30

Soliton Equations as Dynamical Systems

on a Infinite Dimensional Grassmann Manifolds.

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§0.   Introduction.

It is shown that the totality of solutions to Kadomtsev-
Petviashvili (KP) equation

$$3u_{yy} + (-4u_t + u_{xxx} + 6uu_x)_x = 0 \qquad\qquad (0.1)$$

has a natural structure of Grassmann manifold (GM) of infinite
dimension. Evolution of u in variables x,y,t (and also
hidden "higher variables") is now interpreted as dynamical
motion of a point on the GM by the action of 3 (or more)
parameter subgroup of the group GL(∞) of automorphisms of
our GM. Generic points of this GM give generic solutions to
KP equation, whereas points on particular submanifold of GM
give solutions of particular type — such as rational solutions,
(multi-)soliton solutions, (multi-phase) quasi-periodic solu-
tions, or (multi-phase) similarity solutions. (E.g. rational
solutions correspond to points on finite dimensional Grassmann
manifold contained in our infinite GM.) Also, different kinds
of submanifolds of the same GM (and in some cases different

choice of parameter subgroup of GL(∞)) give rise to generic

solutions of other soliton equations such as Korteweg -

de Vries (KdV) equation, modified KdV equation, Boussinesq

equation, Sawada - Kotera equation, Non-linear Schrödinger

equation, Toda lattice, equation of self induced transparency,

Benjamin - Ono equation, etc. as well as to solutions of

particular type of these soliton equations.

The automorphism group GL(∞) of GM naturally plays

the role of group of transformations (or, of "hidden symmetries")

of KP equation. This fact provides us as a consequence with

thourough understanding on transformation property of various

soliton equations.

Multi-component generalization of the theory shows that

other soliton equations such as equation of three wave inter-

action, multi-component non-linear Schrödinger equation, Sine-

Gordon equation, Lund - Regge equation, equation for inter-

mediate long wave, etc. also constitute submanifolds of infinite

dimensional GM.

It may be conjectured that any soliton equation, or

completely integrable system, is obtained in this way.

Classification of soliton equations would then be reduced to

classification of submanifolds of our GM which are stable by

the subgroup of GL(∞) describing space-time evolution.

§1. KP equation as isospectral deformation equation of a micro-differential operator.

Consider a formal micro-differential (or pseudo-differential) operator

$$L \underset{\text{def}}{=} \sum_{n=0}^{\infty} u_n(x)\left(\frac{d}{dx}\right)^{1-n} , \qquad \text{with} \quad u_0(x) = 1,$$

$$= \frac{d}{dx} + u_1(x) + u_2(x)\left(\frac{d}{dx}\right)^{-1} + \dots . \qquad (1.1)$$

Since $u_1(x)$ is easily eliminated by the simple transformation $L \mapsto e^{S(x)} L e^{-S(x)}$ with $S(x) \underset{\text{def}}{=} \int^x u_1(x)dx$, we hereafter set $u_1(x) = 0$ without loss of generality. Let B_n denote the "differential operator" part of the n-th power L^n of L, e.g. $B_0 = 1$, $B_1 = \frac{d}{dx}$, $B_2 = \left(\frac{d}{dx}\right)^2 + 2u_2$, $B_3 = \left(\frac{d}{dx}\right)^3 + 3u_2\left(\frac{d}{dx}\right) + (3u_3 + 3u_{2,x})$,..., then isospectral deformation of the eigenvalue problem

$$L\psi = \lambda\psi , \qquad \lambda = \text{eigenvalue}, \qquad (1.2)$$

is achieved by

$$\frac{\partial}{\partial t_n}\psi = B_n\psi, \quad n = 0,1,2,\dots, \qquad (1.3)$$

where t_0, t_1, t_2,... are deformation parameters on which the quantities $u_2(x)$, $u_3(x)$,... and $\psi(x,\lambda)$ now depend, or equivalently, isospectral deformation of L is given by Lax-type equations

$$\frac{\partial L}{\partial t_n} = [B_n, L] \; ; \quad n = 0,1,2,\dots \qquad (1.4)$$

(1.4) are equivalent to Zakharov-Shabat equations which derive from (1.3):

$$[B_n - \frac{\partial}{\partial t_n}, B_m - \frac{\partial}{\partial t_m}] = 0; \quad n,m = 0,1,2,\dots . \qquad (1.5)$$

The deformation equations ((1.4) or (1.5)) imply in particular that L and hence its coefficients $u_n(x;t)$ do not depend on t_0 and depend on t_1 and x via their sum $x+t_1$, i.e.

$$u_n = u_n(x+t_1, t_2, t_3,\dots), \qquad (1.6)$$

and hence, by replacing $x+t_1$ by t_1, we consistently identify the variable x with t_1 and $\frac{d}{dx}$ with $\frac{\partial}{\partial t_1}$.

The deformation equations now read

$$u_{2,t_2} = 2u_{3,x}+u_{2,xx}, \quad u_{3,t_2} = 2u_{4,x}+u_{3,xx}+2u_2 u_{2,x}$$

$$u_{2,t_3} = 3u_{4,x}+3u_{3,xx}+u_{2,xxx}+6u_2 u_{2,x},\dots \qquad (1.7)$$

which yield in particular the KP equation (0.1) for u_2:

$$3u_{2,t_2 t_2} = (-4u_{2,t_3}+u_{2,xxx}+6u_2 u_{2,x})_x = 0$$

where x,y,t are now written $x(=t_1)$, t_2, t_3. The complete set of equations (1.7) describes complete hierarchy of

extended or higher KP equations involving the complete set of deformation parameters t_1, t_2, t_3, t_4,... .

Within the category of microdifferential operators, our L of (1.1) is transformed to the trivial one, $\frac{d}{dx}$. Namely we can find

$$W = 1 + w_1(x)(\frac{d}{dx})^{-1} + w_2(x)(\frac{d}{dx})^{-2} + \ldots \qquad (1.8)$$

so that

$$L = W \cdot \frac{d}{dx} \cdot W^{-1} \qquad (1.9)$$

holds. Accordingly the eigenfunction ψ is also transformed to the trivial one, $\psi_0 = \text{constant} \times e^{\lambda x + (t_0 + t_1 + {}^2 t_2 + \ldots)}$, yielding

$$(x,t) = W\psi_0(x,t)$$

$$= (1 + w_1(x,t)\lambda^{-1} + w_2(x,t)\lambda^{-2} + \ldots)e^{\lambda x + (t_0 + \lambda t_1 + \lambda^2 t_2 + \ldots)}$$

$$(1.10)$$

except for an arbitrary constant factor.

§2. Generalization to multi-component case

The construction of KP hierarchy given above is generalized to multi-component case as follows. (The number of components will be denoted by r. When r=1 the following formalism reduces to the previous one.)

We introduce formal microdifferential operators with matrix-coefficients of size $r \times r$:

$$L = \sum_{n=0}^{\infty} U_n(x)(\tfrac{d}{dx})^{1-n}, \quad \text{with} \quad U_0(x) = 1, \quad U_1(x) = 0, \quad (2.1)$$

$$C_i = \sum_{n=0}^{\infty} C_{in}(x)(\tfrac{d}{dx})^{-n}, \quad \text{with} \quad C_{i0}(x) = E_{ii} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & \overset{i}{1} & \\ & & & \ddots \\ & & & & 0 \end{pmatrix}, \quad (2.2)$$

$$(i=1,\ldots,r)$$

subject to the conditions (i) L commutes C_i, and (ii) $C_i C_j = \delta_{ij} C_i$. Entries of $U_n(x)$ and $C_{in}(x)$ will play the role of "potentials", i.e. unknown functions to be solved in our multi-component version of KP hierarchy.

Consider the following simultaneous eigenvalue problem of $r \times r$ matrix Ψ

$$\begin{cases} L\Psi = \lambda \Psi, & \lambda = \text{eigenvalue}, & (2.3) \\ \\ C_i \Psi = \Psi E_{ii}, & (i = 1,\ldots r) & (2.4) \end{cases}$$

i.e. the j-th column vector $\Psi^{(j)}$ of Ψ is a simultaneous eigenvector of $L, C_1, \ldots, C_r$:

$$L\Psi^{(j)} = \lambda \Psi^{(j)}, \quad C_i \Psi^{(j)} = \delta_{ij} \Psi^{(j)}, \quad (i=1,\ldots,r). \quad (2.5)$$

and its isospectral deformation equations

$$\frac{\partial}{\partial t_n^{(i)}} = B_n^{(i)} \Psi \quad (i=1,\ldots,r;\ n=1,2,\ldots)$$

36

where $B_n^{(i)}$ are "differential operator" part of $C_i L^n$; e.g.

$$B_1^{(i)} = E_{ii}\frac{d}{dx} + C_{i1}, \quad B_2^{(i)} = E_{ii}(\frac{d}{dx})^2 + C_{i1}\frac{d}{dx} + (2E_{ii}U_2 + C_{i2}), \ldots .$$

Hence the isospectral deformation of L, $C_1, \ldots, C_r$ is given by Lax-type equations

$$\frac{\partial L}{\partial t_n^{(i)}} = [B_n^{(i)}, L], \qquad \frac{\partial C_j}{\partial t_n^{(i)}} = [B_n^{(i)}, C_j], \qquad (2.6)$$

$$(i,j=1,\ldots,r; \quad n=1,2,\ldots),$$

or equivalently, by Zakharov-Shabat type equations

$$[B_n^{(i)} - \frac{\partial}{\partial t_n^{(i)}}, \quad B_m^{(j)} - \frac{\partial}{\partial t_m^{(j)}}] = 0, \qquad (2.7)$$

$$(i,j=1,\ldots,r; \quad n,m=1,2,\ldots).$$

Again, we can find a matrix-coefficiented micro-differential operator

$$W = 1 + W_1(x,t)(\frac{d}{dx})^{-1} + W_2(x,t)(\frac{d}{dx})^{-2} + \ldots \qquad (2.8)$$

so that L, $C_1, \ldots, C_r$ are transformed to trivial ones, $\frac{d}{dx}$, $E_{11}, \ldots, E_{rr}$:

$$L = W \cdot \frac{d}{dx} \cdot W^{-1}, \quad C_i = W \cdot E_{ii} \cdot W^{-1}, \qquad (i=1,\ldots,r), \quad (2.9)$$

and we get

$$\Psi(x,t) = W\Psi_0(x,t)$$

(2.10)

$$= (1+W_1(x,t)\lambda^{-1}+W_2(x,t)\lambda^{-2}+\ldots)\Psi_0(x,t)$$

where

$$\Psi_0(x,t) = \text{constant} \times \begin{pmatrix} e^{\lambda x+\eta(t^{(1)},\lambda)} & & \\ & \ddots & \\ & & e^{\lambda x+\eta(t^{(r)},\lambda)} \end{pmatrix}, \quad (2.11)$$

$$\text{with} \quad \eta(t^{(i)},\lambda) \underset{\text{def}}{=} \sum_{n=1}^{\infty} \lambda^n t_n^{(i)}. \qquad (2.12)$$

§3 Grassmann manifolds of finite and infinite dimensions

Let $V = \mathbb{C}^{m+n}$ be a vector space of dimension $m+n$ over $\mathbb{C}$. (Everything in this note is defined over $\mathbb{C}$). The Grassmann manifold $GM(m,n)$ (or $GM(m,V)$) is by definition the parameter space for the set of m-dimensional subspaces in V. Since such a subspace is spanned by an m-frame $\xi = (\xi^{(1)},\ldots,\xi^{(m)})$ consisting m linearly independent vectors $\xi^{(1)},\ldots,\xi^{(m)} \in V$, we have

$$GM(m,n) = \{m\text{-frames in } V\}/GL(m)$$

(3.1)

$$= GL(m+n)/GL(m,n)$$

where we denote by $GL(m,n)$ the subgroup of $GL(m+n)$ consisting

of elements of the form $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_4 \end{pmatrix}$ with $g_1 \in GL(m)$,

$g_4 \in GL(n)$. We also set

$$\widetilde{GM}(m,n) = (\{m\text{-frames}\} \times GL(1))/GL(m)$$

$$(= \{m\text{-frames}\}/SL(m), \quad \text{if} \quad m > 0) \tag{3.2}$$

$$\Lambda^m(V) = m\text{-th exterior product space of } V \tag{3.3}$$

and have the following situation

$$
\begin{array}{ccc}
\widetilde{GM}(m,n) & \hookrightarrow & \Lambda^m(V) - \{0\} \\
\downarrow {\scriptstyle GL(1)} & & \downarrow {\scriptstyle GL(1)} \\
GM(m,n) & \hookrightarrow & \text{Projective space} \\
& & \text{of dimension } \binom{m+n}{m}-1
\end{array}
\tag{3.4}
$$

where the embedding of the upper line is defined by letting $\widetilde{\xi} \in \widetilde{GM}(m,n)$ represented by an m-frame $\xi = (\xi^{(1)},\ldots,\xi^{(m)})$ correspond to $\xi^{(1)} \wedge \cdots \wedge \xi^{(m)} \in \Lambda^m(V)-\{0\}$. (3.4) gives the standard way of embedding $GM(m,n)$ into a projective space. $\widetilde{GM}(m,n)$ will be called the standard line bundle over $GM(m,n)$. By denoting by $\xi_{\ell_1 \ldots \ell_m}$ the minors of ξ, i.e. the determinants of $m \times m$ matrices consisting of ℓ_1-th,$\ldots$, ℓ_m-th rows of the $(m+n)$ m$\times$matrix ξ, we have

$$\xi^{(1)} \wedge \cdots \wedge \xi^{(m)} = \sum_{1 \leq \ell_1 < \ldots < \ell_m \leq m+n} \xi_{\ell_1 \ldots \ell_m} e_{\ell_1} \wedge \cdots \wedge e_{\ell_m} \tag{3.5}$$

where $e_1,\ldots,e_{m+n}$ denote the unit column vectors $\in V$. $\xi_{\ell_1,\ldots,\ell_m}$ are the Plücker coordinates of $\widetilde{\xi}$, and satisfy

the following Plücker's identities:

$$\sum_{i=1}^{m+1} (-)^i \xi_{\ell_1 \ldots \ell_{m-1} k_i} \xi_{k_1 \ldots \hat{k}_i \ldots k_{m+1}} = 0. \qquad (3.6)$$

In (3.4), GM(m,n) coincides with the intersection of the
quadrics defined by (3.6) in the projective space. We remind
that the derivation of (3.6) relies on the Clifford algebra
structure of $\Lambda(V^*) \cdot \Lambda(V)$.

We denote by V^* the dual vector space of V. The
canonical inner product $< , > : V^* \times V \to \mathbb{C}$ is extended to
$\Lambda^m(V^*) \times \Lambda^m(V) \to \mathbb{C}$ and it takes the form

$$<\bar{\xi}^*, \bar{\xi}> = \det {}^t \xi^* \cdot \xi = \sum_{\ell_1 < \ldots < \ell_m} \xi^*_{\ell_1 \ldots \ell_m} \xi_{\ell_1 \ldots \ell_m} \qquad (3.6)$$

on $\widetilde{GM}(m,V^*) \times \widetilde{GM}(m,V)$. For $m \leq m'$, $n \leq n'$ we regard $V = \mathbb{C}^{m+n}$
as a subspace of $V' = \mathbb{C}^{m'+n'}$ according to the scheme V'
$= \mathbb{C}^{m'-m} \oplus V \oplus \mathbb{C}^{n'-n}$, and let an m-dimensional subspace V_1
of V correspond to the m'-dimensional subspace $\mathbb{C}^{m'-m} \oplus$
$V_1 \oplus 0$ of V'. This process induces the embeddings

$$\begin{array}{ccccccc} \widetilde{GM}(m,n) & \hookrightarrow & \widetilde{GM}(m'n') & & GL(m+n) & \hookrightarrow & GL(m'+n') \\ \cup & & \cup & \text{and} & \cup & & \cup \\ \bar{\xi} & \longmapsto & \bar{\xi}' & & g & \longmapsto & g' \end{array}$$

$$(3.8)$$

where

$$\xi' = \begin{array}{c} m'-m \\ m+n \\ n'-n \end{array} \left(\begin{array}{c|c} \overbrace{1}^{m'-m} & \overbrace{0}^{m} \\ \hline 0 & \xi \\ \hline 0 & 0 \end{array} \right) \quad \text{and} \quad g' = \left(\begin{array}{c|c|c} \overbrace{1}^{m'-m} & \overbrace{0}^{m+n} & \overbrace{0}^{n'-n} \\ \hline 0 & g & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \tag{3.9}$$

If, further, $\bar{\xi}^* \in \widetilde{GM}(m,V^*)$ goes to $\bar{\xi}^{*\prime} \in \widetilde{GM}(m',V'^*)$ then

$$<\bar{\xi}^*, \bar{\xi}> = <\bar{\xi}^{*\prime}, \bar{\xi}'>, \tag{3.10}$$

namely, the inner product is preserved by this embedding process.

The infinite dimensional Grassmann manifold (GM) and its standard line bundle $(\widetilde{GM})$ which we need to parametrize the solutions of KP hierarchy are obtained as the topological closure of the inductive limit of $GM(m,n)$ and $\widetilde{GM}(m,n)$ as m and n tend to ∞. Explicitly, our GM is defined by

$$GM = \{ \mathbb{N}^c\text{-frames} \}/GL(\mathbb{N}^c), \tag{3.11}$$

where by an $\mathbb{N}^c$-frame we mean an infinite-sized matrix

$$\xi = (\xi_{\mu\nu})_{\mu \in \mathbb{Z}, \nu \in \mathbb{N}^c} \tag{3.12}$$

whose rows and columns are labeled by integers $\mathbb{Z}$ and strictly negative integers $\mathbb{N}^c = \mathbb{Z} - \mathbb{N} = \{-1, -2,\ldots\}$, respectively, satisfying the condition that (i) $\exists m \in \mathbb{N}$ such that $\xi_{\mu\nu} = \delta_{\mu\nu}$ for $\mu < -m$, and that (ii) m column vectors for $\nu = -m, -m+1,\ldots,-1$ are linearly independent, while $GL(\mathbb{N}^c)$

consists of

$$h = (h_{\mu\nu})_{\mu,\nu \in \mathbb{N}^c} \qquad (3.13)$$

satisfying the similar condition as above.

§4. Construction of solutions to KP hierarchy

Let Λ be the shift operator defined by $\Lambda = (\delta_{\mu,\nu-1})_{\mu,\nu\in\mathbb{Z}}$. We define τ-function for KP hierachy by

$$\tau(t;\bar{\xi}) = \tau(t_1,t_2,\ldots;\bar{\xi}) \underset{\text{def}}{=} \langle\bar{\xi}_0, e^{\eta(t,\Lambda)}\bar{\xi}\rangle \qquad (4.1)$$

where $\eta(t,\Lambda) \underset{\text{def}}{=} \sum_{n=1}^{\infty} t_n\Lambda^n$, and $\xi_0 \underset{\text{def}}{=} (\delta_{\mu\nu})_{\mu\in\mathbb{Z},\nu\in\mathbb{N}^c}$. It is shown that (4.1) well defines $\tau(t;\bar{\xi})$ under a very mild condition on the growth order of $(t_n)_{n=1,2,\ldots}$ and $(\xi_{\mu\nu})_{\mu\in\mathbb{Z},\nu\in\mathbb{N}^c}$.

From (4.1) we get the following expansion of τ in terms of character polynomials $\chi_Y(t)$ (cf. Appendix):

$$\tau(t;\bar{\xi}) = \sum_Y \xi_Y\chi_Y(t) \qquad (4.2)$$

$$= \xi_\phi + \xi_\square\chi_\square(t) + \xi_{\square\square}\chi_{\square\square}(t) + \xi_\boxminus\chi_\boxminus(t) +\ldots .$$

What distingishes (4.2) from the expansion (A3) of an arbitrary function $f(t)$ is the fact that coefficients ξ_Y in (4.2) are Plücker coordinates of $\bar{\xi} \in \widetilde{GM}$, and as such they satisfy

42

the Plücker identities, e.g.

$$\xi_\phi \xi_\boxplus - \xi_\Box \xi_\boxdot + \xi_\boxminus \xi_\Box = 0, \text{ etc.} \tag{4.3}$$

To any $\bar{\xi} \in \widetilde{GM}$ we associate $\overline{\xi[\lambda]} \in \widetilde{GM}$ defined by

$$(\xi[\lambda])_{\mu\nu} = \begin{cases} \xi_{\mu+1,\nu+1} & \text{if } \nu < -1 \\[2ex] \lambda^{\mu+1} & \text{if } \nu = -1, \end{cases} \tag{4.4}$$

and define $\Psi(t;\lambda;\bar{\xi})$ by

$$\Psi(t;\lambda;\bar{\xi}) = \frac{\tau(t;\overline{\xi[\lambda]})}{\tau(t;\bar{\xi})} . \tag{4.5}$$

Then we have

Theorem.

For any $\bar{\xi} \in \widetilde{GM}$ (4.5) solves KP hierarchy (1.2), (1.3), and vice versa.

Namely, $\bar{\xi} \in \widetilde{GM}$ completely parametrizes the set of solutions to KP hierarchy.

$u_\nu(t)$ are given in terms of τ. For example:

$$u_2(t;\xi) = (\log \tau)_{t_1 t_1}, \quad u_3(t,\xi) = \frac{-1}{2}(\log \tau)_{t_1 t_2} + \frac{1}{2}(\log \tau)_{t_1 t_1 t_1} \cdots .$$

Returning to the τ-functions, we have

$$\tau(t+a,\bar{\xi}) = \tau(t, e^{\eta(a,\Lambda)}\bar{\xi}) \tag{4.6}$$

$$e^{\sum_{n=1}^{\infty} nb_n t_n} \tau(t,\bar{\xi}) = \tau(t, e^{\eta(b,\,{}^t\Lambda)}\bar{\xi}) \qquad (4.7)$$

which are very special cases of the general transformation
formula of the form

$$T_g \tau(t,\bar{\xi}) = \tau(t, g\bar{\xi}) \qquad (4.8)$$

where T_g is a linear operator acting on the functions in t.
Let K^+ and K^- denote the abelian subgroups of $GL(\infty)$
consisting of elements of the form $e^{\eta(a,\Lambda)}$ and $e^{\eta(b,\,{}^t\Lambda)}$,
respectively, and set $K = K^+ \cdot K^-$. (4.6) shows in particular
that $\tau(t,\bar{\xi}) = \tau(0, e^{\eta(t,\Lambda)}\bar{\xi})$ and hence, that the evolution
of τ (and hence, of u_ν) in variables t_1, t_2,... is inter-
preted as the motion of $\bar{\xi} \in \widetilde{GM}$ caused by the action of
$e^{\eta(t,\Lambda)} \in K^+ = K/K^-$. On the other hand, (4.7) implies that
the action of $e^{\eta(b,\,{}^t\Lambda)} \in K^-$ on $\bar{\xi}$ does not change $\Psi(t;\lambda;\bar{\xi})$
and $u_\nu(t;\bar{\xi})$. (This in particular means that what is in 1-1
corresponce with the totality of solutions of KP hierarchy
is the quotient space $K^- \backslash GM$, rather than GM or $\widetilde{GM}$ itself.)

§5. Specializations

If $\bar{\xi} \in \widetilde{GM}$ is generic, its orbit $K\bar{\xi}$ is dense in $\widetilde{GM}$.
If, contrarily, $\bar{\xi}$ lies in a submanifold of $\widetilde{GM}$ which is stable
under K, $\bar{\xi}$ represents a solution of special type.

Let $f(\lambda) \in \mathbb{C}((\lambda^{-1}))$ (= the field of quotients in the
formal power series ring $\mathbb{C}[[\lambda^{-1}]]$). Then, regarding $f(\Lambda)$

as an element of the Lie algebra of K, we have

$f(\Lambda)$ fixes $\bar{\xi}$ $\Longleftrightarrow$ $f(L)$ is a differential operator. (5.1)

Let A be a subalgebra of $\mathbb{C}((\lambda^{-1}))$ such that

$$A \cap \mathbb{C}[[\lambda^{-1}]] = \mathbb{C}, \qquad (5.2)$$

and define

$$\widetilde{GM}^A \underset{\mathrm{def}}{=} \{\bar{\xi} \in \widetilde{GM} \mid f(\Lambda) \text{ fixes } \bar{\xi} \text{ for any } f(\lambda) \in A\}. \quad (5.3)$$

Then we see that $\widetilde{GM}^A$ is the parameter space for solutions of the specialized KP hierarchy satisfying the additional condition:

$f(L)$ is a differential operator for $\forall f(\lambda) \in A$. (5.4)

For example, if A contains an element of the form $\lambda^n + a_1 \lambda^{n-1} + \ldots$ for every sufficiently large n, then $\widetilde{GM}^A$ represents quasi-periodic, soliton or rational solutions attached to the algebraic curve SpecA, while $\widetilde{GM}^{\mathbb{C}[\lambda^2]}$ and $\widetilde{GM}^{\mathbb{C}[\lambda^3]}$ parametrize solutions to KdV hierarchy and Bonssinesq hierarchy, respectively.

Appendix.   Character polynomials of GL

Irreducible tensor representation of GL(n) is classified
by Young diagrams. Let $\chi_Y(g)$ be the irreducible character
of $g \in GL(n)$ corresponding to a Young diagram Y. Since
$\chi_Y(g)$ is a symmetric polynomial of eigenvalues $\varepsilon_1, \ldots, \varepsilon_n$
of g, it is expressed as a polynomial of t_1, $t_2, \ldots$ defined
by

$$t_\nu = \frac{1}{\nu}(\varepsilon_1^\nu + \ldots + \varepsilon_n^\nu) = \frac{1}{\nu}\text{trace } g^\nu. \qquad (A1)$$

The result is well known to be

$$\chi_Y(g) = \chi_Y(t) = \sum_{\nu_1 + 2\nu_2 + \ldots = N} \pi_Y(1^{\nu_1} 2^{\nu_2} \ldots) \frac{t_1^{\nu_1} t_2^{\nu_2} \ldots}{\nu_1! \nu_2! \ldots}, \qquad (A2)$$

where N = size of Y (= rank of tensor), and $\pi_Y(1^{\nu_1} 2^{\nu_2} \ldots)$
denotes the irreducible character of the symmetric permuta-
tion group of N letters corresponding to the Young diagram
Y and to the conjugacy class consisting ν_1 cycles of size 1
ν_2 cycles of size 2, etc. For example we have

$$\chi_\phi(t) = 1, \quad \chi_\square(t) = t_1, \quad \chi_{\square\square}(t) = \frac{t_1^2}{2} + t_2, \quad \chi_{\boxminus}(t) = \frac{t_1^2}{2} - t_2,$$

$$\chi_{\square\square\square}(t) = \frac{t_1^3}{6} + t_1 t_2 + t_3, \quad \chi_{\boxplus}(t) = \frac{t_1^3}{3} - t_3, \quad \chi_{\boxminus}(t) = \frac{t_1^3}{6} - t_1 t_2 + t_3, \ldots .$$

Note that these "character polynomials" $\chi_Y(t)$ are independent
of n as long as $n \geq N$ so that the variables $t_1, \ldots, t_N$
are independent.

46

$\{\chi_Y(t)\}$ constitute a linear base for the polynomial ring $\mathbb{C}[t_1,t_2,\ldots]$. Namely, any polynomial $f(t) \in \mathbb{C}[t_1,t_2,\ldots]$ is uniquely expressed as a finite linear sum of $\chi_Y(t)$'s. Further, any formal power series $f(t) \in \mathbb{C}[[t_1,t_2,\ldots]]$ is uniquely expressed as an infinite formal sum of $\chi_Y(t)$'s:

$$f(t) = \sum_Y f_Y \cdot \chi_Y(t), \quad f_Y \in \mathbb{C}, \qquad\qquad (A3)$$

and vice versa.

It is known that Young diagrams are in 1-1 correspondence with strictly monotone maps $\sigma : \mathbb{N}^c \to \mathbb{Z}$ such that $\sigma(\nu) = \nu$ for almost all $\nu \in \mathbb{N}^c$. On the other hand, such σ is characterized by the strictly increasing series of natural numbers $(\ell_0, \ell_1, \ldots, \ell_{n-1})$ defined by $\ell_\nu = \sigma(-n+\nu)$, $\nu = 0, 1, \ldots, n-1$, where we assume that n is so chosen that $\sigma(\nu) = \nu$ for $\nu < -n$.