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Abelian coverings of links

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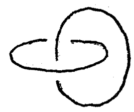
Recently, the Smith Conjecture was proved by W.P. Thurston, H.H. Bass, P. Shalen, W. Meeks - S.T. Yau, and C.McA. Gordon - R.A. Litherland. In fact, they proved the following:

Branched Covering Theorem. Let Σ be a homotopy 3-sphere and let K be a knot in Σ. Then, if the n-fold branched cyclic covering space of Σ branched along K is a homotopy 3-sphere for some $n \geq 2$, K is a trivial knot.

In the Topology Symposium in Sapporo 1979, Professor K. Murasugi showed that Branched Covering Theorem implies the following:

Theorem M. Let S^3 be the 3-sphere. Then if an abelian covering M of a link in S^3 is simply connected, M is S^3.

Furthermore he conjectured the following:

Conjecture M. The only link in S^3, other than the trivial knot, which has a homotopy sphere as an abelian covering is  .

In this paper, we will prove this conjecture.

I would like to express my sincere gratitude to the members of Kobe Topology Seminar for their helpful suggestions and conversations.

§ 1. Reducing the conjecture to another conjecture.

In this section we review Murasugi's proof of Theorem M and reduce Conjecture M to another conjecture.

First, we give a simple proof to a theorem of Murasugi and Mayberry [5] which gives a necessary condition for an abelian covering of a link to be a homology sphere and which is a key lemma to prove Theorem M. To do this, we use a method of [6]. Let $L = K_1 \cup \dots \cup K_\mu$ be an oriented link of μ-components in an oriented homology 3-sphere S, and let $X = S - L$. By Alexander duality the first integral homology group $H_1(X)$ is the free abelian group on μ-generators $t_1, \dots, t_\mu$, where t_i is the meridian of K_i. Let $\tilde{X}_a$ be the universal abelian covering space of X; that is, the covering space of X corresponding to the kernel of the Hurewicz homomorphism $\gamma : \pi_1(X) \to H_1(X)$. Let A be a finite abelian group and $\psi : H_1(X) \to A$ be an epimorphism. Let $\tilde{X}_\psi$ be the covering space of X corresponding to $\mathrm{Ker}(\psi \circ \gamma)$, and M_ψ be the branched covering space of S obtained by the completion of $\tilde{X}_\psi$. We use the symbol q (resp. j) to denote the natural projection $\tilde{X}_a \to \tilde{X}_\psi$ (resp. the inclusion $\tilde{X}_\psi \hookrightarrow M_\psi$). Let $R(\psi)$ be the factor module $H_1(\tilde{X}_\psi)/(j \circ q)_* H_1(\tilde{X}_a)$. Then we have:

<u>Proposition 1.</u> $|R(\psi)| = (\prod_{i=1}^{\mu} n_i)/|A|$, where $|\ |$ denotes the order of a group and n_i is the order of the element $\psi(t_i)$ of A.

Proof. From the definition of $\tilde{X}_\psi$, the following sequence is exact: $1 \longrightarrow \pi_1(\tilde{X}_\psi) \longrightarrow \pi_1(X) \xrightarrow{\psi \circ \gamma} A \longrightarrow 1.$

Factoring this sequence by $q_* \pi_1(\tilde{X}_a)$, we obtain the following exact sequence: $1 \to \pi_1(\tilde{X}_\psi)/q_*\pi_1(\tilde{X}_a) \to H_1(X) \xrightarrow{\psi} A \to 1$.

Since $\pi_1(\tilde{X}_\psi)/q_*\pi_1(\tilde{X}_a)$ is abelian, $H_1(\tilde{X}_\psi)/q_*H_1(\tilde{X}_a) \cong \operatorname{Ker}\psi$. Consider the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & q_*H_1(\tilde{X}_a) & \longrightarrow & H_1(\tilde{X}_\psi) & \xrightarrow{\eta} & \operatorname{Ker}\psi & \to & 0 \\ & & \downarrow & & \downarrow j_* & & \downarrow [j_*] & & \\ 0 & \to & (j\circ q)_* H_1(\tilde{X}_a) & \to & H_1(M_\psi) & \to & R(\psi) & \to & 0 \end{array},$$

where η is a natural map and $[j_*]$ is the homomorphism induced by j_*. Since M_ψ is the completion of $\tilde{X}_\psi$, j_* is onto and $\operatorname{Ker} j_*$ is equal to the branch relation $B \subset H_1(\tilde{X}_\psi)$. Hence, from the kernel and cokernel exact sequence (see [3]), $\operatorname{Ker}[j_*] = \eta(B)$. From the definition of the branch relation B, we can see

$$\eta(B) = < t_i^{n_i} \ (1 \le i \le \mu) > \subset \operatorname{Ker}\psi \subset H_1(X),$$

where $<\ldots>$ denotes the subgroup generated by the elements in $<\ >$. Hence $R(\psi) \cong \operatorname{Ker}\psi / < t_i^{n_i} \ (1 \le i \le \mu) >$. Factoring the exact sequence $0 \to \operatorname{Ker}\psi \to H_1(X) \xrightarrow{\psi} A \to 0$ by $< t_i^{n_i} \ (1 \le i \le \mu) >$, we obtain the following exact sequence:

$$0 \to R(\psi) \to \bigoplus_{i=1}^{\mu} < t_i \mid t_i^{n_i} = 1 > \to A \to 0.$$

Hence $|R(\psi)| \cdot |A| = \prod_{i=1}^{\mu} n_i$. This completes the proof.

<u>Example.</u> Let $M_n(L)$ be the n-fold branched cyclic covering of L; that is, the branched covering space corresponding to $\operatorname{Ker}(p_n \circ \gamma)$, where p_n is the homomorphism $H_1(X) \to < t \mid t^n = 1 >$ defined by the equality $p_n(t_i) = t \ (1 \le i \le \mu)$. Then $|R(p_n)| = n^\mu/n = n^{\mu-1}$ (compare [2] or [6]). In particular, $M_n(L)$ can not be a homology sphere unless $\mu = 1$.

<u>Corollary.</u> (Theorem 11.1 of [5])
If M_ψ is a homology sphere, then $A = \oplus_{i=1}^{\mu} < \psi(t_i) >$. That is, ψ is of the following form:

$$\psi : H_1(X) \to \oplus_{i=1}^{\mu} < t_i \mid t_i^{n_i} = 1 >, \quad \psi(t_i) = t_i \quad (1 \leq i \leq \mu).$$

Now, let us review Murasugi's argument.

<u>Theorem M.</u> Let M_ψ be an abelian covering of a link L in the 3-sphere S^3. Then, if M_ψ is simply connected, M_ψ is S^3.

Proof by Murasugi. From the corollary of Proposition 1, ψ is of the following form:

$$\psi : H_1(X) \to \oplus_{i=1}^{\mu} < t_i \mid t_i^{n_i} = 1 >, \quad \psi(t_i) = t_i \quad (1 \leq i \leq \mu).$$

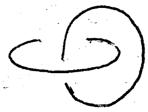
Let ψ_k $(0 \leq k \leq \mu)$ be the homomorphism $H_1(X) \to \oplus_{i=1}^{k} < t_i \mid t^{n_i} = 1 >$ defined by the equality $\psi_k(t_i) = \begin{cases} t_i & (1 \leq i \leq k) \\ 1 & (k+1 \leq i \leq \mu) \end{cases}$.

Then we obtain the following sequence of branched coverings:

$$M_\psi = M_{\psi_\mu} \to M_{\psi_{\mu-1}} \to \dots \to M_{\psi_1} \to M_{\psi_0} = S^3.$$

Note that M_{ψ_i} is the n_i-fold branched cyclic covering space of $M_{\psi_{i-1}}$ branched along $\widetilde{K}_i$, the lift of K_i in $M_{\psi_{i-1}}$. Since M_ψ is simply connected, M_{ψ_i} $(0 \leq i \leq \mu)$ is simply connected. From the example of Proposition 1, $\widetilde{K}_i \subset M_{\psi_{i-1}}$ is connected. Hence, by Branched Covering Theorem, $\widetilde{K}_i \subset M_{\psi_{i-1}}$ is a trivial knot. Since the bottom of the sequence is S^3, every M_{ψ_i} $(1 \leq i \leq \mu)$ is S^3.

From the above argument, Conjecture M is equivalent to the following:

Conjecture $\tilde{M}$. If a link $L = K_1 \cup \ldots \cup K_\mu$ ($\mu \geq 2$) in S^3 has the following property (☆), then $L \subset S^3$ is  .

(☆) There exist integers $n_1, \ldots, n_\mu$ such that

(1) $n_i \geq 2$ $(1 \leq i \leq \mu)$,

(2) $\tilde{K}_i \subset M_{\psi_{i-1}}$ $(1 \leq i \leq \mu)$ is a trivial knot, where ψ_i is the homomorphism defined in the proof of Theorem M.

§ 2. Proof of the conjecture.

Lemma 1. Let $L = K_1 \cup K_2$ be a 2-components link in S^3 with K_1 a trivial knot. Let $\tilde{K}_2$ be the lift of K_2 in $M_n(K_1)$, the n-fold branched cyclic covering of $K_1 \subset S^3$. Let $\tilde{\Delta}(t)$ be the reduced Alexander polynomial of $\tilde{K}_2$. Then, if $\lambda \equiv |lk(K_1, K_2)| \neq 1$ and $n \geq 2$, $\tilde{\Delta}(t) \neq 1$.

Proof. Let $n = p^r m$, g.c.d.$(p,m) = 1$, p a prime, $r > 0$, and $\lambda \neq 0$. Then, by Theorem 1 of Murasugi [4],

$$\rho_\lambda(t)\, \tilde{\Delta}(t) \equiv \left[\prod_{j=0}^{m-1} \Delta(\eta^j, t)\right]^{p^r} \mod p,$$

where $\Delta(t_1, t_2)$ is the Alexander polynomial of L, $\rho_\lambda(t) = 1 + t + \ldots + t^{\lambda-1}$ and η is a primitive m-th root of 1. In particular, $\Delta(1,t)^{p^r}$ divides $\rho_\lambda(t)\,\tilde{\Delta}(t)$ mod p. Since $\rho_\lambda(t)$ divides $\Delta(1,t)$ by the Torres's condition [10], $\rho_\lambda(t)^{p^r - 1}$ divides $\tilde{\Delta}(t)$ mod p. Hence, if $\lambda > 1$, $\tilde{\Delta}(t) \neq 1$. If $\lambda = 0$, $\tilde{K}_2$ is a n-components link; so, $(t-1)^{n-1}$ divides $\tilde{\Delta}(t)$ and $\tilde{\Delta}(t) \neq 1$.

Proposition 2. If $L = K_1 \cup \ldots \cup K_\mu \subset S^3$ has Property (☆), then $\mu = 2$ and $|lk(K_1, K_2)| = 1$.

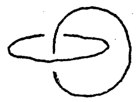
Proof. Suppose that $\mu \geq 3$. Let $\tilde{K}_i$ $(i = 2,3)$ be the lift of K_i in M_{ψ_1}, the n_1-fold branched cyclic covering of $K_1 \subset S^3$. Then $|lk(\tilde{K}_2,\tilde{K}_3)| = n_1 |lk(K_2,K_3)| \neq 1$. Hence the lift $\tilde{\tilde{K}}_3$ of K_3 in M_{ψ_2}, the n_2-fold branched cyclic covering of $\tilde{K}_2 \subset M_{\psi_1}$, can not be a trivial knot from Lemma 1; this is a contradiction. Hence $\mu = 2$ and $|lk(K_1,K_2)| = 1$ by Lemma 1.

Hence we may consider only 2-components links. Using Branched Covering Theorem, we obtain the following:

<u>Proposition 3.</u> Let $L = K_1 \cup K_2$ be a 2-components link in S^3 with K_1 a trivial knot. Suppose L has Property (★); that is, there is an integer $n_1 \geq 2$ such that the lift $\tilde{K}_2$ of K_2 in $M_{n_1}(K_1)$, the n_1-fold branched cyclic covering of $K_1 \subset S^3$, is a trivial knot. Then, for any integer $n \geq 1$, the lift $\tilde{K}_2$ of K_2 in $M_n(K_1)$ is a trivial knot.

Proof. Let $n_2 \geq 1$ be an integer and let ψ be the homomorphism $H_1(S^3-L) \to \oplus_{i=1}^{2} < t_i \mid t_i^{n_i}=1 >$ defined by the equality $\psi(t_i) = t_i$ $(i = 1,2)$. Then M_ψ is the n_2-fold branched cyclic covering space of $M_{n_1}(K_1)$ branched along $\tilde{K}_2$, the lift of K_2. Since $\tilde{K}_2 \subset M_{n_1}(K_1)$ is a trivial knot in S^3, M_ψ is S^3. Let $\tilde{K}_1$ be the lift of K_1 in $M_{n_2}(K_2)$. Then M_ψ is the n_1-fold branched cyclic covering space of $M_{n_2}(K_2)$ branched along $\tilde{K}_1$. Hence, by the proof of Theorem M, $M_{n_2}(K_2)$ is S^3 and $\tilde{K}_1$ is a trivial knot. Repeat the above argument by exchanging the roles of K_1 and K_2, and we obtain the desired result.

Now, the proof of Conjecture $\tilde{M}$ is completed by the following proposition:

<u>Proposition 4.</u> Let $L = K_1 \cup K_2$ be a link in S^3 with K_1 a trivial knot. Let $\tilde{K}_2$ be the lift of K_2 in $M_2(K_1)$, the 2-fold branched cyclic covering of $K_1 \subset S^3$. Then, if $\tilde{K}_2$ is a trivial knot, $L \subset S^3$ is  .

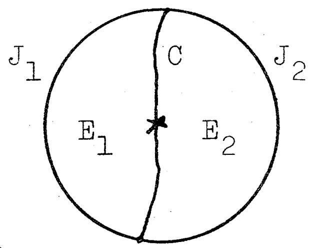
Proof. Let T be the involution of $M_2(K_1)$ generating the covering transformation group. Then the fixed-point set of T is $\tilde{K}_1$, the lift of K_1, and $T(\tilde{K}_2) = \tilde{K}_2$. Let D be a disk in $M_2(K_1)$ bounding $\tilde{K}_2$. Using this disk, we will construct a disk $\underset{\sim}{D}$ in S^3 such that $\partial \underset{\sim}{D} = K_2$ and $\underset{\sim}{D}$ intersects K_1 transversaly in a single point. Let $S(D)$ be the closure of $\mathring{D} \cap T(\mathring{D})$ in $M_2(K_1)$, where $\mathring{D}$ denotes the interior of D. Then we have

<u>Lemma 2.</u> We can deform D, without moving ∂D, so that $\mathring{D}$ is transverse to both $T(\mathring{D})$ and $\tilde{K}_1$, and $S(D)$ is a proper 1-dim. submanifold of D.

Proof. See Lemma 1 of [1]. (Lemma 1 of [1] does not require that the deformation fixes ∂D. But it is not so hard to accomplish the deformation without moving ∂D.)

Hence we may assume that $S(D)$ consists of simple closed curves and arcs with end points on ∂D. If $S(D)$ contains simple closed curves, we can eliminate them by the cut and paste method (see [1], [8]). So we may assume that $S(D)$ consists only of proper arcs. Since $|lk(\tilde{K}_1, \tilde{K}_2)| = |lk(K_1, K_2)| = 1$ by Proposition 2, $D \cap \tilde{K}_1 \neq \phi$. Hence there is a connected component C of $S(D)$ such that $C \cap \tilde{K}_1$

$\neq \phi$. Then D is a union of two disks E_1 and E_2 with $E_1 \cap E_2 = C$. Let $J_i = \partial E_i - C$ for each $i = 1,2$. Suppose there is a component C' other than C such that $C' \cap \tilde{K}_1 \neq \phi$. Without loss of generality we may assume



that $C' \subset E_1$. Then $T(\partial C') \subset T(J_1) = J_2$. On the other hand, since $C' \cap \tilde{K}_1 \neq \phi$, $T(\partial C') = \partial C' \subset J_1$. This is a contradiction. Hence any component of $S(D)$, other than C, does not intersect $S(D)$. From this, we can see that $T(E_1) \cap E_1 = C$. Let $\underset{\sim}{D} = p(E_1)$, where p is the covering projection $M_2(K_1) \to S^3$. Then, from the above argument, $\underset{\sim}{D}$ is a disk with $\partial \underset{\sim}{D} = K_1$ and $\underset{\sim}{D} \cap K_1$ = one point. This completes the proof of Proposition 4.

Thus we have proved Conjecture M.

References

[1] C.McA. Gordon and R.A. Litherland: Incompressible surfaces in branched coverings, preprint.

[2] F. Hosokawa and S. Kinoshita: On the homology group of branched cyclic covering spaces of links, Osaka Math. J. 12(1960), 331-355.

[3] S. MacLane: Homology, Berlin, Goettingen-Heiderberg, Springer-Verlag, 1963.

[4] K. Murasugi: On periodic knots, Comment. Math. Helv. 46 (1971), 162-174.

[5] K. Murasugi and J.P. Mayberry: On representations of abelian groups and the torsion groups of abelian coverings of links, preprint.

[6] M. Sakuma: The homology groups of abelian coverings of links, Math. Sem. Notes, Kobe Univ. 7(1979), 515-530.

[7] P. Shalen: The proof in the case of no incompressible surface, preprint.

[8] Y. Tao: On fixed point free involutions of $S^1 \times S^2$, Osaka Math. J. 14(1962), 145-152.

[9] W.P. Thurston: Hyperbolic structures on 3-manifolds, preprint.

[10] G. Torres: On the Alexander polynomials, Ann. of Math. 57 (1953), 57-89.

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