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7. Weierstrass Points on Curves of Fermat Type

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§1. Introduction

An anlytic function defined on a Riemann surface has some poles if this function is not constant valued. Let g be the genus of the Riemann surface and let P be a point on this surface such that there is an analytic function having a pole of order ν with $\nu \leqq g$ and holomorphic at any other point, then this point P is called the Weierstrass point.

We calculated the Weierstrass points on the curve of Fermat type :

$$X_0^n + X_1^n + X_2^n = 0 ,$$

and obtained all Weierstrass points when $n = 4,5,6,7$, where (X_0, X_1, X_2) is a homogeneous coordinate system of the complex projective plane $\mathbb{P}^2$. We also obtained the gap sequence and the Weierstrass weight of special Weierstrass points on these curves with $n \geqq 4$.

These results are obtained by using the Poincaré residue map which is the relation between the rational 2-forms on $\mathbb{P}^2$ and abelian differentials on the Riemann surface corresponding to the given curve.

In §2 we describe the method to obtaine Weierstrass points by using the Poincaré residue map.

And in §3, we show the algorithm to obtain Weierstrass points and we show the results obtained.

§2. The Poincaré residue map

Let $\mathbb{P}^2$ be the complex projective plane, (X_0, X_1, X_2) be homogeneous coordinates, and let $F(X_0, X_1, X_2)$ be a homogeneous polynomial. The set of all points satisfying the equation

$$F(X_0, X_1, X_2) = 0$$

is called an algebraic curve, and we denote it as C . A point P on C is called a singular point if and only if it satifies the equations

$$\partial F/\partial X_0(P) = \partial F/\partial X_1(P) = \partial F/\partial X_2(P) = 0 .$$

In our work we assume no points on C is singular.

Let Ω^2 be a sheaf of germs of holomorphic 2-forms on $\mathbb{P}^2$, $\Omega^2[C]$ be that of rational 2-forms having a pole of order at most 1 along the curve C , and let $\tilde{\Omega}$ be the quotient sheaf $\Omega^2[C] / \Omega^2$, then we obtain the exact sequence of sheaves :

$$0 \to \Omega^2 \to \Omega^2[C] \to \tilde{\Omega} \to 0 .$$

From this exact sequence, we obtain the exact sequence of cohomology groups :

$$0 \to H^0(\mathbb{P}^2, \Omega^2) \to H^0(\mathbb{P}^2, \Omega^2[C]) \to H^0(\mathbb{P}^2, \tilde{\Omega})$$
$$\to H^1(\mathbb{P}^2, \Omega^2) \to \cdots .$$

It is known that $H^0(\mathbb{P}^2, \Omega^2) = H^1(\mathbb{P}^2, \Omega^2) = 0$ ([1],p.118), so we have the exact sequence :

$$0 \longrightarrow H^0(\mathbb{P}^2, \Omega^2[C]) \xrightarrow{\text{P.R.}} H^0(\mathbb{P}^2, \tilde{\Omega}) \longrightarrow 0 ,$$

where P.R. is the abbreviation of the Poincaré residue map, and the explicit correspondence will be given in (2.1).

Let $x = X_1/X_0$, $y = X_2/X_0$ be affine coordinates and let $f(x,y) = F(1,x,y)$, then if $F(X_0,X_1,X_2)$ is a homogeneous polynomial of degree n and if F does not coincide to X_0 , then $f(x,y)$ is a polynomial of degree n.

Hereafter, we assume $f(x,y)$ is of degree n ($\geqq 4$). Each element of $H^0(\mathbb{P}^2, \Omega^2[C])$ is expressed in the affine coodinates (x,y) as

$$(g/f)dx\wedge dy,$$

where $g(x,y)$ is a polynomial of degree less than or equal to $n - 3$. ([1] p.221). As we are assuming that this curve C has no singular point,

$$\partial f/\partial x(P) \neq 0 \quad \text{or} \quad \partial f/\partial y(P) \neq 0$$

at any point $P = (p,q)$ of C. We may assume without loss of generality that $\partial f/\partial y(P) \neq 0$. Then by the implicit function theorem, there is a function $y = y(x)$ such that

$$f(x,y(x)) \equiv 0$$

and the domain of definition of $y(x)$ is a neighborhood of p.

In the neighborhood of p , the Poincaré residue map is

$$\frac{g}{f}\, dx\wedge dy \longmapsto \frac{g(x,y(x))}{-\partial f/\partial y(x,y(x))} dx . \qquad (2.1)$$

We denote ω the right hand side of this correspondence. It is easy to see that ω is an abelian differential on the curve C.

The order of zero at p of this ω is equal to the number of intersection of C with the curve $g(x,y) = 0$ at this point.

Now we calculate the order of zero at p of ω.

Let

$$g(x,y) = b_{00} + b_{10}x + b_{01}y + \dots + b_{0m}y^m , \qquad (m = n-3)$$

and let

$$y(x) = q + a_1(x-p) + a_2(x-p)^2 + \dots ,$$

then

$$g(x,y(x)) = A_0 + A_1(x-p) + A_2(x-p)^2 + \dots .$$

<u>Proposition</u> ω has zero of order ν at p if and only if

$$A_0 = A_1 = \dots = A_{\nu-1} = 0.$$

Proof. Omitted.

$A_0,\dots,A_{\nu-1}$ are linear forms with variables $b_{00},\dots,b_{0m}$. We denote $C(\nu P)$ the maximum number of linearly independent linear forms among $A_0,\dots,A_{\nu-1}$.

Let $H^0(C,\Omega^1[-\nu P])$ be the set of all abelian differentials having zero at P of order at least ν, then the dimension of $H^0(C,\Omega^1[-\nu P])$ as a vector space over the field of complex numbers $\mathbb{C}$ is :

Proposition

$$\dim_{\mathbb{C}} H^0(C, \Omega^1[-\nu P]) = (n-1)(n-2)/2 - C(\nu P).$$

Proof. The space of all polynomials of degree less than or equal to $n-3$ is of dimension $(n-1)(n-2)/2$, and the dimension of the space of solutions of the equations

$$A_0 = \dots = A_{\nu-1} = 0$$

is $(n-1)(n-2)/2 - C(\nu P)$. Q.E.D.

Let $H^0(C, \mathcal{O}[\nu P])$ be the space of all analytic function having only one pole of order at most ν at P, and let $\ell(\nu P)$ be the dimension of this space, then by the Riemann-Roch theorem

$$\ell(\nu P) = \nu + 1 - g + (n-1)(n-2)/2 - C(\nu P). \qquad ([1] \text{ p.245})$$

Here, g is the genus of the curve C and this coincides with $(n-1)(n-2)/2$ because the curve C is assumed to be nonsingular.

We have

$$\ell(\nu P) - \ell((\nu-1)P) = C((\nu-1)P) - C(\nu P) + 1. \qquad (2.2)$$

From this equation, we see

$$\ell(\nu P) - \ell((\nu-1)P) = 1 \text{ or } 0.$$

If $\ell(\nu P) - \ell((\nu-1)P) = 1$, then there is only one analytic function except multiplication of constant number, such that this function has only one pole of order ν at P. If this order ν is less than or equal to g, then this point P is called the Weierstrass point.

$\ell(\nu P) - \ell((\nu-1)P) = 1$ if and only if $C((\nu-1)P) = C(\nu P)$, and if the linear forms with variables $b_{00}, \ldots, b_{0m}$,

$$A_0, \ldots, A_{g-1}$$

are not linearly independent, there is a number ν ($\leqq g$) such that

$$C(\nu P) = C((\nu-1)P) .$$

From this discussion, we have

<u>Theorem</u> Let $A_0, \ldots, A_{g-1}$ be linear forms obtained from the expansion around P, and let D be the determinant of the matrix consisting of coefficients of the linear forms $A_0, \ldots, A_{g-1}$, then P is a Weierstrass point if and only if $D = 0$.

§3. The algorithm and the results obtained

We use new coordinates (u,v) such that

$$u = x - p \quad \text{and} \quad v = y - q.$$

Let v be a polynomial

$$v = c_1 u + c_2 u^2 + \ldots + c_{g-1} u^{g-1} ,$$

the coefficients $c_1, \ldots, c_{g-1}$ are determined by the equation

$$1 + (u+p)^n + (v+q)^n = 0.$$

We substitute the polynomial

$$q + c_1 u + c_2 u^2 + \ldots$$

for y, and substitute $u + p$ for x in

$$g(x,y) = b_{00} + b_{10}x + b_{01}y + \ldots + b_{0m}y^m .$$

Then we obtain a polynomial

$$A_0 + A_1 u + A_2 u^2 + \dots + A_{g-1} u^{g-1} .$$

Let M be a $g \times g$ matrix (m_{ij}) such that

$$m_{ij} = \text{the coefficient of } b_j \text{ in } A_{i-1} ,$$

where we renamed the variables $b_{00}, \dots, b_{0m}$ as b_1, $\dots, b_g$.

Finally we calculate the determinant of M which we denote as AX. Each root of the equation

$$AX = 0$$

gives a x-coordinate of the Weierstrass point.

The polynomial AX in the table at the last of this section determines the Weierstrass points on the curve

$$x_0^7 + x_1^7 + x_2^7 = 0.$$

The case $n = 4,5,6$ are omitted.

The point $(1,0,z)$ with $z = \sqrt[n]{-1}$, is on the curve

$$x_0^n + x_1^n + x_2^n = 0 .$$

<u>Theorem</u> ([2])

(a) $(1,0,z)$ is a Weierstrass point.

(b) The Weierstrass weight of this point is

$$\sum_{i=1}^{3} (1+i)(n-2-i)(n-1-i)/2 .$$

(c) The gap sequence at $(1,0,z)$ is

$$\underbrace{1,2,\dots,n-2}_{n-2},\underbrace{n+1,\dots,2n-3}_{n-3},\dots,\underbrace{kn+1,\dots,(k+1)n-k-2}_{n-k-2},\dots,$$

$$\underbrace{(n-3)n+1}_{1} .$$

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\begin{aligned}
AX = {} & (Z-1)^3*(2*Z+1)^3*(Z+2)^3*(9444843*Z^{36} + 170007174*Z^{35} - 8406573066*Z^{34} - 159770786877*Z^{33} - 242468001517 9*Z^{32} \\
& - 2443833181000 00*Z^{31} - 1246220406363 84*Z^{30} - 1998574581555 08*Z^{29} + 1519823955156461* \\
& Z^{28} + 13516123799328326*Z^{27} + 59277489469031722*Z^{26} + 173796016162807051*Z^{25} + \\
& 354163522740082544*Z^{24} + 457491209325473008*Z^{23} + 148744997471980283*Z^{22} - \\
& 920224057351991726*Z^{21} - 2695918832393204176*Z^{20} - 4464275587034399447*Z^{19} - \\
& 5215161393220145251*Z^{18} - 4464275587034399447*Z^{17} - 2695918832393204176*Z^{16} - \\
& 920224057351991726*Z^{15} + 148744997471980283*Z^{14} + 457491209325473008*Z^{13} + \\
& 354163522740082544*Z^{12} + 173796016162807051*Z^{11} + 59277489469031722*Z^{10} + \\
& 13516123799328326*Z^{9} + 1519823955156461*Z^{8} - 1998574581555 08*Z^{7} - 1246220406363 84*Z^{6} \\
& - 2443833181000 00*Z^{5} - 242468001517 9*Z^{4} - 159770786877*Z^{3} - 8406573066*Z^{2} + 170007174 \\
& *Z + 9444843)
\end{aligned}
$$

Table. ($z = p^7$)

References

[1] Griffiths,P. and Harris,J.,"Principles of Algebraic Geometry," New York : John Wiley & Sons, 1978.

[2] Kobayashi,H.,"Weierstrass gap sequences at points on curves of Fermat type," to appear.

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