数理解析研究所講究録
484巻 1983年 1-10

ON THE CANONICAL MODULES

愛媛大理 青山陽一 (Yôichi Aoyama)

A ring will always mean a commutative noetherian ring with unit. Let R be a ring, M a finitely generated R-module and N a submodule of M. We denote by $\mathrm{Min}_R(M)$ the set of minimal elements in $\mathrm{Supp}_R(M)$ and put $U_M(N) = \bigcap Q$ where Q runs through all the primary components of N in M such that $\dim M/Q = \dim M/N$. Let T be an R-module and $\underline{a}$ an ideal of R. $E_R(T)$ denotes an injective envelope of T and $H^i_{\underline{a}}(T)$ is the i-th local cohomology module of T with respect to $\underline{a}$. We denote by $\hat{}$ the Jacobson radical adic completion over a semi-local ring. For a ring R, $Q(R)$ denotes the total quotient ring of R. Throughout this note A denotes a local ring of dimension d and with maximal ideal $\underline{m}$.

<u>Definition</u>([7, Definition 5.6]). An A-module K is called a canonical module of A if $K \otimes_A \hat{A} \cong \mathrm{Hom}_A(H^d_{\underline{m}}(A), E_A(A/\underline{m}))$.

For elementary properties of canonical modules, we refer the reader to [6, §6], [7, 5 Vortrag und 6 Vortrag] and [2, §1]. It is not obvious that the localization of a canonical module is a canonical module of the localization ring, which was known only for local rings with dualizing complexes, and Ogoma [9] showed that there is a non-acceptable (hence without dualizing complex) local ring with canonical module. Our purposes are to prove that $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ for every $\underline{p}$ in $\mathrm{Supp}_A(K)$ (A is a local ring with canonical module K) and to consider endomorphism rings of canonical modules.

<u>Lemma 1</u>(Corollary to [5, Theorem 1]). Let B be a faithfully flat local A-algebra with maximal ideal $\underline{n}$. Then:

(1) If $B/\underline{m}B$ is an artinian Gorenstein ring, then $E_A(A/\underline{m}) \otimes_A B \cong E_B(B/\underline{n})$.

(2) If T is an A-module such that $T \otimes_A B \cong E_B(B/\underline{n})$, then $T \cong E_A(A/\underline{m})$ and $B/\underline{m}B$ is an artinian Gorenstein ring.

<u>Theorem 2</u>([4]). Assume that A has a canonical module K and let B be a faithfully flat local A-algebra. Then the following are equivalent:

(a) $B/\underline{m}B$ is a Gorenstein ring.

(b) $K \otimes_A B$ is a canonical module of B and $B/\underline{m}B$ is a Cohen-Macaulay ring.

(Proof) Suppose that $B/\underline{m}B$ is a Cohen-Macaulay ring and let $y_1, \dots, y_r$ be a system of elements in $\underline{n}$, the maximal ideal of B , which is a maximal $B/\underline{m}B$-regular sequence ($r = \dim B/\underline{m}B$). Let $R = A[X_1, \dots, X_r]_{(\underline{m}, X_1, \dots, X_r)}$ with indeterminates $X_1, \dots, X_r$ over A and let f be the natural A-algebra homomorphism from R to B such that $f(X_i) = y_i$ for $i = 1, \dots, r$. Then f is a flat local homomorphism. By [7, Korollar 5.12], $C = K \otimes_A R$ is a canonical module of R . Hence we may assume that $B/\underline{m}B$

is artinian. Furthermore we may assume that A and B are both complete. In this case it is shown that $K \otimes_A B$ is a canonical module of B if and only if $E_A(A/\underline{m}) \otimes_A B \cong E_B(B/\underline{n})$ ([2, Proof of Proposition 4.1]). Hence the assertion follows from Lemma 1. (Q.E.D.)

Suppose that A has a canonical module K. Let M be a finitely generated A-module and h_M the natural map from M to $\mathrm{Hom}_A(\mathrm{Hom}_A(M,K),K)$.

<u>Proposition 3</u>([2, (1.11)]). The following are equivalent:

(a) The map h_M is an isomorphism.

(b) $\hat{M}$ is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}_A(M)$.

<u>Corollary 4</u>([1, Proposition 2]). $A \cong \mathrm{Hom}_A(K,K)$ if and only if $\hat{A}$ is (S_2).

Next we show some elementary properties of the endomorphism ring of a canonical module. Assume that A has a canonical module K and put $H = \mathrm{End}_A(K)$.

<u>Theorem 5</u>([2, Theorem 3.2]). The following statements hold for H:

(1) H is a semi-local ring which is a finitely generated A-module and $A/U \subseteq H \subseteq Q(A/U)$ where $U = U_A(0) = \mathrm{ann}_A(K)$.

(2) Every maximal chain of prime ideals in H is of length d.

(3) $\hat{H}$ is (S_2).

(4) For every maximal ideal $\underline{n}$ of H, $K_{\underline{n}}$ is a canonical module of $H_{\underline{n}}$. (K is an H-module by the usual way.)

(5) $\dim_A \mathrm{Coker}(A \to H) \leq d-2$.

(Proof) We may assume that $\mathrm{ann}_A(K) = U_A(0) = 0$.

(1) Let $\underline{p}$ be a prime ideal of A with $\dim A/\underline{p} = d$ and $\underline{q}$ a minimal prime ideal of $\underline{p}\hat{A}$. Then $\dim \hat{A}/\underline{q} = d$ and $\hat{K}_{\underline{q}}$ is a canonical module of $\hat{A}_{\underline{q}}$. Since $\dim \hat{A}_{\underline{q}} = 0$, $\hat{K}_{\underline{q}} \cong E_{\hat{A}}(\hat{A}/\underline{q})$. Since $K_{\underline{p}} \otimes_{A_{\underline{p}}} \hat{A}_{\underline{q}} \cong \hat{K}_{\underline{q}}$, $K_{\underline{p}} \cong E_A(A/\underline{p})$ by Lemma 1(2). Let $\mathrm{Ass}(A) = \{\underline{p}_1, \ldots, \underline{p}_t\}$ and $S = A \setminus \bigcup_{i=1}^{t} \underline{p}_i$, the set of non-zerodivisors of A . Since K is torsion free, so is H and the natural map $H \longrightarrow S^{-1}H$ is injective. Since $S^{-1}K \cong \bigoplus_{i=1}^{t} K_{\underline{p}_i} \cong \bigoplus_{i=1}^{t} E_A(A/\underline{p}_i)$, $S^{-1}H \cong \mathrm{Hom}_A(S^{-1}K, S^{-1}K) \cong \bigoplus_{i=1}^{t} A_{\underline{p}_i} \cong Q(A)$.

(2) Because A is unmixed.

(3) Because $\hat{K}$ is (S_2).

(4) The map $h_K : K \longrightarrow \mathrm{Hom}_A(H,K)$ is an isomorphism by Proposition 3. Hence the assertion follows from [7, Satz 5.12] and (3).

(5) We may assume that A is complete. Let $\underline{p}$ be a primt ideal such that height $\underline{p} \leq 1$. Then $A_{\underline{p}}$ is Cohen-Macaulay and $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ because A is complete and $U_A(0) = 0$. Hence $A_{\underline{p}} = H_{\underline{p}}$, that is, $\mathrm{Coker}(A \to H)_{\underline{p}} = 0$, which means $\dim_A \mathrm{Coker}(A \to H) \leq d-2$. (Q.E.D.)

<u>Theorem 6</u>([2, Theorem 4.2]). Let $(A,\underline{m}) \longrightarrow (B,\underline{n})$ be a flat local homomorphism and M an A-module. If $M \otimes_A B$ is a canonical module of B , then M is a canonical module of A .

<u>Corollary 7</u>([2, Corollary 4.3]). Assume that A has a canonical module K and let $\underline{p}$ be an element of $\mathrm{Supp}_A(K)$. Then $K_{\underline{p}}$ is a canonical module of $A_{\underline{p}}$ and $\hat{A}_{\underline{q}}/\underline{p}\hat{A}_{\underline{q}}$ is a Gorenstein ring for every minimal prime ideal $\underline{q}$ of $\underline{p}\hat{A}$.

Before proving Theorem 6, we show two lemmas.

Lemma 8. Assume that A is complete. Let T be a finitely generated (S_2) A-module such that $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}_A(T)$ and $H^d_{\underline{m}}(T) \cong E_A(A/\underline{m})$. Then T is a canonical module of A. In this case A is (S_2).

(Proof) By Proposition 3, the map h_T is an isomorphism. Since $\mathrm{Hom}_A(T,K) \cong \mathrm{Hom}_A(H^d_{\underline{m}}(T), E_A(A/\underline{m})) \cong \mathrm{Hom}_A(E_A(A/\underline{m}), E_A(A/\underline{m})) \cong A$, $T \cong \mathrm{Hom}_A(A,K) \cong K$, a canonical module of A. (Q.E.D.)

Lemma 9. Let R be a finite over-ring of A such that $\dim_A R/A \le d-2$ and $\dim R_{\underline{p}} = d$ for every maximal ideal $\underline{p}$ of R. If T is a finitely generated R-module such that $T_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ for every maximal ideal $\underline{p}$ of R, then T, as an A-module, is a canonical module of A.

(Proof) We may assume that A is complete. For every maximal ideal $\underline{p}$ of R, $\mathrm{Hom}_A(R,K)_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ by [7, Satz 5.12] (K is a canonical module of A). Hence $T_{\underline{p}} \cong \mathrm{Hom}_A(R,K)_{\underline{p}}$ for every maximal ideal $\underline{p}$ of R and therefore $T \cong \mathrm{Hom}_A(R,K)$. Since $\dim_A R/A \le d-2$, we have $\mathrm{Hom}_A(R/A,K) = 0$ and $\mathrm{Ext}^1_A(R/A,K) = 0$ (cf. [2, (1.10)]). Hence, from the exact sequence $0 \to A \to R \to R/A \to 0$, we have $\mathrm{Hom}_A(R,K) \cong \mathrm{Hom}_A(A, K) \cong K$, a canonical module of A. (Q.E.D.)

(Proof of Theorem 6) We may assume that A and B are both complete and $\underline{m}B$ is $\underline{n}$-primary. Let K (resp. L) be a canonical module of A (resp. B).

(I) The case that B is (S_2): Since B is (S_2), $B \cong \mathrm{Hom}_B(L,L)$, i.e., $H^d_{\underline{n}}(L) \cong E_B(B/\underline{n})$. Since $H^d_{\underline{m}}(M) \otimes_A B \cong H^d_{\underline{n}}(M \otimes_A B) \cong H^d_{\underline{n}}(L) \cong E_B(B/\underline{n})$, $H^d_{\underline{m}}(M) \cong E_A(A/\underline{m})$ by Lemma 1(2). Since L is (S_2), so is M . Since $\mathrm{Ass}_B(L) = \{ \underline{q} \in \mathrm{Spec}(B) \mid \dim B/\underline{q} = d \}$, $\mathrm{Ass}_A(M) = \{ \underline{p} \in \mathrm{Spec}(A) \mid \dim A/\underline{p} = d \}$. Hence we have $M \cong K$ by Lemma 8.

(II) The general case: Since $\mathrm{Ass}_A(M) = \{ \underline{p} \in \mathrm{Spec}(A) \mid \dim A/\underline{p} = d \}$ and $M_{\underline{p}} \cong E_A(A/\underline{p})$ for every $\underline{p}$ in $\mathrm{Ass}_A(M)$ (cf. Proof of Theorem 5(1)), we have $\mathrm{ann}_A(M) = U_A(0)$. Hence we may assume that $U_A(0) = 0$ and $U_B(0) = 0$. Put $R = \mathrm{End}_A(M)$ and $S = \mathrm{End}_B(L)$. Since $R \otimes_A B \cong S$ is a finite over-ring of B , R is a finite over-ring of A . For every maximal ideal $\underline{p}$ of R , $\dim R_{\underline{p}} = d$ because A is unmixed. We have $\dim_A R/A \leq d-2$ because $\dim_B S/B \leq d-2$. Let $\underline{p}$ be a maximal ideal of R and $\underline{q}$ a maximal ideal of S lying over $\underline{p}$. Since $M_{\underline{p}} \otimes_{R_{\underline{p}}} S_{\underline{q}} \cong L_{\underline{q}}$ is a canonical module of $S_{\underline{q}}$ by Theorem 5(4) and $S_{\underline{q}}$ is (S_2) by Theorem 5(3), $M_{\underline{p}}$ is a canonical module of $R_{\underline{p}}$ by the case (I). Hence we have that M is a canonical module of A by Lemma 9. (Q.E.D.)

<ins>Remark</ins>. Goto (Nihon University) proved the following lemma and gave another proof of Theorem 6. ([3, Appendix])

> <ins>Lemma</ins>. Let $(A,\underline{m}) \longrightarrow (B,\underline{n})$ be a flat local homomorphism such that $\underline{m}B$ is $\underline{n}$-primary. If there is a finitely generated A-module T such that $T \otimes_A B$ is a canonical module of B , then $B/\underline{m}B$ is a Gorenstein ring.

By virtue of Corollary 7, we can prove the following proposition by induction on $\dim A$ (cf. [1, Proof of Proposition 2]). Assume that A has a canonical module K . For a finitely generated A-module M , h_M denotes the natural map from M to

$\mathrm{Hom}_A(\mathrm{Hom}_A(M,K),K)$.

Proposition 10([2, Proposition 4.4]). The following are equivalent:

(a) The map h_M is an isomorphism.

(b) $\hat{M}$ is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}_A(M)$.

(c) M is (S_2) and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}_A(M)$.

Corollary 11([9, Proposition 4.2] and [4]). The following are equivalent:

(a) $A \cong \mathrm{Hom}_A(K,K)$.

(b) $\hat{A}$ is (S_2).

(c) A is (S_2).

Remark. The implication (c) ⇒ (a) was first proved by Ogoma (Kochi University), not by induction. (See [9, §4]. cf. [3, (二)])

Corollary 12([4]). Assume that A has a canonical module and $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}(A)$. Then the (S_2)-locus of A is open in $\mathrm{Spec}(A)$.

Corollary 13([4]). Assume that A has a canonical module. Let $(A,\underline{m}) \longrightarrow (B,\underline{n})$ be a flat local homomorphism such that $B/\underline{m}B$ is a Gorenstein ring.

(1) Let M be a finitely generated (S_2) A-module such that $\dim A/\underline{p} = d$ for every $\underline{p}$ in $\mathrm{Min}_A(M)$. Then $M \otimes_A B$ is (S_2) and $\dim B/\underline{q} = \dim B$ for every $\underline{q}$ in $\mathrm{Min}_B(M \otimes_A B)$.

(2) If A is (S_2), then B is also (S_2).

Next we show that the endomorphism ring of a canonical module is characterized by the properties described in Theorem 5.

Theorem 14([4]). Assume that A has a canonical module K. Let R be a ring satisfying the following conditions:

(i) R is a finite (S_2) over-ring of $A/U_A(0)$,

(ii) For every maximal ideal $\underline{n}$ of R, $\dim R_{\underline{n}} = d$, and

(iii) $\dim_A \operatorname{Coker}(A \to R) \leq d-2$.

Then $R \cong \operatorname{End}_A(K)$ as A-algebras.

(Proof) We may assume that $U_A(0) = 0$. Put $L = \operatorname{Hom}_A(R,K)$. Then $L_{\underline{n}}$ is a canonical module of $R_{\underline{n}}$ for every maximal ideal $\underline{n}$ of R. By Lemma 9, we have $L \cong K$. From this isomorphism, we have an A-algebra isomorphism $\operatorname{End}_A(K) \xrightarrow{\sim} \operatorname{End}_A(L)$. Since $\operatorname{End}_A(K)$ is commutative, so is $\operatorname{End}_A(L)$ and $\operatorname{End}_A(L) = \operatorname{End}_R(L)$. Since R is (S_2), $R \cong \operatorname{End}_R(L)$. Hence we have $R \cong \operatorname{End}_A(K)$ as A-algebras. (Q.E.D.)

In the following we assume that A has a canonical module K, $d \geq 2$ and $U_A(0) = 0$. Put $H = \operatorname{End}_A(K)$ and $\underline{c} = A :_A H$, the conductor. Let T be the $\underline{c}$-transform of A, i.e., $T = \{ x \in Q(A) \mid \underline{c}^t x \subseteq A \text{ for some } t \}$. Let $\underline{q}$ be a prime ideal of $\hat{A}$ containing $\underline{c}\hat{A}$ and $\underline{p}$ an associated prime ideal of $\hat{A}_{\underline{q}}$. Since $U_{\hat{A}}(0) = U_A(0)\hat{A} = 0$ and height $\underline{c} \geq 2$, we have $\dim \hat{A}_{\underline{q}}/\underline{p} \geq 2$. Hence by [8, Proposition(2.7)] we have:

(15.1) T is a finitely generated A-module.

The following two assertions are obvious:

(15.2) $\dim_A T/A \leq d-2$.

(15.3) T is (S_2).

Hence, from Theorem 14, we obtain the following

Proposition 16([4]). $T \cong H$ as A-algebras.

We denote by A^g the global transform of A, i.e., $A^g = \{ x \in Q(A) \mid \underline{m}^t x \subseteq A$ for some $t \}$. Since $U_A(0) = 0$ and $d \geq 2$, A^g is a finitely generated A-module by [8, Proposition (2.3)].

Corollary 17([4]). $A^g \cong H$ as A-algebras if and only if depth $A_{\underline{p}} \geq \min\{2, \dim A_{\underline{p}}\}$ for every non-maximal prime ideal $\underline{p}$ of A. In particular, if $H^i_{\underline{m}}(A)$ is of finite length for $i \neq d$, $A^g \cong H$ as A-algebras.

Ehime University

References

[1] Y. Aoyama : On the depth and the projective dimension of the canonical module, Japan. J. Math. 6 (1980) 61 - 66.

[2] Y. Aoyama : Some basic results on canonical modules, J. Math. Kyoto Univ. 23 (1983) 85 - 94.

[3] Y. Aoyama : On canonical modules with appendix by S. Goto (in Japanese), R.I.M.S. Kokyuroku 465, 46 - 53, Kyoto Univ., 1982.

[4] Y. Aoyama and S. Goto : Note on endomorphism rings of canonical modules, in preparation.

[5] H.-B. Foxby : Injective modules under flat base change, Proc. A.M.S. 50 (1975) 23 - 27.

[6] A. Grothendieck : Local cohomology, Lect. Notes Math. 41, Springer Verlag, 1967.

[7] E. Kunz , J. Herzog et al. : Der kanonische Modul eines Cohen-Macaulay-Rings, Lect. Notes Math. 238, Springer Verlag, 1971.

[8] J. Nishimura : On ideal transforms of noetherian rings, I and II, J. Math. Kyoto Univ. 19 (1979) 41 - 46 and 20 (1980) 149 - 154.

[9] T. Ogoma : Existence of dualizing complexes, to appear.