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The Classical Incompressible Navier-Stokes Limit of the Boltzmann Equation

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1. Introduction

It has been recently recognized by several mathematicians, Sone (So) (for the first time), Bardos, Golse and Levermore (BGL1), and De Masi, Esposito and Lebowitz (DMEL), that the incompressible Navier-Stokes equationn can be obtained as the limit of the Boltzmann equation, when both the Mach number and the Knudsen number go to zero. It is the only macroscopic limit with a finite Reynold number and therefore the only case where global in time solutions exist. In (BGL2) the relation between the global weak solutions of the incompressible Navier-Stokes equation due to Leray (L) and the renormarized solutions of the Boltzmann equation introduced by Diparna and Lions (DL) is discussed. However, the proof cannot be completed without some additional assumptions, namely,

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-- that the corresponding renormalized solution of Diperna and Lions satisfies the equation of the conservation of momentum,

$$\partial_t\langle v f^{\varepsilon}\rangle + \nabla_x\langle v \quad v f^{\varepsilon}\rangle = 0,$$

-- and that some concentration phenomenas are avoided.

On the other hand it is known that with initial data small enough (in the Sobolev space $H^{\ell}(R^n_x)$ with $\ell>n/2$) with respect to the viscosity, the incompressible Navier-Stokes equation has a classical solution. Similar results concerning the Boltzmann equation were proved by Ukai (U1).

In the present paper a rigorous proof of the connection between these two points of view is done, precising and completing some results which were announced in (BGL1). The point to be stressed is that exactly the same type of hyposesis are made on the initial data for the Navier-Stokes equation and the Boltzmann equation. Such a result is obtained by sharp estimates of the linearized operator (cf. Lemmas 2.1-3 below, and references (U1) and (U2) for the same technique).

Let $\varepsilon > 0$ be the Knudsen Number (=Mach number/Reynolds number). According to (BGL1), the Boltzmann equation gives the incompressible Navier-Stokes equation in the limit $\varepsilon \to 0$ if the time scale is measured with ε and if the solution f^{ε} remains near an abolute Maxwellian $M = M(v)$ with the distance of order ε. Thus, the scaled Boltzmann equation

(1.1) $$\varepsilon f^{\varepsilon}_t + v\cdot\nabla_x f^{\varepsilon} = \frac{1}{\varepsilon}Q(f^{\varepsilon}, f^{\varepsilon})$$

is to be solved assuming $f^{\varepsilon} = M + \varepsilon M^{1/2} g^{\varepsilon}$, that is, the solution to the equation

$$\varepsilon g_t^{\varepsilon} + v \cdot \nabla_x g^{\varepsilon} = \frac{1}{\varepsilon} L g^{\varepsilon} + \Gamma(g^{\varepsilon}, g^{\varepsilon}) \tag{1.2}$$

is sought (the definition of the operators L and Γ follows [U1]). Let $\langle\ ,\ \rangle$ denote the inner product of $L^2(R_v^n)$. The following theorem is due to [BGL1], with the assumption slightly modified.

Theorem 1.1 ([BGL1]). Suppose that as $\varepsilon \to 0$,

$$g^{\varepsilon} \to g^0 \quad \text{in } \mathcal{D}'_{t,x,v} \ \text{(distribution sense)}, \tag{1.3}$$

$$\langle\psi, g^{\varepsilon}\rangle \to \langle\psi, g^0\rangle, \qquad \langle\psi, \Gamma(g^{\varepsilon}, g^{\varepsilon})\rangle \to \langle\psi, \Gamma(g^0, g^0)\rangle \quad \text{in } \mathcal{D}'_{t,x},$$

for any $\psi \in L^2(R_v^n)$, with some limit g^0. Then g^0 must have the form

$$g^0 = \{\rho + v\cdot u + \tfrac{1}{2}(|v|^2 - n)\theta\} M(v)^{1/2}, \tag{1.4}$$

and the coefficients ρ, v and θ must satisfy the equations,

$$\nabla(\rho + \theta) = 0, \tag{1.5}$$

$$\frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \tag{1.6}$$

$$\nabla\cdot u = 0,$$

$$\frac{\partial\theta}{\partial t} - \kappa\Delta\theta + u\cdot\nabla\theta = 0. \tag{1.7}$$

where ν and κ are positive constants determined by the operator L.

The aim of this paper is to prove the convergence (1.3) for the

Cauchy problem to (1.2) with the initial condition

(1.8) $\quad g^{\varepsilon}|_{t=0} = g_0,$

assuming that the initial g_0 is independent of ε. Throughout the paper Grad's cutoff hard potential ([G]) is assumed. Then, it is known ([U1]) that for each fixed $\varepsilon>0$, the Cauchy problem to (1.2) has a unique (strong) solution globally in t if g_0 is small in some norm.

In the below we shall show the following.

-- The smallness condition on g_0 for the global existence does not depend on $\varepsilon \in (0, 1]$, and

-- g^{ε} converges to some limit g^0 as $\varepsilon \to 0$ strongly enough to substantiate the assumption (1.3).

We can go farther:

-- In general, the convergence is not uniform near $t=0$.

-- However, the limit g^0 is strongly continuous up to $t=0$.

-- The coefficient u in (1.4) is a unique global strong solution to the Cauchy problem for the incompressible Navier-Stokes equation (1.6) with the initial condition

(1.9) $\quad u|_{t=0} = Pu_0,$

while it holds that

(1.10) $\quad \rho + \theta = 0$

and θ is a unique global strong solution to the heat convective equation (1.7) with the initial condition

(1.11) $\quad \theta|_{t=0} = \frac{1}{2}(\theta_0-\rho_0).$

Here, P is the projection to the divergence-free subspace and ρ_0, u_0 and θ_0 are given respectively by

(1.12) $$\rho_0 = <M^{1/2}, g_0>, \quad u_0 = <vM^{1/2}, g_0>, \quad \theta_0 = \frac{1}{n}<(|v|^2-n)M^{1/2}, g_0>.$$

The breakdown of the uniform convergence near t=0 is the initial layer to the Boltzmann equation (1.2). But we can show:

-- The necessary and sufficient condition for the uniform convergence up to t=0 is that the initial g_0 has the form

(1.13) $$g_0 = \{\rho_0 + u_0 \cdot v + \frac{1}{2}(|v|^2-n)\theta_0\} M(v)^{1/2},$$

with the coefficients ρ_0, u_0, θ_0 satisfying

(1.14) $$\nabla \cdot u_0 = 0, \quad \rho_0 + \theta_0 = 0.$$

To state our result precisely, we need some function spaces. Let $C(\Omega;X)$ and $L^\infty(\Omega;X)$ denote the spaces of functions continuous and bounded on $\Omega \subset \mathbf{R}^m$ with values in a Banach space X, respectively. As usual, X is omitted when $X = \mathbf{C}$. Denote the norm of the Sobolev space $H^\ell = H^\ell(\mathbf{R}^n_x)$ by $\| \ \|_\ell$ and define the spaces

$$H_{\ell,\beta} = \{f=f(x,v) \mid (1+|v|^\beta) f \in L^\infty(\mathbf{R}^n_v; H^\ell),$$

$$\sup_{|v|>R} (1+|v|)^\beta \| f(\cdot, v) \|_\ell \to 0 \ (R \to 0)\},$$

$$X_{\ell,\beta} = C([0,\infty); H_{\ell,\beta}) \cap L^\infty(0,\infty; H_{\ell,\beta}),$$

$$L^{\infty,\beta} = \{u = u(v) \mid (1+|v|)^\beta |u(v)| \in L^\infty(\mathbf{R}^n)\}.$$

The norm of $H_{\ell,\beta}$ is defined by

$$\|f\|_{\ell,\beta} = \sup_v (1+|v|)^{\beta} \|f(\cdot,v)\|_{\ell}.$$

Our results are as follows.

Theorem 1.2. Let $\ell>n/2$ and $\beta>n/2+1$. Then there are positive constants a_0 and a_1 such that for any $\varepsilon\in(0,1]$ and for any $g_0 \in H_{\ell,\beta}$ with $\|g_0\|_{\ell,\beta} \leq a_0$, there exists a unique global solution g^{ε} to (1.2) and (1.8) satisfying

(1.15) $\quad g^{\varepsilon} \in X_{\ell,\beta}$,

(1.16) $\quad \|g^{\varepsilon}\|_{\ell,\beta} \leq a_1 \|g_0\|_{\ell,\beta}$,

Remark 1.3. Under the additional condition $g_0 \in H_{\ell,\beta} \cap L^1(R^n_x; L^2(R^n_v))$, we can have the decay

$$\|g(t)\|_{\ell,\beta} \leq a_1 (1+t)^{-n/4} (\|g_0\|_{\ell,\beta} + \|g_0\|_{L^{1,2}}).$$

Theorem 1.4. Let g^{ε} be as in Theorem 1.2. Then, as $\varepsilon \to 0$,

(1.17) $\quad g^{\varepsilon} \to g^0$ weakly* in $L^{\infty}(0,\infty;H_{\ell,\beta})$ and

strongly in $C([\delta,T]\times K; L^{\infty,\beta})$,

for any $T>\delta>0$ and any compact $K \subset R^n_x$, with the limit

(1.18) $\quad g^0 \in X_{\ell,\beta}$.

Remark 1.5. (i) (1.17) is strong enough to satisfy the convergence assumptions (1.3). In particular, (1.17) assures that

$$\Gamma(g^{\varepsilon}, g^{\varepsilon}) \to \Gamma(g^{0}, g^{0}) \text{ weakly* in } L^{\infty}(0,\infty;H_{\ell,\beta-1}).$$

(ii) Accordingly, g^0 must be of the form (1.4) but the initial g_0 is not necessarily of the form (1.13).

(iii) (1.18) says that g^0 is continuous up to $t=0$. This does not come automatically from (1.15) and (1.17) if $\delta \neq 0$.

Since $\{1, v, |v|^2-n\}M^{1/2}$ forms an orthogonal system, ρ, u and θ in (1.4) are given by (1.12) with g_0 replaced by g^0. Define ρ^{ε}, u^{ε} and θ^{ε} by (1.12) with g_0 replaced by g^{ε}.

Theorem 1.6. (i) As $\varepsilon \to 0$,

(1.19) $\quad (\rho^{\varepsilon}, u^{\varepsilon}, \theta^{\varepsilon}) \to (\rho, u, \theta)$ weakly* in $L^{\infty}(0,\infty;H^{\ell})$ and strongly in $C([\delta,T]\times K)$,

for any $T>\delta>0$ and any compact $K \subset \mathbf{R}^n_x$, and

(1.20) $\quad \rho,\ u,\ \theta \in C([0,\infty);H^{\ell}) \cap L^{\infty}(0,\infty;H^{\ell})$.

(ii) (1.10) holds, and u is a unique global solution to (1.6) and (1.9) while so is θ to (1.7) and (1.11).

(iii) It holds that

(1.21) $\quad g^0|_{t=0} = \{a + b\cdot v - \frac{a}{2}(|v|^2-n)\}M^{1/2}$,

where

(1.22) $\quad a = \frac{1}{2}(\rho_0-\theta_0), \quad b = Pu_0$,

ρ_0, u_0, θ_0 being given by (1.12) in terms of g_0.

Remark 1.7. (i) Because of (1.18), (1.5) is equivalent to (1.10).
(ii) A theorem similar to Theorem 1.2 is known for the Cauchy problem to the Navier-Stokes equation (1.6): Let $\ell > n/2$. Then there are positive constants a_0 and a_1 such that if $b \in H^\ell$ with $\|b\|_\ell \leq a_0$ and $\nabla \cdot b = 0$, (1.6) has a unique global solution u belonging to the class (1.20) and satisfing $u|_{t=0} = b$. If, in addition, $b \in H^\ell \cap L^1(R^n_x)$, then,

$$\|u(t)\|_\ell \leq a_1 (1+t)^{-n/4} (\|b\|_\ell + \|b\|_{L^1}).$$

(iii) Write the right hand side of (1.21) as $P^0 g_0$. Then, P^0 is the projection on $H_{\ell,\beta}$ onto its subspace consisting of functions g_0 of the form (1.13) with the coefficients satisfying (1.14).

Theorem 1.8. $\delta = 0$ is allowed in (1.17), and hence in (1.19), if and only if $g_0 = P^0 g_0$.

2. Outline of the Proof

First, we write (1.2) in the form of the evolution equation,

$$\text{(2.1)} \qquad \frac{dg}{dt} = B^\varepsilon g + \frac{1}{\varepsilon} \Gamma(g, g), \qquad g|_{t=0} = g_0,$$

where

(2.2) $$B^{\varepsilon} = \frac{1}{\varepsilon^2} (-\varepsilon v \cdot \nabla_x + L).$$

Let $U^{\varepsilon}(t)$ denote the semigroup generated by B^{ε};

(2.3) $$U^{\varepsilon}(t) = e^{tB^{\varepsilon}}.$$

Then, (2.1) can be reduced to an integral equation

(2.4) $$g = N^{\varepsilon}(g),$$

where N^{ε} is a map define by

(2.5) $$N^{\varepsilon}(g)(t) = U^{\varepsilon}(t) g_0 + \frac{1}{\varepsilon} \int_0^t U^{\varepsilon}(t-\tau) \Gamma(g(\tau), g(\tau)) d\tau.$$

We will prove Theorem 1.2 by showing that the map N^{ε} is a contraction for all $\varepsilon \in (0, 1)$ in a ball of $X_{\ell,\beta}$ with radius independent of ε. This requires

-- uniform estimates of $U^{\varepsilon}(t)$ and the integral

(2.6) $$\Psi^{\varepsilon}(g,h)(t) = \frac{1}{\varepsilon} \int_0^t U^{\varepsilon}(t-\tau) \Gamma(g(\tau), g(\tau)) d\tau.$$

We will derive the estimates

-- by using the spectral representation ([U1]) of $U^{\varepsilon}(t)$ in the space $H_{\ell} = L^2(R^n_v; H^{\ell}_x)$, and

-- by following Grad's idea [G] to derive the estimates in $H_{\ell,\beta}$ from those in H_{ℓ}.

The unbounded factor $1/\varepsilon$ $(\varepsilon \to 0)$ in (2.6) is controlled by using again the spectral representaion and the fact ([G]) that

(2.7) $\Gamma(g,h)$ is orthogonal to $Ker(L)$, for any g, h.

The following two lemmas gives the desires estimates.

Lemma 2.1. Let $\ell \in \mathbf{R}$ and $\beta > n/2+1$. For each $\varepsilon \in (0,1)$, $U^{\varepsilon}(t)$ is a strongly continuous semigroup on $H_{\ell,\beta}$, and

(2.8) $\exists C_1 > 0,\ \forall \varepsilon \in (0,1),$

$$\|U^{\varepsilon}(t) g_0\|_{\ell,\beta} \le C_1 \|g_0\|_{\ell,\beta}.$$

Lemma 2.2. Let $\ell > n/2$ and $\beta > n/2 + 1$. For each $\varepsilon \in (0,1)$, Ψ^{ε} is a bilinear symmetric continuous map from $X_{\ell,\beta} \times X_{\ell,\beta}$ into $X_{\ell,\beta}$, and

(2.9) $\exists C_2 > 0,\ \forall \varepsilon \in (0,1),$

$$\|\Psi^{\varepsilon}(g,h)\|_{\ell,\beta} \le C_2 \|g\|_{\ell,\beta} \|h\|_{\ell,\beta}.$$

Proof of Theorem 1.2. Owing to Lemmas 2.1 and 2.2, N^{ε} maps $X_{\ell,\beta}$ into itself, and it holds that

$$\|N^{\varepsilon}(g)\|_{\ell,\beta} \le C_1 \|g_0\|_{\ell,\beta} + C_2 \|g\|^2_{\ell,\beta},$$

$$\|N^{\varepsilon}(g) - N^{\varepsilon}(h)\|_{\ell,\beta} \le C_2 (\|g\|_{\ell,\beta} + \|h\|_{\ell,\beta}) \|g-h\|_{\ell,\beta}.$$

This implies that there exist constants a_0, $a_1 > 0$ determined only by C_1 and C_2 such that N^{ε} is a contraction on a ball in $X_{\ell,\beta}$ of radius a_1 if $\|g_0\|_{\ell,\beta} \le a_0$. This proves Theorem 1.1 because C_1, C_2 are independent of ε and so are a_0, a_1.

The proof of Theorem 1.4 depends on

-- the existence of limits of $U^{\varepsilon}(t)$ and Ψ^{ε} as $\varepsilon \to 0$, which is establised by a stationary phase method of (U2) applied to the spectral representation of $U^{\varepsilon}(t)$, and

-- a compactness argument.
These are stated in the following two lemmas.

Lemma 2.3. Let $\ell > n/2$ and $\beta > n/2+1$. There exists a linear operator $V(t)$ having the following properties (2.10-13).

(2.10) For any $g_0 \in H_{\ell,\beta}$, $V(t)g_0 \in X_{\ell,\beta}$ and

$$\|V(t)g_0\|_{\ell,\beta} \le C_1 \|g_0\|_{\ell,\beta},$$

with the same constant C_1 as in (2.8).

(2.11) $V(0) = P^0,$

where P^0 is as in Remark 1.6 (iii).

(2.12) $U^{\varepsilon}(t)g_0 \to V(t)g_0 \quad (\varepsilon \to 0)$

strongly in $C((\delta,\infty)\times R^n; L^{\infty,\beta})$ for any $\delta>0$.

(2.13) $\delta=0$ if and only if $g_0 = P^0 g_0$.

Lemma 2.4. Let $\ell > n/2$ and $\beta > n/2+1$. Then, there exists a bilinear symmetric operator Θ having the following properties (2.14-18).

(2.14) If $g, h \in X_{\ell,\beta}$, then $\Theta(g,h) \in X_{\ell,\beta}$, and

$$\|\Theta(g,h)\|_{\ell,\beta} \le C_2 \|g\|_{\ell,\beta} \|h\|_{\ell,\beta},$$

with the same constant C_2 as in (2.9).

(2.15) $\Theta(g,h)|_{t=0} = 0.$

(2.16) $\exists\,\sigma>0,\ \forall\,T>0,\ \exists\,C_T>0,$

$$|\Psi^{\varepsilon}(g,h) - \Theta(g,h)|_{T,\beta} \le C_T\,\varepsilon^{\sigma}\,\|g\|_{\ell,\beta}\,\|h\|_{\ell,\beta}$$

where $|\ |_{T,\beta}$ is the norm of the space $L^{\infty}((0,T)\times R^n_x;L^{\infty,\beta}_v)$.

(2.17) For any bounded set $\{g_k\}\subset X_{\ell,\beta}$, $\{\Theta(g_k,g_k)\}$ is compact in $C((0,T)\times K;L^{\infty,\beta})$ for any $T>0$ and any compact $K\subset R^n$.

(2.18) If $\{g_k\}$ is such that $g_k\to g_0$ weakly* in $L^{\infty}(0,\infty;H_{\ell,\beta})$, and strongly in $C((\delta,T)\times K;L^{\infty,\beta})$ for any $T>\delta>0$, $K\subset R^n$, then,

$$\Theta(g_k,g_k)\to\Theta(g_0,g_0)\ \text{weakly* in}\ L^{\infty}(0,\infty;H_{\ell,\beta}).$$

Proof of Theorem 1.4. Let $g=g^{\varepsilon}$ be the solution of Theorem 1.2. In view of (1.16), $\{g^{\varepsilon}\}$ is bounded in $X_{\ell,\beta}$, so that we have, going to a subsequence,

$$g^{\varepsilon}\to g^{0}\ \text{weakly* in}\ L^{\infty}(0,T;H_{\ell,\beta}),$$

with a limit $g^0\in L^{\infty}(0,\infty;H_{\ell,\beta})$. Write (2.4) as

$$\begin{aligned} g^{\varepsilon} &= V(t)g_0 + (U^{\varepsilon}(t)-V(t))g_0 + \Theta(g^{\varepsilon},g^{\varepsilon}) + \\ &\quad + (\Psi^{\varepsilon}(g^{\varepsilon},g^{\varepsilon})-\Theta(g^{\varepsilon},g^{\varepsilon})) \\ &\equiv g_1 + g_2^{\varepsilon} + g_3^{\varepsilon} + g_4^{\varepsilon}. \end{aligned}$$

Apply (2.12) to g_2^{ε}, (2.16) to g_4^{ε} and (2.17) to g_3^{ε}. Then, again passing to a subsequence, (2.19) $\quad g^{\varepsilon} \to g^0$ strongly in $C((\delta, T] \times K; L^{\infty,\beta})$,

with $g^0 = g_1 + g_3^0$, where g_3^0 is a limit of a convergent subsequence of $\{g_3^{\varepsilon}\}$. But, then, by (2.18), we get $g_3^0 = \Theta(g^0, g^0)$, and hence,

(2.20) $\quad g^0 = V(t) g_0 + \Theta(g^0, g^0)$.

Since the constants C_1, C_2 are the same in Lemmas 2.1 – 4, we can apply the same contraction mapping principle to (2.20) as to N^{ε}. Hence the limit g_0 is a unique solution to (2.20) in $X_{\ell,\beta}$. Thus g^0 belongs to the class (1.18), and the uniqueness of g^0 implies that (1.17) is true for the whole sequence $\{g^{\varepsilon}\}$, not only for a subsequence. This proves Theorem 1.4.

Proof of Theorem 1.6. (1.19) comes from (1.17), and (1.20) from (1.18). Putting t=0 in (2.20) and applying (2.11) and (2.15) prove (iii) of the theorem. Now, (1.21) and (1.22) imply the initial conditions (1.9) and (1.11). This completes the proof of the theorem.

Proof of Theorem 1.8. This is evident from (2.13) applied to (2.19).

The proof of Lemmas 2.1 to 2.4 will be given elsewhere.

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