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AFFINE SEMIGROUP RINGS AND HODGE ALGEBRAS

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A Hodge algebra is a commutative algebra having a special basis which allows one to determine many features of its structure by a relatively simple combinatorial study of its generators and relations.

Many interesting examples, such as coordinate rings of Grassmannian varieties, determinantal varieties and varieties of complexes, turn out to be Hodge algebras governed by "good" ideals.

In this paper we study some Hodge algebra structures of affine semigroup rings.

§1. Definition of Hodge algebras.

Let H be a finite set and $\mathbb{N}$ the set of non-negative integers. We denote by $\mathbb{N}^H$ the set of maps from H to $\mathbb{N}$. A <u>monomial</u> M on H is an element of $\mathbb{N}^H$. If M and N are monomials, then the product is defined by $(MN)(x)=M(x)+N(x)$ for all $x \in H$. We say that N divides M if $N(x) \leqq M(x)$ for all $x \in H$. The support of M is the set $\mathrm{Supp}(M) = \{x \in H;\ M(x) \neq 0\}$. An <u>ideal of monomials</u> is a subset $\Sigma \subset \mathbb{N}^H$ such that $M \in \Sigma$ and $N \in \mathbb{N}^H$ imply $MN \in \Sigma$. A monomial M is called <u>standard</u> with respect to Σ if $M \notin \Sigma$. A <u>generator</u> of an ideal Σ is an element of Σ which is not divisible by any other element of Σ.

If A is a commutative ring and an injection $\varphi : H \hookrightarrow A$ is given, then to each monomial M on H we may associate $\varphi(M) = \prod_{x \in H} \varphi(x)^{M(x)} \in A$. We will usually identify H with $\varphi(H)$ and write $M \in A$ for $\varphi(M) \in A$.

Now let R be a commutative ring and let A be a commutative R-algebra. Suppose that H is a finite partially ordered set, called a poset, with an injection $\varphi: H \hookrightarrow A$, and that Σ is an ideal of monomials on H.

We call A a Hodge algebra governed by Σ and generated by H if the following axioms are satisfied:

(Hodge-1) A is a free R-module admitting the set of standard monomials (w.r.t. Σ) as a basis.

(Hodge-2) If $N \in \Sigma$ is a generator and

$$(*) \quad N = \sum_i r_{N,i} M_{N,i}, \qquad 0 \neq r_{N,i} \quad R,$$

is the unique expression for $N \in A$ as a linear combination of distinct standard monomials guaranteed by (Hodge-1), then for each $x \in \mathrm{Supp}(N)$ and each $M_{N,i}$ there is $y_{N,i} \in \mathrm{Supp}(M_{N,i})$ which satisfies $y_{N,i} < x$.

The relations (*) are called the straightening relations for A.

If we put the right-hand sides of all the straightening relations to be 0, then we can construct the "simplest" Hodge algebra, called the discrete Hodge algebra, which is isomorphic to $A_0 = R[H]/\Sigma R[H]$, where R[H] is the polynomial ring over R whose indeterminates are the elements of H.

A Hodge algebra A is called square-free if Σ is generated by square-free monomials. We say that A is ordinal if Σ is generated by the products of the pairs of elements which are incomparable in the partial order on H, then Σ consists of all monomials whose supports are not totally ordered, and is, of course, square-free.

All the examples treated in [2] are ordinal or square-free. It seems to be interested to construct more general Hodge algebras.

Let A (resp. A') be a Hodge algebra governed by an ideal Σ (resp. Σ') and generated by a poset H (resp. H') over a base ring R. The tensor product $A^*=A\otimes_R A'$ turns out to be a Hodge algebra in the following way. We make $H^*=H\cup H'$ (disjoint union) a poset by preserving the ordering of H and H', and by setting $\alpha<\alpha'$ for all $\alpha\in H$ and $\alpha'\in H'$. We inject H^* to A^* by sending $\alpha\in A$ (resp. $\alpha'\in A'$) to $\alpha\otimes 1$ (resp. $1\otimes\alpha'$) $\in A^*$. We regard $\mathbb{N}^H$ (resp. $\mathbb{N}^{H'}$) as the subset

$$\{N^*\in\mathbb{N}^{H^*};\ \mathrm{Supp}(N^*)\subset H\ (\text{resp. } H')\}$$

of $\mathbb{N}^{H^*}$, and define Σ^* to be an ideal of $\mathbb{N}^{H^*}$ which is generated by $\Sigma\cup\Sigma'$. Now it is easy to see that $A^*=A\otimes_R A'$ is a Hodge algebra governed by Σ^* and generated by H^* over R, and that A^* is ordinal if and only if both A and A' are ordinal.

§ 2. Affine semigroup rings.

Let k be a field, $S\subset\mathbb{N}^r$ $(r>0)$ an affine semigroup, i.e., a finitely generated additive semigroup with identity, and $k[T]=k[T_1,\dots,T_r]$ the polynomial ring in r variables over k. We denote by $k[S]$ the affine semigroup ring of S over k

$$k[T^w;\ w\in S]\subset k[T],$$

where $T^w=T_1^{w_1}\cdots T_r^{w_r}$ if $w=(w_1,\dots,w_r)$.

Our first result is the following:

<u>THEOREM 1.</u> Any affine semigroup ring has a structure of Hodge algebras.

Proof: Suppose that S is generated by $h_1,\dots,h_p$ (write $S=\langle h_1,\dots,h_p\rangle$). We put H to be $\{T^{h_1},\dots,T^{h_p}\}$, and make H a poset by setting

$$T^{h_1}<\dots<T^{h_p},$$

namely H is a chain (totally ordered set). We define

$$(2.1)\ \Sigma=\left\{ N\in\mathbb{N}^H \;\middle|\; \begin{array}{l} \text{There exists } M\in\mathbb{N}^H \text{ such that} \\ 1)\ \sum_{i=1}^{p} N(T^{h_i})h_i = \sum_{i=1}^{p} M(T^{h_i})h_i, \\ 2)\ (N(T^{h_1}),\dots,N(T^{h_p}))<(M(T^{h_1}),\dots,M(T^{h_p})) \\ \text{in the lexicographic order in } \mathbb{N}^p. \end{array} \right\}$$

Obviously Σ is an ideal, and the axiom (Hodge-1) follows immediately from the definition of Σ. If $N\in\Sigma$ is a generator and

$$\prod_{i=1}^{p}(T^{h_i})^{N(T^{h_i})} = \prod_{i=1}^{p}(T^{h_i})^{M(T^{h_i})}$$

is the straightening relation guaranteed by (Hodge-1), then

$$\mathrm{Supp}(N)\cap\mathrm{Supp}(M)=\emptyset$$

since N is a generator. Accordingly

$$(N(T^{h_1}),\dots,N(T^{h_p}))<(M(T^{h_1}),\dots,M(T^{h_p}))$$

show the axiom (Hodge-2). ▨

Let A be a Hodge algebra governed by an ideal Σ and generated by a poset H, and H' another poset with an order preserving bijection $\lambda:H\to H'$. If we identify $\alpha\in H$ with $\lambda(\alpha)\in H'$, then A turns out to be a Hodge algebra generated by H'. In particular, if we take a chain H' which consists of the same number of elements as H, then we have an order preserving bijection $\lambda:H\to H'$. Consequently, any Hodge algebra turns out to be a Hodge algebra generated by a chain. This is a key point in the proof of THEOREM 1.

§ 3. Cohen-Macaulayness of Hodge algebras.

Let A be a Hodge algebra governed by an ideal Σ and generated by a poset H over a base ring R. We denote by A_0 the corresponding discrete Hodge algebra $R[H]/\Sigma R[H]$.

The following result, which is obtained in [2], is a fundamental theorem in the theory of Hodge algebras.

<u>If A_0 is a Cohen-Macaulay (resp. Gorenstein) ring, then $A_{\mathfrak{P}}$ is a Cohen-Macaulay (resp. Gorenstein) ring for every prime ideal $\mathfrak{P}$ of A which contains H.</u>

The converse of the above theorem is false. A counter example of the Gorenstein case is well-known, while that of the Cohen-Macaulay case does not seem to be known (see [2] P.38). In the following we construct a Gorenstein ring A such that A_0 is not a Cohen-Macaulay ring.

<u>EXAMPLE.</u> Let $S=\langle h_1,h_2,h_3,h_4\rangle \subset \mathbb{N}^2$ be an affine semigroup with $h_1=(2,0)$, $h_2=(2,1)$, $h_3=(1,2)$ and $h_4=(0,2)$. Firstly if we make $k[S]$ a Hodge algebra so that the total order of $H=\{T^{h_1},T^{h_2},T^{h_3},T^{h_4}\}$ is $T^{h_1}<T^{h_2}<T^{h_3}<T^{h_4}$, then $\Sigma=((T^{h_2})^2,(T^{h_3})^2)$. Since the corresponding discrete Hodge algebra

$$k[X,Y,Z,W]/(Y^2,Z^2)$$

(write X,Y,Z,W for $T^{h_1},T^{h_2},T^{h_3},T^{h_4}$) is a Gorenstein ring, $k[S]$ is a Gorenstein ring. Secondly if the total order of H is $T^{h_2}<T^{h_3}<T^{h_1}<T^{h_4}$, then $\Sigma=((T^{h_1})^2T^{h_4},T^{h_1}(T^{h_4})^2,(T^{h_3})^4,T^{h_1}(T^{h_3})^2)$ and the corresponding discrete Hodge algebra is

$$k[X,Y,Z,W]/(X^2W,XW^2,Z^4,XZ^2)$$
$$=(k[X,Z,W]/(X^2W,XW^2,Z^4,XZ^2))[Y].$$

By the primary decomposition

$$(X^2W,XW^2,Z^4,XZ^2) = (X,Z^4)\cap(X^2,Z^2,W^2)\cap(X^2,W),$$

we have

$$\operatorname{depth}(k[X,Z,W]/(X^2W,XW^2,Z^4,XZ^2))_{(X,Z,W)} = 0,$$

so $k[X,Z,W]/(X^2W,XW^2,Z^4,XZ^2)$ is not a Cohen-Macaulay ring. Accordingly, $k[X,Y,Z,W]/(X^2W,XW^2,Z^4,XZ^2)$ is not a Cohen-Macaulay ring.

§ 4. Cohen-Macaulayness of affine semigroup rings and the corresponding discrete Hodge algebras.

In this section we study the converse of the fundamental theorem (§3) in the case of affine semigroup rings.

An affine semigroup ring is a subring of a polynomial ring over a field k, which is generated by a finite number of monomials, while a discrete Hodge algebra is a residue ring by an ideal which is generated by a finite number of monomials. Both of them are investigated as natural examples of commutative rings. It is interesting that these two classes of commutative rings are associated under the concept of Hodge algebras.

PROPOSITION 1. Let $S=\langle f_1,\dots,f_n,g_1,\dots,g_m\rangle\subset\mathbb{N}^r$ $(r>0)$ be an affine semigroup ($n=\dim k[S]$). We assume that T^{f_i},T^{f_j} is a $k[S]$-sequence for all i,j $(i\neq j)$. If we make $k[S]$ a Hodge algebra so that the total order of $H=\{T^{f_1},\dots,T^{f_n},T^{g_1},\dots,T^{g_m}\}$ satisfies

$$T^{f_s} < T^{g_t} \qquad (\forall s,t)$$

then the corresponding discrete Hodge algebra is a Cohen-Macaulay ring.

For the proof of PROPOSITION 1, we need some lemmas. By the definition of affine semigroup rings, $k[S]=k[T^{h_1},\ldots,T^{h_p}]$ $(S=\langle h_1,\ldots,h_p\rangle)$ is a subring of $k[T]=k[T_1,\ldots,T_r]$, the polynomial ring in r variables over k. Then,

<u>LEMMA 1.</u> If a monomial M in T (in usual sense) is contained in $k[S]$, then M is a monomial in $T^{h_1},\ldots,T^{h_p}$.

By using LEMMA 1, we prove the following

<u>LEMMA 2.</u> If two monomials M and N in $T^{h_1},\ldots,T^{h_p}$ satisfy $M \in (N)$, then there exists a monomial N' in $T^{h_1},\ldots,T^{h_p}$ such that $M=NN'$. Here we denote by (N) the principal ideal in $k[S]$ which is generated by $N\in k[S]$.

Proof of PROPOSITION 1: To begin with we show that $\mathrm{Supp}(N) \subset \{T^{g_1},\ldots,T^{g_m}\}$ if N is a generator of Σ. Suppose that $\mathrm{Supp}(N) \not\subset \{T^{g_1},\ldots,T^{g_m}\}$, and that T^{f_i} is a minimal element in Supp(N). If the straightening relation of (Hodge-2) is

$$\prod_{s=1}^{n}(T^{f_s})^{N(T^{f_s})}\prod_{t=1}^{m}(T^{g_t})^{N(T^{g_t})} = \prod_{s=1}^{n}(T^{f_s})^{M(T^{f_s})}\prod_{t=1}^{m}(T^{g_t})^{M(T^{g_t})},$$

then Supp(M) must contain T^{f_j} which satisfies $T^{f_j}<T^{f_i}$. Suppose that $N=T^{f_i}N'$, $M=T^{f_j}M'$ as the elements of $\mathbb{N}^H$, then we have $T^{f_i}N' = T^{f_j}M'$ in $k[S]$. Since T^{f_i}, T^{f_j} is a $k[S]$-sequence, we have $M'\in(T^{f_i})$.

Therefore, by using LEMMA 2 there exists $M'' \in \mathbb{N}^H$ such that $M' = T^{f_i} M''$ in $k[S]$. Thus $N' = T^{f_j} M''$. Now by the minimality of T^{f_i} and $T^{f_j} < T^{f_i}$, Supp(N') does not contain T^{f_s} which is $\leqq T^{f_j}$. Accordingly by the definition (2.1) of Σ, N' must be contained in Σ. This contradicts to the fact that N is a generator of Σ. Consequently, we have $\mathrm{Supp}(N) \subset \{T^{g_1}, \ldots, T^{g_m}\}$.

In this case, the corresponding discrete Hodge algebra is

$$k[H]/\Sigma k[H] = (k[H_2]/\Sigma k[H_2])[H_1],$$

where $H_1 = \{T^{f_1}, \ldots, T^{f_n}\}$, $H_2 = \{T^{g_1}, \ldots, T^{g_m}\}$. Since $\dim k[S] = n$, the dimension of the corresponding discrete Hodge algebra is also n ([2], Theorem 6.1.). Thus $\dim k[H_2]/\Sigma k[H_2] = 0$, and it follows that $k[H_2]/\Sigma k[H_2]$ is a Cohen-Macaulay ring. Accordingly, $k[H]/\Sigma k[H]$ is also a Cohen-Macaulay ring. ▨

<u>COROLLARY.</u> If $S \subset \mathbb{N}^r$ satisfies the assumption of PROPOSITION 1, then $k[S]$ is a Cohen-Macaulay ring.

We call an affine semigroup $S \subset \mathbb{N}^r$ a <u>simplicial monoid</u> if there exist $f_1, \ldots, f_n \in S$ $(0 < n \leqq r)$ such that

(4.1) $f_1, \ldots, f_n$ are linearly independent over $\mathbb{Q}$ and

(4.2) $S \subset \mathbb{Q}_+ f_1 + \cdots + \mathbb{Q}_+ f_n$,

where $\mathbb{Q}$ and $\mathbb{Q}_+$ denote the set of rationals and of non-negative rationals, respectively.

Note that $n = \dim k[S]$, and that the assumption above is always satisfied when $r \leqq 2$.

The following lemma concerning simplicial monoids is apparent intuitively.

LEMMA 3. If $S=\langle h_1,\dots,h_p\rangle$ is a simplicial monoid, then $f_1,\dots,f_n$ ($n=\dim k[S]$) which satisfy (4.1) and (4.2) can be selected from $h_1,\dots,h_p$.

Now we get to our second result in this paper.

THEOREM 2. Let $S=\langle h_1,\dots,h_p\rangle$ be a simplicial monoid. If $k[S]$ is a Cohen-Macaulay ring, then for one of the Hodge algebra structures given in THEOREM 1, the corresponding discrete Hodge algebra is a Cohen-Macaulay ring.

Proof: By using LEMMA 3, we can select $f_1,\dots,f_n \in S$ ($n=\dim k[S]$) which satisfy (4.1) and (4.2) from $h_1,\dots,h_p$. We may arrange $h_1,\dots,h_p$ into

$$f_1,\dots,f_n,g_1,\dots,g_m \qquad (p=n+m)$$

Let $\mathfrak{m}=(T^{f_1},\dots,T^{f_n},T^{g_1},\dots,T^{g_m})$ be the relevant maximal ideal in $k[S]=k[T^{f_1},\dots,T^{f_n},T^{g_1},\dots,T^{g_m}]$. By (4.2) $\mathfrak{m}$ is nilpotent modulo $(T^{f_1},\dots,T^{f_n})$, and we have $\dim k[S]=\mathrm{ht}\,\mathfrak{m}=n$. Therefore, $T^{f_1},\dots,T^{f_n}$ is a system of parameters in $\mathfrak{m}$. Since $k[S]$ is a Cohen-Macaulay ring, $T^{f_1},\dots,T^{f_n}$ is a $k[S]$-sequence in any order. Now we can apply the PROPOSITION 1, and this completes the proof of our theorem. ▨

Supplement: By using the COROLLARY of PROPOSITION 1, we can give another proof of the following result when S is a simplicial monoid:

If $k[S]$ satisfies the Serre's condition (S_2), then $k[S]$ is a Cohen-Macaulay ring.

Prof. S.Goto and K.Watanabe remarked the author that the above result is essentially in [3] and [4].

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