokes operator developed by Giga [4], we shall construct a *strong solution* of (S) in any dimension for f small enough. In section 2, we shall prove Theorem 2. Let $w \in D(A_r)$ be the solution in Theorem 1. Setting $u(t) = v(t) - w$, we have the following equation:

$$\left.\begin{aligned} &\frac{du}{dt} + A_r u + B_r u + P_r u\cdot\nabla u = P_r g, \quad t > 0, \\ &u(0) = a, \end{aligned}\right\} \text{(N. S)''}$$

where $B_r u = P_r(w\cdot\nabla u + u\cdot\nabla w)$. Then, the stability problem for (S) can be reduced to obtaining the time-decay estimates for the solution of (N. S)". In order to solve (N, S)" *globally in time*, we make some modifications of the argument in Giga & Miyakawa [5]. This requires the analysis of the perturbed operator $A_r + B_r$. From our view-point, the result of [5] may be regarded as the stability theorem in $\mathbb{L}^r(\Omega)$ around the rest fluid motion, i.e., $w \equiv 0$ in Ω. To characterize the domains of the fractional powers of the perturbed operators plays an important role in our case.

1. Proof of Theorem 1

In what follows, different positive constants might be denoted by the letter C. Since A_r has the bounded inverse A_r^{-1} in X_r, (S)' is equivalent to the following equation in X_r:

$$w + A_r^{-1}P_r w\cdot\nabla w = A_r^{-1}P_r f. \tag{S}''$$

We consider $D(A_r)$ as a Banach space with the norm $\|\cdot\|_{D(A_r)}$, given by $\|u\|_{D(A_r)} := \|A_r u\|_r$ for $u \in D(A_r)$. Without loss of generality, we may assume $f \in X_r$, i.e., $P_r f = f$. For $f \in X_r$ and $w \in D(A_r)$, we define

$$F(f,w) := w + A_r^{-1}P_r w\cdot\nabla w - A_r^{-1}f.$$

Then we have:

Proposition 1.1. *Let* $r > \max(n/3, 1)$. *Then,*
(1) $F: (f,w) \longmapsto F(f,w)$ *is continuous from* $X_r \times D(A_r)$ *into* $D(A_r)$.
(2) *For each* $f \in X_r$, *the map* $F(f,\cdot): D(A_r) \ni w \longmapsto F(f,w) \in D(A_r)$ *is of class* C^1.

Proof. We choose $\theta = \theta(n,r)$ and $\rho = \rho(n,r)$ satisfying $0 < \theta < 1$, $1/2 < \rho < 1$ and $\theta + \rho = n/2r + 1/2$. By Giga & Miyakawa (5, Lemma 2.2), we have

$$\|P_r u\cdot\nabla v\|_r \le C\|A_r^\theta u\|_r\|A_r^\rho v\|_r \le C\|A_r u\|_r\|A_r v\|_r, \quad u,\ v \in D(A_r). \tag{1.1}$$

Hence $F(f,w) \in D(A_r)$ for all $f \in X_r$ and $w \in D(A_r)$. Since $\|F(f_1,w) - F(f_2,w)\|_{D(A_r)} = \|P_r(f_1 - f_2)\|_r$ for $f_i \in X_r$, $i = 1, 2$, and $w \in D(A_r)$, part (1) will follow if we can show part (2). For each $w \in D(A_r)$, we define a linear operator K_w by

$$K_w u = u + A_r^{-1} P_r (w \cdot \nabla u + u \cdot \nabla w) \quad \text{for} \quad u \in D(A_r).$$

By (1.1), K_w is in the space $\mathbb{B}(D(A_r))$ of all bounded operators in $D(A_r)$. Moreover, for each $f \in X_r$, we have

$$\|F(f, w + u) - F(f, w) - K_w u\|_{D(A_r)}$$

$$= \|A_r^{-1} u \cdot \nabla u\|_r \leq C\|A_r u\|_r^2 = C\|u\|_{D(A_r)}^2.$$

This shows that the Fréchet derivative $D_w F(f, w)$ at $(f, w) \in X_r \times D(A_r)$ is equal to K_w. Since again by (1.1), the inequality

$$\|K_{w_1} v - K_{w_2} v\|_{D(A_r)} \leq C\|A_r v\|_r \|A_r (w_1 - w_2)\|_r$$

holds for all w_i, $v \in D(A_r)$, $i = 1, 2$, we see that the map $w \longmapsto K_w$ is continuous from $D(A_r)$ into $\mathbb{B}(D(A_r))$. This completes the proof of Proposition 1.1. □

By the proof of this proposition, we have $F(0, 0) = 0$, $D_w F(0, 0) = K_0 =$ identity on $D(A_r)$. Therefore it follows from the implicit function theorem that there is a unique *continuous* mapping w from a neighborhood $U_\lambda = \{f \in X_r;\ \|f\|_r < \lambda\}$ of 0 into $D(A_r)$ such that

$$w(0) = 0, \quad F(f, w(f)) = 0 \quad \text{for} \quad f \in U_\lambda. \tag{1.2}$$

(1.2) shows that $w(f)$ is a unique solution of (S)".

2. Proof of Theorem 2

We define the operator B_r by $B_r u = P_r(w\cdot\nabla u + u\cdot\nabla w)$ for $u \in D(B_r) := D(A_r^{\sigma})$, where w is the solution obtained in Theorem 1. Then it follows that $D(A_r) \subset D(B_r)$ and

$$\|B_r u\|_r \le C\|A_r w\|_r \|A_r^{\sigma} u\|_r, \quad u \in D(B_r). \tag{2.1}$$

Indeed, by the choice of σ, we have $1/2 < \sigma < 1$ and $1 + \sigma \ge n/2r + 1/2$. Then (2.1) follows from Giga & Miyakawa [5, Lemma 2.2]. The following propositions play an important role in this section.

Proposition 2.1. *Let* $L_r := A_r + B_r$ *with domain* $D(L_r) = D(A_r)$. *There is a positive constant* $C_* = C_*(\Omega, n, r)$ *such that if* $\|A_r w\|_r \le C_*$, *then* $\Sigma_+ := \{\lambda \in \mathbb{C};\ \mathrm{Re}\lambda \ge 0\} \subset \rho(-L_r)$ *(the resolvent set of* $-L_r$*) and*

$$\|(\lambda + L_r)^{-1}\|_{\mathbb{B}(X_r)} \le M_r(1 + |\lambda|)^{-1} \quad \textit{for all} \quad \lambda \in \Sigma_+ \tag{2.2}$$

with a positive constant M_r *independent of* λ.

Proof. It follows from Giga [3] (see also Wahl [15, Chapter III]) that $\Sigma_+ \subset \rho(-A_r)$ and $\|(A_r + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \le N_r(1 + |\lambda|)^{-1}$ for all $\lambda \in \Sigma_+$ with $N_r > 0$ independent of λ. Since $L_r + \lambda = (1 + B_r(A_r + \lambda)^{-1})(A_r + \lambda)$ for $\lambda \in \Sigma_+$, it is sufficient to prove that there is a constant $k_r \in (0, 1)$ such that $\|B_r(A_r + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \le k_r$ for all $\lambda \in \Sigma_+$. Indeed, by (2.1) and the moment inequality (see Tanabe [13, Proposition 2.3.3]), we have

$$
\begin{aligned}
\|B_r(A_r + \lambda)^{-1} f\|_r &\le C\|A_r w\|_r \|A_r^{\sigma}(A_r + \lambda)^{-1} f\|_r \\
&\le C\|A_r w\|_r \|A_r(A_r + \lambda)^{-1} f\|_r^{\sigma} \|(A_r + \lambda)^{-1} f\|_r^{1-\sigma} \\
&\le C\|A_r w\|_r ((N_r + 1)\|f\|_r)^{\sigma} (N_r(1 + |\lambda|)^{-1}\|f\|_r)^{1-\sigma} \\
&\le C(N_r + 1)\|A_r w\|_r \|f\|_r \qquad (2.3)
\end{aligned}
$$

for all $\lambda \in \Sigma_+$ and all $f \in X_r$. Therefore taking C_* so that $0 < C_* < 1/C(N_r + 1)$ and $k_r := C(N_r + 1)C_*$, we see, under the condition $\|A_r w\|_r \le C_*$, that $\|B(A_r + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \le k_r < 1$. □

An immediate consequence of this proposition is as follows.

Corollary 2.2. *Let* $\|A_r w\|_r \le C_*$. *Then,* $-L_r$ *generates a uniformly bounded holomorphic semi-group* $\{e^{-tL_r}\}_{t\ge0}$ *of class* C_0 *in* X_r.

Moreover, we can define the fractional power L_r^{α} of L_r for any $\alpha \in \mathbb{R}$. Concerning the domains of fractional powers L_r^α and A_r^α, we have the following:

Proposition 2.3. *Suppose that* $\|A_r w\|_r \le C_*$ *(see Proposition 2.1).*
(1) For $0 < \alpha < 1$, *the identity* $D(A_r^\alpha) = D(L_r^\alpha)$ *holds and there is a constant* $K = K(\alpha, r)$ *such that*

$$K^{-1}\|L_r^\alpha u\|_r \le \|A_r^\alpha u\|_r \le K\|L_r^\alpha u\|_r \quad \textit{for all} \quad u \in D(L_r^\alpha). \tag{2.4}$$

(2) For $\kappa > 0$ *with* $\kappa + \sigma \le 1$, *there is a constant* $K' = K'(\kappa, \sigma, r)$ *such that*

$$\|L_r^{-\kappa} u\|_r \le K'\|A_r^{-\kappa} u\|_r \quad \textit{for all} \quad u \in X_r. \tag{2.5}$$

Proof. (1) We first prove that $D(A_r^\alpha) \subset D(L_r^\alpha)$. For simplicity, we write $A = A_r$, $B = B_r$ and $L = L_r$. Note that

$$\begin{aligned} A^{-\alpha} &= \pi^{-1}\sin\pi\alpha \int_0^\infty \lambda^{-\alpha}(A + \lambda)^{-1}d\lambda \\ &= \pi^{-1}\sin\pi\alpha \int_0^\infty \lambda^{-\alpha}(A + B + \lambda)^{-1}(A + B + \lambda)(A + \lambda)^{-1}d\lambda \\ &= \pi^{-1}\sin\pi\alpha \int_0^\infty \lambda^{-\alpha}(L + \lambda)^{-1}(1 + B(A + \lambda)^{-1})d\lambda \\ &= L^{-\alpha} + S_\alpha, \end{aligned} \tag{2.6}$$

where $S_\alpha = \pi^{-1}\sin\pi\alpha \int_0^\infty \lambda^{-\alpha}(L + \lambda)^{-1}B(A + \lambda)^{-1}d\lambda$. Suppose that $u \in D(A^\alpha)$. Setting $v = A^\alpha u$, we have by (2.6) $u = L^{-\alpha}v + S_\alpha v$. Therefore it is enough to show $S_\alpha v \in D(L^\alpha)$. By (2.2), (2.3), Krein [9, p.115 (5.15)] and $CC_*(N_r + 1) < 1$, we have

$$\|L^\alpha(L + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \leq M(1 + \lambda)^{\alpha-1}, \quad \|B(A + \lambda)^{-1}\|_{\mathbb{B}(X_r)} \leq (1 + \lambda)^{\sigma-1}$$

for all $\lambda \geq 0$. This gives

$$\int_0^\infty \|L^\alpha\lambda^{-\alpha}(L + \lambda)^{-1}B(A + \lambda)^{-1}v\|_r d\lambda$$

$$\leq \int_0^\infty \lambda^{-\alpha}\|L^\alpha(L + \lambda)^{-1}\|_{\mathbb{B}(X_r)}\|B(A + \lambda)^{-1}\|_r d\lambda$$

$$\leq M \int_0^\infty \lambda^{-\alpha}(1 + \lambda)^{\sigma+\alpha-2}d\lambda\|v\|_r.$$

Since $\sigma < 1$, the last integrand above converges and we obtain $S_\alpha v \in D(L^\alpha)$. We next prove that $D(L^\alpha) \subset D(A^\alpha)$. Similarly we have $L^{-\alpha} = A^{-\alpha} + T_\alpha$, where $T_\alpha = -\pi^{-1}\sin\pi\alpha \int_0^\infty \lambda^{-\alpha}(A + \lambda)^{-1}B(A + B + \lambda)^{-1}d\lambda$.
Hence it suffices to show that $T_\alpha v \in D(A^\alpha)$ for $v \in X_r$. By the proof of Proposition 2.1, we see that $1 + B(A + \lambda)^{-1}$ is invertible and $\|(1 + B(A + \lambda)^{-1})^{-1}\|_{\mathbb{B}(X_r)} \leq (1 - k_r)^{-1}$ for all $\lambda \geq 0$.
Therefore

$$\|B(A+B+\lambda)^{-1}\|_{\mathbb{B}(X_r)} = \|B(A+\lambda)^{-1}(1+B(A+\lambda)^{-1})^{-1}\|_{\mathbb{B}(X_r)}$$
$$\le \|B(A+\lambda)^{-1}\|_{\mathbb{B}(X_r)}\|(1+B(A+\lambda)^{-1})^{-1}\|_{\mathbb{B}(X_r)}$$
$$\le (1-k_r)^{-1}(1+\lambda)^{\sigma-1}$$

for all $\lambda \ge 0$ and we get as before

$$\int_0^\infty \|A^\alpha \lambda^{-\alpha}(A+\lambda)^{-1}B(A+B+\lambda)^{-1}v\| d\lambda$$
$$\le \int_0^\infty \lambda^{-\alpha}\|A^\alpha(A+\lambda)^{-1}\|_{\mathbb{B}(X_r)}\|B(A+B+\lambda)^{-1}v\|_r d\lambda$$
$$\le N_r(1-k_r)^{-1}\int_0^\infty \lambda^{-\alpha}(1+\lambda)^{\sigma+\alpha-2}d\lambda\|v\|_r < \infty.$$

This shows that $T_\alpha v \in D(A^\alpha)$ for all $v \in X_r$. After all we obtain $D(A^\alpha) = D(L^\alpha)$. Since $0 \in \rho(A) \cap \rho(L)$, (2.4) is an immediate consequence of this identity.

(2) By (2.6), it suffices to show

$$\|S_\kappa u\|_r \le C\|A^{-\kappa}u\|_r \quad \text{for all } u \in X_r,$$

with $C > 0$ independent of u. For this purpose, we prove $\|S_\kappa A^\kappa v\|_r \le C\|v\|_r$ for all $v \in D(A^\kappa)$. By (2.1) and Krein [9, p.115 (5.15)], we have

$$\|B(A+\lambda)^{-1}A^\kappa v\|_r \le C\|Aw\|_r\|A^\sigma(A+\lambda)^{-1}A^\kappa v\|_r$$

$$= C\|Aw\|_r \|A^{\sigma+\kappa}(A+\lambda)^{-1}v\|_r$$

$$\le CC_*(N_r+1)(1+\lambda)^{\sigma+\kappa-1}\|v\|_r \le (1+\lambda)^{\sigma+\kappa-1}\|v\|_r$$

for all $v \in D(A^\kappa)$. Hence it follows from (2.2) that

$$\|S_\kappa A^\kappa v\|_r \le \pi^{-1}\sin\pi\kappa \int_0^\infty \lambda^{-\kappa}\|(L+\lambda)^{-1}\|_{\mathbb{B}(X_r)}\|B(A+\lambda)^{-1}A^\kappa v\|_r d\lambda$$

$$\le M\pi^{-1}\sin\pi\kappa \int_0^\infty \lambda^{-\kappa}(1+\lambda)^{\kappa+\sigma-2}d\lambda\|v\|_r,$$

as required. □

Now, we solve (N.S)". We first construct a solution of the following integral equation:

$$u(t) = e^{-tL_r}a + \int_0^t e^{-(t-s)L_r}P_r(g(s) - u\cdot\nabla u(s))ds. \tag{I.E}$$

In order to solve (I.E), we use the implicit function theorem similar to Kozono [8]. Let r, γ and δ be as in Theorem 2. We define function spaces $\mathfrak{X} = \mathfrak{X}^r_{\gamma,\delta}$ and $\mathfrak{Y} = \mathfrak{Y}^r_\gamma$ by

$$\mathfrak{X}^r_{\gamma,\delta} = \{f;\ \text{measurable functions on } (0,\infty) \text{ with values in } X_r,\ t^{1-\gamma-\delta}L_r^{-\delta}f \in BC([0,\infty);\ X_r)\},$$

$$\mathfrak{Y}^r_\gamma = \{u \in BC([0,\infty);D(L_r^\gamma)) \cap C^0((0,\infty);D(L_r^{(1+\gamma)/2}));$$

$$\sup_{t>0} t^{(1-\gamma)/2} \|L_r^{\gamma} u(t)\|_r < \infty\},$$

respectively. Then $\mathfrak{X}^r_{\gamma,\delta}$ and $\mathfrak{Y}^r_{\gamma}$ are Banach spaces with norms

$$\|f\|_{\mathfrak{X}} = \|f\|_{\mathfrak{X}^r_{\gamma,\delta}} := \sup_{t>0} t^{1-\gamma-\delta} \|L_r^{-\delta} f(t)\|_r,$$

$$\|u\|_{\mathfrak{Y}} = \|u\|_{\mathfrak{Y}^r_{\gamma}} := \sup_{t>0} \|L_r^{\gamma} u(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} u(t)\|_r,$$

respectively. Without loss of generality, we may assume $P_r g = g$. For $(a,g,u) \in D(L_r^{\gamma}) \times \mathfrak{X} \times \mathfrak{Y}$, we define

$$G(a,g,u)(t) := u(t) - e^{-tL_r} a - \int_0^t e^{-(t-s)L_r} (g(s) - P_r u\cdot\nabla u(s))\, ds.$$

Then we have:

Proposition 2.4. *Suppose that* $\|A_r w\|_r \le C_*$ *(see Proposition 2.1).*
(1) $G: (a,g,u) \longmapsto G(a,g,u)$ *is continuous from* $D(L_r^{\gamma}) \times \mathfrak{X} \times \mathfrak{Y}$ *into* $\mathfrak{Y}$.
(2) For each $(a,g) \in D(L_r^{\gamma}) \times \mathfrak{X}$, *the map*
$G(a,g,\cdot): \mathfrak{Y} \ni u \longmapsto G(a,g,u) \in \mathfrak{Y}$ *is of class* C^1.

Proof. We first show that $G(a,g,u) \in \mathfrak{Y}$ for $(a,g,u) \in D(L_r^{\gamma}) \times \mathfrak{X} \times \mathfrak{Y}$. By the moment inequality (Tanabe [13, Propotition 2.3.3]), we have

$$\|L_r^{\alpha} u\|_r \le C_{\alpha,\gamma} \|L_r^{\gamma} u\|_r^{(1+\gamma-2\alpha)/(1-\gamma)} \|L_r^{(1+\gamma)/2} u\|_r^{2(\alpha-\gamma)/(1-\gamma)}$$

for $\gamma \le \alpha \le (1+\gamma)/2$ and $u \in D(L_r^{(1+\gamma)/2})$ with $C_{\alpha,\gamma}$ independent of u. Therefore it follows that

$$\|L_r^{\alpha} u(t)\|_r \le C_{\alpha,\gamma}\|u\|_{\mathcal{Y}} t^{\gamma-\alpha}, \quad t > 0, \ \gamma \le \alpha \le (1+\gamma)/2 \tag{2.7}$$

for $u \in \mathcal{Y}$. Now, we set $v_0(t) = e^{-tL_r}a$, $v_1(t) = \int_0^t e^{-(t-s)L_r} g(s)\,ds$ and $v_2(t) = \int_0^t e^{-(t-s)L_r} P_r u\cdot\nabla u(s)\,ds$. Note that by Corollary 2.2, the inequality

$$\|L_r^{\alpha} e^{-tL_r}\|_{\mathbb{B}(X_r)} \le C_\alpha t^{-\alpha} \quad \text{for all } \alpha \ge 0,\ t > 0,$$

holds with C_α independent of t. Hence $v_0 \in \mathcal{Y}$ since $a \in D(L_r^{\gamma})$. Moreover, we have

$$\begin{aligned}
\|L_r^{\alpha} v_1(t)\|_r &\le \int_0^t \|L_r^{\alpha} e^{-(t-s)L_r} g(s)\|_r\,ds \\
&\le \int_0^t \|L_r^{\alpha+\delta} e^{-(t-s)L_r}\|_{\mathbb{B}(X_r)} \|L_r^{-\delta} g(s)\|_r\,ds \\
&\le C_{\alpha+\delta} \int_0^t (t-s)^{-\alpha-\delta} \|g\|_{\mathfrak{X}} s^{\gamma+\delta-1}\,ds \\
&\le C_{\alpha+\delta}\, B(1-\alpha-\delta, \gamma+\delta) \|g\|_{\mathfrak{X}} t^{\gamma-\alpha}
\end{aligned} \tag{2.8}$$

for $\alpha < 1-\delta$, where $B(\cdot,\cdot)$ is the beta function. Since γ, $(1+\gamma)/2 < 1-\delta$, there is a positive constant B such that

$$\sup_{t>0} \|L_r^{\gamma} v_1(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} v_1(t)\|_r \le B\|g\|_{\mathfrak{X}}. \tag{2.9}$$

Hence $v_1 \in \mathfrak{Y}$. Taking $\rho = (1+\gamma)/2 - \delta/2$, we have $\rho > 0$, $\rho + \delta > 1/2$ and $\delta + 2\rho = 1 + \gamma \ge n/2r + 1/2$. Since $\delta + \sigma < 1$, we obtain, by Proposition 2.3, Giga & Miyakawa [5, Lemma 2.2] and (2.7),

$$\begin{aligned}
\|L_r^{\alpha} v_2(t)\|_r &\le \int_0^t \|L_r^{\alpha+\delta} e^{-(t-s)L_r}\|_{\mathbb{B}(X_r)} \|L_r^{-\delta} P_r u\cdot\nabla u(s)\|_r ds \\
&\le C_{\alpha+\delta} K' \int_0^t (t-s)^{-\alpha-\delta} \|A_r^{-\delta} P_r u\cdot\nabla u(s)\|_r ds \\
&\le C_{\alpha+\delta} K' \int_0^t (t-s)^{-\alpha-\delta} \|A_r^{\rho} u(s)\|_r^2 ds \\
&\le C_{\alpha+\delta} K' K_\rho^2 \int_0^t (t-s)^{-\alpha-\delta} \|L_r^{\rho} u(s)\|_r^2 ds \\
&\le C_{\alpha+\delta} K' K_\rho^2 \int_0^t (t-s)^{-\alpha-\delta} \|u\|_{\mathfrak{Y}}^2 s^{2\gamma-2\rho} ds \\
&= C_{\alpha+\delta} K' K_\rho^2 \|u\|_{\mathfrak{Y}}^2 \int_0^t (t-s)^{-\alpha-\delta} s^{\gamma+\delta-1} ds \\
&= C_{\alpha+\delta} K' K_\rho^2 B(1-\alpha-\delta, \gamma+\delta) \|u\|_{\mathfrak{Y}}^2 t^{\gamma-\alpha}
\end{aligned} \tag{2.10}$$

for $\gamma \le \alpha < 1 - \delta$. Hence there is a constant $B' > 0$ such that

$$\sup_{t>0} \|L_r^{\gamma} v_2(t)\|_r + \sup_{t>0} t^{(1-\gamma)/2} \|L_r^{(1+\gamma)/2} v_2(t)\|_r \le B' \|u\|_{\mathfrak{Y}}^2 \tag{2.11}$$

and we have $v_2 \in \mathfrak{Y}$. After all we see that G maps $D(L_r^{\gamma}) \times \mathfrak{X} \times \mathfrak{Y}$ into $\mathfrak{Y}$. In view of Corollary 2.2 and (2.9), part (1) will follow if we can show part (2). For $u, v \in \mathfrak{Y}$, we put

$$(T_u v)(t) := v(t) + \int_0^t e^{-(t-s)L_r} P_r(u\cdot\nabla v(s) + v\cdot\nabla u(s))ds$$

In the same way as in (2.11), we see that $T_u \in \mathbb{B}(X_r)$ for $u \in \mathfrak{Y}$ and that $u \longmapsto T_u$ is continuous from $\mathfrak{Y}$ into $\mathbb{B}(X_r)$. Moreover,

$$\|G(a,g,u+v) - G(a,g,u) - T_u v\|_{\mathfrak{Y}} \le B'\|v\|_{\mathfrak{Y}}^2 \tag{2.12}$$

for $(a,g,u) \in D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{Y}$ and $v \in \mathfrak{Y}$. Indeed, in the same way as in (2.10), we have

$$\sum_{\alpha=\gamma,\,(\gamma+1)/2} t^{\alpha-\gamma}\|L_r^\alpha(G(a,g,u+v) - G(a,g,u) - T_u v)\|_r$$

$$= \sum_{\alpha=\gamma,\,(\gamma+1)/2} t^{\alpha-\gamma}\|\int_0^t L_r^\alpha e^{-(t-s)L_r} P_r v\cdot\nabla v(s)ds\|_r \le B'\|v\|_{\mathfrak{Y}}^2$$

for all $t > 0$.

(2.12) shows that the Fréchet derivative $D_u G(a,g,u)$ at $(a,g,u) \in D(L_r^\gamma) \times \mathfrak{X} \times \mathfrak{Y}$ is equal to T_u. This completes the proof. □

Since $G(0,0,0) = 0$, $D_u G(0,0,0) =$ identity on $\mathfrak{Y}$, it follows from the implicit function theorem that there is a unique continuous map u from a neighborhood $V_{\eta'} = \{(a,g) \in D(L_r^\gamma) \times \mathfrak{X};\ \|L_r^\gamma a\|_r + \|g\|_{\mathfrak{X}} < \eta'\}$ of $(0,0)$ into $\mathfrak{Y}$ such that

$$u(0,0) = 0, \quad G(a,g,u(a,g)) = 0 \quad \text{for} \quad (a,g) \in V_{\eta'}. \tag{2.13}$$

This shows that $u(a,g)$ is a unique solution of (I.E) for (a,g).

Using the same method as in Giga & Miyakawa [5, Theorem 2.5], we see that $P_r u\cdot\nabla u$ for such a solution u is Hölder continuous on $(0,\infty)$ with values in X_r. Then it follows from Tanabe [13, Theorem 3.3.4] that u satisfies the *differential equation* (N.S)". □

Remark. By Proposition 2.3, we can choose η in Theorem 2 so small that $(a,g) \in V_{\eta'}$. Since the map $w : U_\lambda \ni f \longmapsto w(f) \in D(A_r)$ is continuous (see Proposition 1.1), we can take λ' $(\le \lambda)$ so that $\|A_r w\|_r \le C_*$ if $\|f\|_r \le \lambda'$.

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