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GENERAL ELEMENTS OF IDEALS IN LOCAL RINGS

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In many situations arising in the theory of local rings, it is necessary to make use of elements $x_1,\ldots,x_s$ of ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ which are sufficiently general in some sence, depending on the particular situation involved. The purpose of this lecture is to describe a general set-up in which such general elements can be defined which satisfy the required conditions in most such situations and to give an illustration of its application.

We suppose that $(Q, \mathfrak{m}, k)$ is a local ring of dimension d. We first construct the general extension Q_g of Q. Let $X_1, X_2, \ldots$ be a countable sequence of indeterminates over Q. Then Q_g is the localisation of $Q[X_1,X_2,\ldots]$ at the prime ideal $\mathfrak{m}[X_1,X_2,\ldots]$. It follows from a general result of Grothendieck that Q_g is noetherian (alternatively one can prove that if $\mathfrak{a}$ is a finitely generated ideal of Q_g, then $\bigcap_{n=1}^{\infty}(\mathfrak{a} + \mathfrak{m}_g^n) = \mathfrak{a}$, and then, observing that the completion of Q_g is noetherian, use the above to show that if $\mathfrak{a}\overline{Q}_g = \mathfrak{a}'\overline{Q}_g$ where $\mathfrak{a}'$ is a finitely generated ideal of Q_g contained in $\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a}'$.)

Now suppose that $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$ are ideals of Q, and that $\mathfrak{a}_i$ has a basis $a_{i1},\ldots,a_{im_i}$. Write $M_i = m_1 + \ldots + m_i$. Then we term $x_1,\ldots,x_s$ an independent set of general elements of $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$ if there exists an automorphism T of Q_g over Q such that

$$T(x_i) = \sum_{j=1}^{m_i} X_{M_{i-1}+j}\, a_{ij} \qquad (i = 1,\ldots,s).$$

It is a simple matter to prove that this definition is independent of the choice of bases of $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$. It also follows that the ideal $(x_1,\ldots,x_s)\cap Q$ of Q and the Q-algebra $Q_g/(x_1,\ldots,x_s)$ (to within isomorphism as a Q-algebra) depend only on the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$. I will only consider the first in the case when the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$ are all equal to $\mathfrak{a}$. Let $a(\mathfrak{a})$ denote the analytic spread of $\mathfrak{a}$, and $v(\mathfrak{a})$ the minimal number of generators of $\mathfrak{a}$. Then

i) if $s < a(\mathfrak{a})$, the ideal $(x_1,\ldots,x_s)\cap Q$ is nilpotent;

ii) if $s = a(\mathfrak{a})$, $(x_1,\ldots,x_s)$ is a reduction of $\mathfrak{a}Q_g$ and hence $(x_1,\ldots,x_s) \supseteq \mathfrak{a}^n Q_g$ for n large, and hence $(x_1,\ldots,x_s)\cap Q$ contains a power of $\mathfrak{a}$;

iii) if $s \geqslant v(\mathfrak{a})$, we have $(x_1,\ldots,x_s)\cap Q = \mathfrak{a}$.

Now we consider the second. In this case we will be concerned with the case when $s = d-1$ or d, and the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_s$ are all $\mathfrak{m}$-primary. Let N be any integer and define Q_N to be the ring $Q[Y_1,\ldots,Y_N]$ localised at $\mathfrak{m}[Y_1,\ldots,Y_N]$, $Y_1,\ldots,Y_N$ being indeterminates over Q. If we replace Y_i by X_i, it is clear that we can consider Q_N as a subring of Q_g. Now suppose that $\mathfrak{a}$ is any ideal of Q_g. Then for some N, $\mathfrak{a}$ is generated by elements of the sub-ring Q_N of Q_g and therefore $\mathfrak{a} = (\mathfrak{a}\cap Q_N)Q_g$. Now we have an isomorphism of $(Q_g)_N \to Q_g$ in which X_i maps to X_{N+i} and $Y_i \to X_i$ for $i = 1,\ldots,N$. It follows that $Q_g/\mathfrak{a}$ is isomorphic to

$(Q_g)_N/\mathfrak{a}^*$, where $\mathfrak{a}^*$ is an ideal of $(Q_g)_N$ meeting Q_g in $(Q\cap\mathfrak{a})Q_g$. The case that will concern us is when $\mathfrak{a}$ is generated by general elements $x_1,\ldots,x_{d-1}$ of $\mathfrak{m}$-primary ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_{d-1}$ of Q. For simplicity of exposition, we will restrict ourselves to the case when Q is a domain. Then $Q_g/(x_1,\ldots,x_{d-1})$ is a local ring of dimension 1. Now suppose y_i,z_i $(i = 1,\ldots,d-1)$ is a set of independent general elements of the ideals $\mathfrak{a}_1, \mathfrak{a}_1,\ldots, \mathfrak{a}_{d-1}, \mathfrak{a}_{d-1}$. Now choose N so that the elements y_i,z_i $(i = 1,\ldots,d-1)$ are all contained in the sub-ring Q_N of Q_g. Then it is not difficult to prove that the elements $w_i = y_i - X_{N+i}z_i$ $(i = 1,\ldots,d-1)$ form a set of independent general elements of $\mathfrak{a}_1,\ldots,\mathfrak{a}_{d-1}$. We further note that for each i, the elements $y_i,z_1,\ldots,z_{d-1}$ generate an $\mathfrak{m}Q_g$-primary ideal of Q_g. We now quote a general result which will be proved in an appendix:

Let Q be a local domain of dimension d, and let y_i,z_i $(i = 1,\ldots,d-1)$ be elements of Q such that $y_i,z_1,\ldots,z_{d-1}$ generate an $\mathfrak{m}$-primary ideal for each i. Then, if B is the ring $Q[y_1/z_1,\ldots,y_{d-1}/z_{d-1}]$,

i) $B/\mathfrak{m}B$ is isomorphic to $k[X_1,\ldots,X_{d-1}]$, where $k = Q/\mathfrak{m}$, and $X_1,\ldots,X_{d-1}$ are indeterminates over k;

ii) if L denotes B localised at the prime ideal $\mathfrak{m}[y_1/z_1,\ldots,y_{d-1}/z_{d-1}]$, and Q(X) denotes the ring $Q[X_1,\ldots,X_{d-1}]$ localised at $\mathfrak{m}[X_1,\ldots,X_{d-1}]$, where $X_1,\ldots,X_{d-1}$ are indeterminates over Q, then the kernel of the homomorphism of Q(X) onto L in which $X_i \to y_i/z_i$ $(i = 1,\ldots,d-1)$ is a prime ideal $\mathfrak{P}$ containing the ideal $\mathfrak{X} = (y_1-z_1X_1, \ldots , y_{d-1}-z_{d-1}X_{d-1})$ and $\mathfrak{P}/\mathfrak{X}$ is annihilated by a power of $\mathfrak{m}$.

Applying this result, we see that, replacing Q by Q_g and giving y_i, z_i their original meaning, the ring L obtained in this situation is isomorphic to $Q_g/(x_1,\ldots,x_{d-1}):\mathfrak{m}^n$ if n is large enough.

It follows that we can consider L in two ways, first as a homomorphic image of Q_g, and second as a local ring containing Q_g and contained in its field of fractions F_g. Further the maximal ideal of L is $\mathfrak{m}L$ and $\mathfrak{m}L \cap Q_g = \mathfrak{m}Q_g$. Now L is 1-dimensional. Hence, by the Krull-Akizuki theorem, the integral closure L^* of L in F_g is the intersection of a finite set of discrete valuation rings. Let the associated valuations be $V_1,\ldots,V_q$ and let their restriction to the field of fractions F of Q be $v_1,\ldots,v_q$. Then $v_1,\ldots,v_q$ are independent of the choice of the elements y_i, z_i.

Now we must digress to consider valuations on Q_g. Suppose that V is a valuation $\geqslant 0$ on Q_g, and > 0 on $\mathfrak{m}Q_g$, and taking integer values. If K_V is the residue field of V, then K_V is an extension of k_g, and an old result of Zariski states that $\text{tr.deg}_{k_g} K_V \leqslant d-1$. Now let v be the restriction of V to F. Then it is quite easy to prove that

$$\text{tr.deg}_{k_g} K_V \geqslant \text{tr.deg}_k K_v.$$

Now I recall another old result, due in this case to Northcott. Let K denote the residue field of L (which is a pure transcendental extension of k_g of transcendence degree $d-1$). Now the valuations V_i already referred to have an extension to the completion $\bar{L}$ of L which we denote by $\bar{V}_i$, and each such extension $\bar{V}_i$ takes the value ∞ on a minimal prime ideal $\mathfrak{P}_i$ of $\bar{L}$. Let δ_i denote the length of the primary component of (0) in $\bar{L}$ with associated prime $\mathfrak{P}_i$. Then if $x \in L$,

$$e(xL) = \ell(L/xL) = \sum_{i=1}^{q} \delta_i [K_{V_i}:K]\ V_i(x)$$

where $e(\cdot)$ is the multiplicity.

Now we turn to multiplicities and degree functions. Following Teissier, we will use mixed multiplicities. Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_d$ be d $\mathfrak{m}$-primary ideals of Q, and let M be a finitely generated Q-module. Then we define $e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d;\ M)$ as $e(x_1,\ldots,x_d;M)$ where $x_1,\ldots,x_d$ are independent general elements of $\mathfrak{a}_1,\ldots,\mathfrak{a}_d$. Then we have the result that if L is as described earlier,

$$e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d) = e(x_dL) = e(\mathfrak{a}_dL),$$

the latter following since x_dL is a reduction of $\mathfrak{a}_dL$. Further this latter remark also implies that, if V_i, v_i have the meanings given earlier, then $V_i(x_d) = v_i(\mathfrak{a}_d)$ where the latter denotes the minimum value of $v_i(x)$ on $\mathfrak{a}_d$. We further note that $e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d;\ M)$ is a symmetric function of $\mathfrak{a}_1,\ldots,\mathfrak{a}_d$ and, if $\mathfrak{a}'_d$ is another $\mathfrak{m}$-primary ideal of Q, then

$$e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d\mathfrak{a}'_d\ ;\ M) = e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d\ ;\ M) + e(\mathfrak{a}_1,\ldots,\mathfrak{a}'_d;M)$$

we can now write down a formula for the multiplicity symbol

$$e(\mathfrak{a}_1,\ldots,\mathfrak{a}_d\ ;\ Q) = \sum_{i=1}^{q} \delta_i [K_{V_i}:\ K]\ v_i(\mathfrak{a}_d)$$

and similar formulae arising from the symmetry of the symbol. However this formula attains its full force if we introduce degree functions. We define the degree function $d(\mathfrak{a}_1,\ldots,\mathfrak{a}_{d-1};x)$ where x is an element of Q to be $e(\mathfrak{a}_1',\ldots,\mathfrak{a}_{d-1}';Q')$ where $Q'=Q/xQ$ and $\mathfrak{a}_i' = (\mathfrak{a}_i + xQ)/xQ$. If Q is a domain, this can also be written as $e(x_1,\ldots,x_{d-1},x;Q)$ and we obtain the expression

$$d(\mathfrak{a}_1,\ldots,\mathfrak{a}_{d-1};x) = \sum_{i=1}^{q} \delta_i [K_{V_i}:K]\ v_i(x).$$

APPENDIX

First we prove a lemma which is well known.

LEMMA. Let B be a noether domain, y,z elements of B such that (y,z) has height 2. Let B' be the ring B[y/z] and let $\mathfrak{P}$ be the kernel of the map $B[Y] \to B'$ in which $Y \to y/z$. Then $\mathfrak{P}$ contains $w = zY - y$, and

$$wB[Y]:(z^m,y^m) = \mathfrak{P}$$

if m is sufficiently large. Further, if $\mathfrak{m}$ is any prime ideal of B containing (y,z), then $B'/\mathfrak{m}B' \cong (B/\mathfrak{m})[X]$, where X is an indeterminate over $B'/\mathfrak{m}B'$.

Proof. Let f(Y) be a polynomial of degree r over B such that $f(y/z) = 0$. Then we can write $f(Y) = F(Y,1)$ where F(Y,Z) is a homogeneous polynomial over B of degree r such that $F(y,z) = 0$. Then

$$\begin{aligned} z^r F(Y,Z) &= F(zY,zZ) = F(yZ+(zY-yZ),zZ) \\ &= F(yZ,zZ) + (zY-yZ)G(Y,Z) \qquad \text{by Taylor's Theorem} \\ &= Z^r F(y,z) + (zY-yZ)G(Y,Z) \end{aligned}$$

whence, by putting $Z = 1$, we see that $z^r f(Y) \in wB[Y]$. Also,

$$y^r f(Y) = (y^r - z^r Y^r) f(Y) + Y^r z^r f(Y) \in wB[Y].$$

But as the ascending sequence of ideals $wB[Y]:(y^r,z^r)$ becomes stationary for large r, it follows that

$$\mathfrak{P} = wB[Y]:(y^m,z^m) \qquad m \text{ large.}$$

Hence $\mathfrak{P}$ is the radical of $wB[Y]$ and since $y,z \in \mathfrak{m}$, $w \in \mathfrak{m}B[Y]$, i.e. $\mathfrak{P} \subset \mathfrak{m}B[Y]$, which proves the result.

We now come to the main result of this appendix.

THEOREM. Let $(Q,\mathfrak{m},k)$ be a local domain of dimension $d \geqslant 2$, and let y_i,z_i $(i = 1,\dots,d-1)$ be elements of $\mathfrak{m}$ such that $(y_i,z_1,\dots,z_{d-1})$ is $\mathfrak{m}$-primary for $i = 1,\dots,d-1$. Let $u_i = y_i/z_i$ and $B = Q[u_1,\dots,u_{d-1}]$. Then

$$B/\mathfrak{m}B \cong k[X_1,\dots,X_{d-1}]$$

where $X_1,\dots,X_{d-1}$ are indeterminates over k, implying that $\mathfrak{m}B$ is prime.

Further let $L = B_{\mathfrak{m}B}$ and let Q_{d-1} denote $Q[X_1,\dots,X_{d-1}]$ localised at $\mathfrak{m}[X_1,\dots,X_{d-1}]$. Let $\mathfrak{P}$ denote the kernel of the homomorphism $Q_{d-1} \to L$ in which $X_i \to u_i$. Let $w_i = z_iX_i - y_i$ and let $\mathfrak{X}$ be the ideal $(w_1,\dots,w_{d-1})$. Then for r large,

$$\mathfrak{m}^r\mathfrak{P} \subset \mathfrak{X}.$$

Proof. The proof will be by induction on d, the case d=2 following from the lemma. Now suppose that $d > 2$. Write Q' for $Q[u_{d-1}]$ localised at $\mathfrak{m}[u_{d-1}]$, which is prime by the lemma. We first prove that $(y_i,z_1,\dots,z_{d-2})Q'$ is $\mathfrak{m}Q'$-primary for $i = 1,\dots,d-2$. Now, by the lemma, $Q' \cong Q(X_{d-1})/\mathfrak{P}'$, where $Q(X_{d-1})$ denotes $Q[X_{d-1}]$ localised at $\mathfrak{m}[X_{d-1}]$, and $\mathfrak{P}'$ is the radical of $w_{d-1}Q(X_{d-1})$. Hence it will be sufficient to show that $(w_{d-1},y_i,z_1,\dots,z_{d-2})$ is $\mathfrak{m}Q(X_{d-1})$-primary. Write

$$C_i = y_i Q(X_{d-1}) + z_1 Q(X_{d-1}) + \dots + z_{d-2} Q(X_{d-1}).$$

Then the minimal prime ideals of C_i are generated by elements of Q and so can only contain w_{d-1} if it contains y_{d-1}, z_{d-1}. Since $C_i + z_{d-1}Q(X_{d-1})$ is $\mathfrak{m}$-primary, $\dim C_i = 1$, and since w_{d-1} belongs to no minimal prime of C_i, the result now follows.

Now we consider the first statement of the theorem. It is clearly equivalent to the statement that if $f(X_1,\dots,X_{d-1})$ is a polynomial over Q such that $f(u_1,\dots,u_{d-1}) = 0$, then all the coefficients of f belong to $\mathfrak{m}$. Suppose there is a coefficient of f not in $\mathfrak{m}$. Then if we consider the polynomial $f(X_1,\dots,X_{d-2},u_{d-1})$ as a polynomial with coefficients in Q', then the lemma implies that this has a coefficient not in $\mathfrak{m}Q'$. But Q' has dimension $d-1$ and the conditions of the theorem apply. Hence by our inductive hypothesis $f(u_1,\dots,u_{d-1}) \neq 0$.

We are now in a position to construct L. Consider the homomorphism $Q_{d-1} \to L$. This can be factored as the product of the homomorphism $Q_{d-1} \to Q'_{d-2}$ in which $X_{d-1} \to u_{d-1}$ and the homomorphism $Q'_{d-2} \to L$. Denote by $\mathfrak{a}$ the kernel of the homomorphism $Q_{d-1} \to Q'_{d-2}$. Applying the inductive hypothesis to the second factor, we see that, for r large,

$$\mathfrak{m}^r \mathfrak{P} \subset \mathfrak{a} + (w_1,\dots,w_{d-2})$$

while, by the lemma,

$$(y_{d-1}{}^m,\ z_{d-1}{}^m)\,\mathfrak{a} \subset w_{d-1}Q_{d-1}.$$

Hence

$$(y_{d-1}{}^m,\ z_{d-1}{}^m)\,\mathfrak{m}^r \mathfrak{P} \subset (w_1,\dots,w_{d-1}) = \mathfrak{X}.$$

But by reordering the suffixes $1,\dots,d-1$, we can replace $d-1$ on the left hand side by i $(i = 1,\dots,d-2)$. Hence if m,r are large enough,

$$(y_1^m, \ldots, y_{d-1}^m, z_1^m, \ldots, z_{d-1}^m)\mathfrak{m}^r\mathfrak{p} \subset \mathfrak{X}$$

and the result follows since the first factor is $\mathfrak{m}$-primary.