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On absorbing sets for evolution equations in fluid mechanics

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§1. Introduction.

We consider an evolution equation generated by a subdifferential operator and show the existence of an absorbing set for this equation. As for examples, we deal with the Navier-Stokes equation and the heat convection equation in a time-dependent domain $\Omega(t)$ in R^2.

In the case of a fixed domain $\Omega(t) \equiv \Omega$, Constantin- Foias-Temam [1] and Foias-Manley-Temam [2] studied the Navier-Stokes equation and the heat convection equation respectively. They discussed the existence of absorbing sets and attractors in their papers. (See also Temam [6].)

§2. Abstract equation.

Let H and V be two Hilbert spaces with $V \subset H$. We identify H with a subspace of V', where V' stands for the duall of V. We note $V \subset H \subset V'$.

Let φ^t be a proper lower semicontinuous convex function and $\partial\varphi^t$ be a subdifferential of φ^t. We assume $D(\varphi^t) \subset V$. We notice that $D(\partial\varphi^t) \subset D(\varphi^t)$ holds.

We make some assumptions on φ^t :

Assumption I. (Poincaré type inequality)
There exist a positive constant $C_1 > 0$ such that

(1) $\varphi^t(u) \geq C_1\|u\|_V^2 \geq C_1\|u\|_H^2$ for any $u \in D(\varphi^t)$.

Assumption II. There exist positive constants τ_0, C_2 and C_3 such that the next properties hold:

For every $t_0 \in [0,T]$ and $u_0 \in D(\varphi^{t_0})$, There exists an H-valued absolutely continuous function $v(t)$ on a closed interval $I(t_0) \equiv [\max\{t_0 - \tau_0, 0\}, \min\{t_0 + \tau_0, T\}]$ satisfying

(2) $\|v(t) - u_0\|_H \leq C_2 \cdot |t - t_0| \cdot \varphi^{t_0}(u_0)^{1/2}$ for each $t \in I(t_0)$

and

(3) $\varphi^t(v(t)) \leq \varphi^{t_0}(u_0) + C_3 \cdot |t-t_0| \cdot \varphi^{t_0}(u_0)$ for each $t \in I(t_0)$

Assumption III. (Green's formula type)

For any $u \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(u)$, the following holds :

(4) $(g, u)_H = 2\varphi^t(u)$.

Remark. The constant 2 in the right hand side of (4) is not essential.

Now, we introduce a bilinear operator B^t mapping $V \times V$ into V' and $D(\partial\varphi^t) \times D(\partial\varphi^t)$ into H such that

(B0) $(B^t(u, v), v)_H = 0$ for any $u, v \in V$,

(B1) $\|B^t(u, v)\|_H \leq C_4\|u\|_H^{\theta_4} \cdot \|u\|_V^{1-\theta_4} \cdot \|v\|_V^{1-\theta_4} \cdot \|g\|_H^{\theta_4}$

for $u \in V$, $v \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(v)$,

(B2) $\|B^t(u, v)\|_H + \|B^t(v, u)\|_H \leq C_5\|u\|_V \cdot \|v\|_V^{1-\theta_5} \cdot \|g\|_H^{\theta_5}$

for $u \in V$, $v \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(v)$, (B3) $|(B^t(u, v), w)_H| \leqq C_6 \|u\|_H^{\theta_6} \cdot \|u\|_V^{1-\theta_6} \cdot \|v\|_V \cdot \|w\|_V^{\theta_6} \cdot \|w\|_H^{1-\theta_6}$

for $u, v, w \in V$,

where C_i $(i = 4, 5, 6)$ are positive constants and $\theta_i \in [0,1)$ $(i = 4, 5, 6)$.

Next, we are given a linear operator $R(t)$ mapping V into V' and $D(\partial\varphi^t)$ into H such that

(R1) $\|R(t)u\|_H \leqq C_7 \|u\|_V^{1-\theta_7} \cdot \|g\|_H^{\theta_7}$ for $u \in D(\partial\varphi^t)$, $g \in \partial\varphi^t(u)$,

(R2) $|(R(t)u, v)_H| \leqq C_8 \|u\|_V^{1+\theta_8} \cdot \|u\|_H^{1-\theta_8}$ for $u \in V$,

where C_i $(i=7,8)$ are positive constants and $\theta_i \in [0,1)$ $(i=7,8)$.

Then, we consider the nonlinear evolution equation (E) in H as follows :

(E) $\dfrac{du}{dt} + \partial\varphi^t(u) + B^t(u, u) + R(t)u \ni f$,

where f is given in H.

Here, we define a solution of (E) and an absorbing set for (E) in V.

Definition 1. Let $u : [0,T] \to H$. Then u is called a solution of (E) on $[0,T]$ if it satisfies the following properties (i), (ii) and (iii) :

(i) $u \in C([0,T] ; H)$,

(ii) $u(t)$ is absolutely continuous in t on $[0,T]$ and

$\dfrac{du}{dt} \in L^2(0, T ; H)$,

(iii) $u(t) \in D(\partial\varphi^t)$ for a.e. $t \in [0,T]$ and there is a function

$g \in L^2(0,T;H)$ satisfying $g(t) \in \partial\varphi^t(u(t))$ and

$$\frac{du}{dt} + g(t) + B^t(u(t), u(t)) + R(t)u(t) = f(t)$$

for a.e. $t \in [0,T]$.

Definition 2. Let $u(t)$ be a solution of (E) satisfing an initial condition $u(0) = u_0 \in D(\varphi^0)$. Then a subset $\mathcal{A}$ of V is called an absorbing set for (E) in V if for an arbitrary bounded set $\mathcal{B} \subset V$, there exists a positive number $t(\mathcal{B})$ such that if the initial data u_0 is in $\mathcal{B}$ then $u(t) \in \mathcal{A}$ holds all $t \geqq t(\mathcal{B})$.

§3. Examples.

We will give some examples in fluid mechanics.

Example 1. Let us consider the Navier-Stokes equation in $\Omega(t) \subset R^2$. We assume that there exists a bounded set B in R^2 satisfying $\Omega(t) \subset B$ and the boundary $\partial\Omega(t)$ is sufficiently smooth with respect to (x, t). Moreover, suppose that a function $\beta(\cdot,t)$ on $\partial\Omega(t)$ is the boundary value of a smooth solenoidal function $b(\cdot,t)$ which is defined on $\Omega(t)$. Then we consider

$$\text{(NS)} \begin{cases} u_t + (u\cdot\nabla)u = \nu\Delta u - \nabla p + f & \text{in } \Omega(t) , \\ \operatorname{div} u = 0 & \text{in } \Omega(t) , \\ u|_{\partial\Omega(t)} = \beta \quad , \quad u|_{t=0} = a & , \end{cases}$$

where u means the velocity and p is the pressure. Putting $u = \hat{u} + b$ and abbreviating the roof ^, we introduce the followings :

$$\varphi_B(u) = \begin{cases} \frac{\nu}{2}\int_B |\nabla u|^2 dx & \text{if } u \in H^1_\sigma(B) , \\ + \infty & \text{if } u \in H_\sigma(B) \setminus H^1_\sigma(B) , \end{cases}$$

and

$$I_{K(t)}(u) = \begin{cases} 0 & \text{if } u \in K(t) , \\ +\infty & \text{if } u \in H_\sigma(B) \setminus K(t) , \end{cases}$$

where $K(t) = \{u \in H_\sigma(B) \mid u = 0 \text{ a.e. in } B \setminus \Omega(t)\}$.

Then we put

$$\varphi^t(u) = \varphi_B(u) + I_{K(t)}(u) \qquad \text{for each } t \in [0,T].$$

Moreover, we define

$$B^t(u(t), u(t)) = P_\sigma(B)(u\cdot\nabla)u \qquad \text{for } u \in H^1_\sigma(B),$$

$$R(t)u(t) = P_\sigma(B)\{(u\cdot\nabla)b + (b\cdot\nabla)u\} \quad \text{for } u \in H^1_\sigma(B),$$

where $P_\sigma(B)$ is the projection $L^2(B) \longrightarrow H_\sigma(B)$. We notice that (B0) $\sim$ (B3), (R1) and (R2) are all satisfied if $\theta_4 = \theta_5 = \theta_6 = \theta_7 = \frac{1}{2}$ and $\theta_8 = 0$.

Remark. (B1) dose not hold in the case R^3.

Then, we can reduce the equation (NS) to the abstract Navier-Stokes equation (ANS) as follows :

$$\text{(ANS)} \quad \frac{du}{dt} + \partial\varphi^t(u(t)) + B^t(u(t), u(t)) + R(t)u(t) \ni P_\sigma(B)\bar{f}(t),$$

where $\bar{f} = \tilde{f} - b_t - (b\cdot\nabla)b + \nu\Delta b$ and $\tilde{f}$ is an extension of f putting zero on the outside of $\Omega(t)$.

Example 2. Consider the heat convection equation in $\Omega(t) \subset R^2$ with the boundary $\partial\Omega(t) = \partial\Omega_0 \cup \partial\Omega_1(t)$. The domain $\Omega(t)$ is included in a bounded set B and the boundary $\partial\Omega(t)$ is sufficiently smooth in (x,t). u, p and β stand for the same as in Example 1. While θ denotes the temperature. Then the heat convection equation is as follows :

$$
\text{(HC)}\begin{cases} u_t + (u\cdot\nabla)u = -\dfrac{\nabla p}{\rho} + \{1-\alpha(\theta - T_0)\}g + \nu\Delta u & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta & \text{in } \Omega(t), \\ u|_{\partial\Omega(t)} = \beta \ , \quad \theta|_{\partial\Omega_0} = T_0 > 0 \ , \quad \theta|_{\partial\Omega_1(t)} = 0 & \\ u|_{t=o} = a \ , \quad \theta|_{t=o} = h \ . & \end{cases}
$$

After suitable changes of variables (some scalings and translations), we define for $U = {}^t(u, \theta)$

$$
\varphi_B(U) = \begin{cases} \frac{1}{2}\int_B(|\nabla u|^2 + \frac{\kappa}{\nu}|\nabla\theta|^2)dx & \text{if } U \in H^1_\sigma(B) \times \mathring{W}^1_2(B) \ , \\ +\infty & \text{if } U \in \{H_\sigma(B) \times L^2(B)\} \setminus \{H^1_\sigma(B) \times \mathring{W}^1_2(B)\} \ , \end{cases}
$$

and

$$
I_{k(t)}(U) = \begin{cases} 0 & \text{if } u \in K(t) \ , \\ +\infty & \text{if } U \in \{H_\sigma(B) \times L^2(B)\} \setminus K(t) \ , \end{cases}
$$

where $K(t) = \{U \in H_\sigma(B) \times L^2(B) \ ; \ U = 0 \ \text{ a.e. in } B \setminus \Omega(t)\}$.

We put

$$
\varphi^t(U) = \varphi_B(U) + I_{K(t)}(U) \quad \text{for each } t \in [0,T].
$$

Moreover, we set

$$
B^t(U(t), U(t)) = {}^t(P_\sigma(B)(u\cdot\nabla)u, (u\cdot\nabla)\theta)
$$

$$
\text{for } u \in H^1_\sigma(B) \times \mathring{W}^1_2(B) \ ,
$$

$$
R(t)U(t) = {}^t(P_\sigma(B)((u\cdot\nabla)b + (b\cdot\nabla)u + R\theta), \ (u\cdot\nabla)\tilde{\bar{\theta}} + (b\cdot\nabla)\theta),
$$

and

$$\tilde{f}(t) = (-b_t + (b\cdot\nabla)b + \Delta b + \frac{d^3}{\nu^2}g - R_a(\tilde{\bar{\theta}} - \frac{\kappa}{\nu}), -(b\cdot\nabla)\tilde{\bar{\theta}}),$$

where $\bar{\theta}$ is a solution of a linear heat equation in $\Omega(t)$ with $\bar{\theta} = T_0$ on $\partial\Omega_0$ and $\bar{\theta} = 0$ on $\partial\Omega_1(t)$. Furthermore, $R_a = \alpha g T_0 d^3/\kappa\nu$ and 2d is the diameter of B. Notice that (B0) ~ (B3), (R1) and (R2) are satisfied if $\theta_4 = \theta_5 = \theta_6 = \theta_7 = \frac{1}{2}$ and $\theta_8 = 0$.

Remark. (B1) does not hold if $\Omega(t) \subset R^3$.

Then, we introduce the abstract heat convection equation (AHC) as follows :

(AHC) $\frac{du}{dt} + \partial\varphi^t(u(t)) + B^t(u(t), u(t)) + R(t)u(t) \ni P(B)\tilde{f}(t).$

§4. Some Lemmas.

First we mention lemmas on B^t.

Lemma 1. Let u, v $\in D(\partial\varphi^t)$ and put w = u - v, then

(5) $\|B^t(u, u) - B^t(v, v)\| \leqq$

$$C_4(\|w\|_H^{\theta_4}\|w\|_V^{1-\theta_4}\|u\|_V^{1-\theta_4}\|\partial\varphi^t(u)\|_H^{\theta_4}+\|v\|_H^{\theta_4}\|v\|_V^{1-\theta_4}\|w\|_V^{1-\theta_4}\|\partial\varphi^t(w)\|_H^{\theta_4}).$$

Lemma 2. Let u, v $\in$ V and put w = u - v, then

(6) $(B(u, u) - B(v, v), w)_H = (B(w,u), w)_H$,

(7) $|(B(u, u) - B(v, v), w)_H|$

$$\leqq C_6\|w\|_H^{\theta_6}\cdot\|w\|_V^{1-\theta_6}\cdot\|u\|_H\cdot\|w\|_V^{\theta_6}\cdot\|w\|_H^{1-\theta_6} .$$

Next we prepare a lemma on φ^t.

Lemma 3. Suppose Assumption II holds. Let u : [0,T] $\longrightarrow$ H and $\varphi^t(u(\cdot))$: [0,T] $\longrightarrow$ [0,+∞) be absolutely continuous on [0,T]. Let L ≡ {t ∈ (0,T):du/dt, $d\varphi^t(u(t))/dt$ exist and $u(t) \in D(\partial\varphi^t)$}.

Then, there exist positive constants C_2 and C_3 such that

(8) $|\frac{d}{dt}\varphi^t(u(t))-(g, \frac{d}{dt}u(t))_H| \leqq C_2\cdot\|g\|_H\cdot\varphi^t(u(t))^{1/2} + C_3\cdot\varphi^t(u(t))$

holds for every $t \in L$ and $g \in \partial\varphi^t(u(t))$, where C_2 and C_3 are positive constants in Assumption II.

Here we consider the following linear abstract evolution equation (E').

(E') $\frac{du}{dt} + \partial\varphi^t(u(t)) \ni f(t)$, $t \in [0,T]$.

Then we have

Lemma 4. Suppose that Assumption I ~ II hold. Let $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi^0)$. Then, there exist a unique pair of functions $u \in C([0,T];H)$ and $g \in L^2(0,T;H)$ such that the following properties (i) ~ (V) hold :

(i) $u(t)$ is strongly absolutely continuous on $[0,T]$ and $du/dt \in L^2(0,T;H)$,

(ii) $u(t) \in D(\partial\varphi^t)$ and $g(t) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0,T]$,

(iii) $\frac{dt}{du} + g(t) = f(t)$ for a.e. $t \in [0,T]$,

(iV) $u(t) \in D(\varphi^t)$ for every $t \in [0,T]$ and $\varphi^t(u(t))$ is absolutely continuous on $[0,T]$,

(V) $u(0) = u_0$ in H.

Now, we return to the equation (E). In what follows in this section, let $\varphi(t)$ satisfy Assumption I ~ III, operators B^t and $R(t)$ have the properties (B0) ~ (B3) , (R1) and (R2), respectively. Then we have

Lemma 5. Let $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi^0)$. Then a solution u of (E) with $u(0) = u_0$ is atmost one if it exists.

Lemma 6. Let $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi^0)$. Then there exists a positive number τ_0 (depending on f and u_0) such that a solution of (E) with $u(0) = u_0$ exist on $[0,\tau_0]$.

Lemma 7. Let u be a local solution of (E) on $[0,\tau_0]$ obtained in Lemma 6. Then $\varphi^t(u(t))$ is absolutely continuous in t on $[0,\tau_0]$.

Here we mention the case that the operator $\partial\varphi^t + R(t)$ is coercive on V.

Lemma 8. Suppose that we can take a positive number α' independent of $T > 0$ such that for any $u \in D(\partial\varphi^t)$ and $g \in \partial\varphi^t(u(t))$

(9) $(g, u)_H + (R(T)u, u)_H \geqq \alpha' \|u\|_V^2$

holds for each $t \in [0,T]$. Let u be solution of (E) with $u(0) = u_0 \in D(\varphi^0)$. Then the followings hold :

(i) If $f \in L^2(0,T;H)$, then

(10) $\|u(t)\|_H^2 \leqq e^{-\alpha' t}\|u(0)\|_H^2 + \frac{1}{\alpha'}\|f\|^2_{L^2(0,T;H)}$.

(i)' If $f \in L^\infty(o,T;H)$, then

(11) $\|u(t)\|_H^2 \leqq e^{-\alpha' t}\|u(0)\|_H^2 + \frac{1}{(\alpha')^2}\|f\|^2_{L^\infty(o,T;H)} \cdot (1 - e^{-\alpha' t})$.

Moreover, if a solution u exists on $[0,\infty)$, then we have

(ii) If $f \in L^2(o,\infty;H)$, then

(12) $\|u\|^2_{L^\infty(o,\infty;H)} \leqq \|u(0)\|_H^2 + \frac{1}{\alpha'}\|f\|^2_{L^2(o,\infty;H)}$,

(13) $\limsup_{t \to \infty} \|u(t)\|_H^2 \leqq \frac{1}{\alpha'}\|f\|^2_{L^2(o,\infty;H)}$.

(ii)' If $f \in L^\infty(o,\infty;H)$, then we can get two inequalities obtained by replacing $\|f\|^2_{L^2(o,\infty;H)}/\alpha'$ with $\|f\|^2_{L^\infty(o,\infty;H)}/(\alpha')^2$ in (12) and (13).

Remark. We can show that in the Navier-Stokes equation the operator $\partial\varphi^t + R(t)$ is coercive in $H^1_\sigma(B)$.

Now, the following two lemmas play important roles in proving the existence of an absorbing set. Note that we does not assume $\partial\varphi^t + R(t)$ is coercive in the lemmas below.

Lemma 9. Let f be in $L^\infty(0,\infty;H)$ or in $L^2(0,\infty;H)$. Let u(t) be a solution of (E) on [0,T]. Suppose that

(i) u(t) satisfies the following estimate :

(14) $\|u(t)\|^2_H \leqq A_0 + A_0'\|u(0)\|^2_H$ for any $t \in [0,T]$,

where A_0 and A_0' are two positive constants independent of T. and that

(ii) $\varphi^t(u(t))$ is absolutely continuous with repect to t on [0,T].

Then, for any δ in $(0,T)$, there exist positive constants $a_1(\delta)$, $a_2(\delta)$ and $a_3(\delta)$, independent of T, depending on δ, such that

(15) $\varphi^t(u(t)) \leqq (\frac{a_2(\delta)}{\delta} + a_3(\delta))e^{a_1(\delta)}$

holds for every $t \in [\delta,T]$.

Corollary of Lemma 9. Let f be in $L^\infty(0,\infty;H)$ or in $L^2(0,\infty;H)$. If the local solution u on $[0,\tau_0]$ obtained in Lemma 6 satisfies the estimate (14) in Lemma 9, then u can be extended globally to $[0,\infty)$.

Lemma 10. Let $f \in L^\infty(0,\infty;H)$ or $f \in L^2(0,\infty;H)$. Suppose that a global solution u of (E) on $[0,\infty)$ has the following properties :

(i) There exists a constant $A > 0$ such that for any $\varepsilon > 0$ we can take $t_0 > 0$ (depending on u(0) and ε) satisfying

(16) $\|u(t)\|_H^2 \leqq A + \varepsilon$ for any $t \geqq t_0$.

(ii) $\varphi^t(u(t))$ is absolutely continuous in t on [0,T].

Then the followings hold :

(i) There exist positive constants $a_1(\varepsilon)$, a_2 and $a_3(\varepsilon)$ (those are depending on u(0)) such that

(17) $\varphi^t(u(t)) \leqq (a_2 + a_3(\varepsilon))e^{a_1(\varepsilon)}$ for any $t \geqq t_0 + 1$.

(ii) There exists an absorbing set for (E) in V.

§5. Applications.

Example 1. We consider the abstract Navier-Stokes equation (ANS) introduced in §3. Note that $\Omega(t) \subset R^2$.

Then we have

Theorem 1. If $\tilde{f} - b_t - (b\cdot\nabla) + \nu\Delta b \in L^\infty(0,\infty;L^2(B))$ or $L^2(0,\infty;L^2(B))$, then there exists an absorbing set for (ANS) in $V = H_\sigma^1(B)$.

Outline of the proof. As we mentioned in Remark after Lemma 8, the operator $\partial\varphi^t + R(t)$ is coercive in $V = H_\sigma^1(B)$. Therefore, Lemma 8 is applicable to (ANS). Hence, by virture of Lemma 9, Corollary of Lemma 9 and Lemma 10, we can show the existence of an absorbing set. Q.E.D.

Example 2. We consider the abstract heat convection equation (AHC) introduced in §3. Note that $\Omega(t) \subset R^2$. Then we have

Theorem 2. If $\tilde{f} = (-b_t - (b\cdot\nabla)b + \Delta b + d^3g/\nu^2 - R_a(\tilde{\bar{\theta}} -$

κ/ν), $-(b\cdot\nabla)\tilde{\bar{\theta}}) \in L^{\infty}(0,\infty; L^2(B) \times L^2(B))$, then there exists an absorbing set for (AHC) in $V = H^1_\sigma(B) \times \mathring{W}^1_2(B)$.

Outline of the proof. In the equation (AHC), the operator $\partial\varphi^t + R(t)$ is not coercive. But an a priori estimate on $\theta(t)$ holds(see Lemma 11 below). Thanks to Lemma 11, we can use Lemma 9, its corollary and Lemma 10. Therefore we have established the theorem. Q.E.D.

Finally, we mention Lemma 11.

Lemma 11. Let $U = {}^t(u,\theta)$ be a strong solution of (AHC). Then there exist functions θ_1 and θ_2 such that

(18) $\theta(\cdot, t) = \theta_1(\cdot, t) + \theta_2(\cdot, t)$ for a.e. (x, t),

(19) $-\kappa/\nu \leqq \theta_1(\cdot, t) \leqq \kappa/\nu$ for a.e. (x, t),

(20) $\|\theta_2(t)\|_{L^2(B)} \leqq \{\|(\theta-\kappa/\nu)_+(0)\|_{L^2(B)} + \|(\theta+\kappa/\nu)_-(0)\|_{L^2(B)}\}$
$\times \exp(-2\kappa t/\nu)$ for $t \geqq 0$.

References

[1] Constantin, P., Foias, C. and Temam, R., Attractors representing turbulent flows, Mem. Am. Mat. Soc., 53, No. 314(1985).

[2] Foias, C., Manley, O., and R. Temam., Attractors for the Bénard problem : Existence and physical bounds on their fractal dimension, Nonlinear Anal. T.M.A., 11, 939-967 (1987).

[3] Ōeda, K., On the initial value problem for the heat convection equation of Boussinesq approximation in a time-

dependent domain, Proc. Japan Acad., 64A, 143-146(1988).

[4] Ōeda, K., Weak and strong solutions of the heat convection equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo, Sect. IA, 36, 491-536(1989).

[5] Ōeda, K., Remarks on the stability of certain periodic solutions of the heat convection equations, Proc. Japan Acad., 66A, 9-12(1990).

[6] Temam, R., Infinite-dimensional dynamical systems in mechanics and physics, Springer-Verlag, 1988.