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Some results in the divisor problems

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In what follows, ε denotes any positive number and c, with or without suffix, denotes a positive absolute constant.

§1. Introduction and historical survey.

Let $B_k(x)$ denote the k-th Bernoulli polynomial, $[x]$ the integral part of x, $\{x\} := x - [x]$ the fractional part of x, $P_k(x) := B_k(\{x\})$ the k-th periodic Bernoulli polynomial, $\sigma_r(n) := \sum_{d|n} d^r$ the sum of r-th powers of divisors of n, and define the basic functions $G_{a,k}(x)$ by

$$G_{a,k}(x) := \sum_{n\le\sqrt{x}} n^a P_k\left(\frac{x}{n}\right)$$

for real a and $k \in \mathbb{N}$. It was observed by Landau [28] as early as 1920 that the asymptotic relation

$$(1.1) \qquad \Delta(x) = -2G_{0,1}(x) + O(1)$$

had been implicit in Dirichlet's work on the divisor problem, where the error term $\Delta(x)$ is defined by

$$\sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + \Delta(x),$$

$d(n) = \sigma_0(n)$ and $\gamma = 0,5772...$ being the Euler (- Mascheroni) constant. By (1.1), the Dirichlet divisor problem, viz. the problem of establishing the estimate

$$(1.2) \qquad \Delta(x) = O\left(x^{1/4+\varepsilon}\right)$$

is equivalent to that of obtaining the estimate

$$(1.3) \qquad G_{0,1}(x) = O\left(x^{1/4+\varepsilon}\right)$$

The best estimate of $\Delta(x)$ known to date is due to Kolesnik [26]:

$$(1.4)\qquad \Delta(x) = O\left(x^{35/108+\varepsilon}\right),$$

while the best known Ω_+(resp. Ω_-)-result for $\Delta(x)$ is due to Hafner [14] (resp. to Corrádi and Kátai [11]):

$$(1.5)\quad \Delta(x) = \Omega_+\{(x\log x)^{1/4}(\log\log x)^{(3+2\log 2)/4}\exp(-c\sqrt{\log\log\log x})\}$$

$$(1.5')\quad \Delta(x) = \Omega_-\{x^{1/4}\exp(c(\log\log x)^{1/4}(\log\log\log x)^{-3/4})\}\ ,$$

and the Ω_+-result of Hafner is again obtained in [16] as a consequence of a general omega theorem.

As a generalization of (1.3), Chowla and Walum [9] conjectured that

$$(1.6)\qquad G_{a,k}(x) = O\left(x^{a/2+1/4+\varepsilon}\right)$$

and proved the special case $a=1,\ k=2$ without ε-factor:

$$(1.7)\qquad G_{1,2}(x) = O\left(x^{3/4}\right).$$

This is rather an astonishing result, as is seen from the following interpretation of Nowak [34]: If on could estimate the discrepancy $D_M(\omega)$ mod 1 of the sequence $\omega = (N/n)$, $n = 1,\dots,$ $M: = |\sqrt{N}|$ by

$$(1.8)\qquad D_M(\omega) = O\left(N^{-1/4+\varepsilon}\right),$$

then (1.3) would follow immediately from Koksma's inequality (cf. Hlawka [17], p. 107). Assuming the validity of (1.8) for every integer $M \leq \sqrt{N}$, one could easily infer (1.7).

Another special case $a=0,\ k=2$ of (1.6), which is much harder than (1.7), was again stated by Chowla [10]. Since (1.7) was proved, several attempts have been made in the case $k \geq 2$:

$$(1.9) \qquad G_{a,k}(x) = \begin{cases} O\left(x^{a/2+1/4}\right) & \text{if} \quad a > \frac{1}{2}, \\ O\left(x^{1/2}\log x\right) & \text{if} \quad a = \frac{1}{2}, \\ O\left(x^{(4a+3)/10}\right) & \text{if} \quad 0 \le a < \frac{1}{2}, \end{cases}$$

the last estimate being improved as

$$(1.10) \qquad G_{a,k}(x) = O\left(x^{2/7}\log x\right) \qquad \text{if} \quad 0 \le a < \frac{1}{2},$$

where (1.9) is due to Kanemitsu and Sita Rama Chandra Rao [22], and generalizes Walfisz's estimate [52], [53] of $G_{0,2}(x)$ to the case $0 \le a < \frac{1}{2}$, which improved upon the previous results of Wigert [54], Landau [27], Ramanujan [41] and Landau [29], and where (1.10) is due to Nowak [35], and generalizes Peng's estimate [40] of $G_{0,2}(x)$ (MacLeod [32],[33] mistakingly attributes [40] to Buchštab).

It should be noted that the papers by MacLeod [32],[33] and Suryanarayana [50], which are relevant to conjecture (1.6), contain errors due to a fallacious estimate given by Segal [44], which, however, Segal [45] himself pointed out to be false: Defining $E_1(x)$ by $\sum_{n\le x} \sigma(n) = \frac{\zeta(2)}{2}x^2 + E_1(x)$ $(\sigma(n) = \sigma_1(n))$, we find that

$$(1.11) \quad \sum_{n\le x} E_1(n) = \frac{1}{4}(\zeta(2) - 1)x^2 + \frac{x^{5/4}}{2^{3/2}\pi^2}\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^{7/4}}\sin\left(4\pi\sqrt{nx} - \frac{\pi}{4}\right) + O(x\log x)$$

in [44], and that the second term is bounded above by $O\left(x^{5/4}\right)$, which is not the case because the series is not absolutely convergent, and (1.11) gives the fallacious estimate used both in [32],[33] and [50]:

$$x\,G_{0,2}(x) + G_{2,2}(x) = O\left(x^{5/4}\right).$$

From the other direction, i.e. regarding Ω-results, the following has been established or just stated (cf. Kanemitsu and

Sita Rama Chandra Rao [23],[24]):

$$(1.12)\quad G_{a,2}(x) = \begin{cases} \Omega_+\left[x^{a/2+1/4}(\log x)^{1/4-a/2}\right] & \text{if} \quad 0 \le a < \frac{1}{2} \\ \Omega_-\left[x^{a/2+1/4}\exp\left(c\,\dfrac{(\log\log x)^{1/4-a/2}}{(\log\log\log x)^{3/4-a/2}}\right)\right] & \\ & \text{if} \quad 0 < a < \frac{1}{2}\,, \end{cases}$$

$$(1.13)\qquad \liminf_{x\to\infty} x^{-1/4}G_{0,2}(x) = -\infty\,,$$

$$(1.14)\qquad G_{1/2,2}(x) = \Omega_+(x^{1/2}\log x)\,,$$

and

$$(1.15)\qquad G_{a,2}(x) = \Omega_\pm(x^{a/2+1/4}) \quad \text{if} \quad \tfrac{1}{2} < a < \tfrac{3}{2} \quad a \ne 1.$$

Among these, (1.15) follows from a theorem of Chandrasekharan and Narasimhan [7] together with our Theorem 1 below, and, combined with (1.9), it provides a complete solution of conjecture (1.6) in the case $k = 2$, $\frac{1}{2} < a < \frac{3}{2}$, $a \ne 1$. The Ω_+-results in (1.12) and (1.14) are consequences of general theoremsof Hafner [16], which also give $\Omega_\pm$-results for $G_{a,2}(x)$ when $-\frac{1}{2} < a < 0$, when combined with Theorem 1.

In this note we present, without proof, some general omega -theorems which yield, as a simple corollary, Ω_--result in (1.12) and $\Omega_\pm$-results for $G_{a,2}(x)$ when $-2 < a < -\frac{1}{2}$.

Speaking rather dogmatically, I dare say that in analytic number theory there are two very different kinds of arithmetic functions, the difference arising from whether the generating Dirichlet series (if any) satisfy the functional equation with $A > 0$ or not (for details, see §2 below). When the generating Dirichlet series satisfy the functional equation with $A > 0$, there are two different but not mutually independent ways of obtaining omega-theorems. The one starts from Ingham [18] and has been developed in Corrádi and Kátai [11], Gangadharan [13], Joris [19],[20], Redmond [43], which give, in general, estimates of type (1.15'), and to which our main theorems belong also; the other has been developed in the works of Szegö, Berndt [4], [5],[6], Steinig [49], Hafner [16] (see this paper for more details). The general reference is Chandrasekharan and Narasimhan [7].

Now we refer to conjecture (1. 6) on average. The following average results have been obtained:

$$(1.16)\qquad \frac{1}{x}\int_1^x G_{a,2}(t)^2 dt = O\left(x^{a+1/2}\right) \quad \text{if} \quad |a| < \frac{1}{2}$$

(Kanemitsu and Sita Rama Chandra Rao [23]), and this has been extended to the more general setting by Balakrishnan and Srinivasan [2] and Srinivasan [45] whose theorems provide, in particular,

$$(1.16')\qquad \frac{1}{x}\int_1^x G_{a,k}(t)^2 dt = O\left(x^{a+1/2}\right) \quad \text{if} \quad a > -\frac{1}{2} \quad \text{and} \quad k \geq 2$$

We hope, however, to get an asymptotic formula for $\int_1^x G_{a,k}(t)^2 dt$ similar to that obtained by Cramér [12] for $\int_1^x \Delta^2(t)\,dt$, and hope to accomplish this through the error function $R(x,r)$ which is closely related to $G_{a,k}(x)$. Cf. also Bellman [3; 12.5], Chandrasekharan and Narasimhan [8] and Redmond [42].

§2. Statement of results

2.1. To state Theorem 1 let us first introduce some further notation. For $b > 0$, $\rho > 0$ let

$$P^\rho(x,r,b) := \frac{1}{\Gamma(\rho+1)} \sum_{n \leq x}{}' (x^b - n^b)^\rho \sigma_r(n) - S^\rho((\pi x)^b, r, b) ,$$

$$R(x,r) := P^1(x,r,r,) ,$$

where the prime on the summation sign means that if $\rho = 0$ and $n = x$, the term $\sigma_{-r}(n)$ must be halved, and

$$S^{\rho}(x,r,b) := \sum_{\xi} \operatorname*{Res}_{s=\xi} \frac{\Gamma(s)\pi^{-bs}\zeta(bs)\zeta(bs+r)}{\Gamma(s+\rho+1)} x^{s+\rho},$$

where ξ runs through all the poles of the function described above in the half-plane $\operatorname{Re}\xi > -\rho - 1 - k$, and k is such that

$$k > \left|\frac{-r+1/2}{2b}\right|,$$

so that, for $r \neq 0, \pm 1$,

$$R(x,r) = \sum_{n\le x} (x^r - n^r)\sigma_{-r}(n) - \left(\frac{r}{1-r}\zeta(1-r)x + \frac{r}{1+r}\zeta(1+r)x^{r+1} - \frac{\zeta(r)}{2}x^r + \frac{\zeta(-r)}{2}\right),$$

$$R(x,1) = \sum_{n\le x} (x-n)\sigma_{-1}(n) - \left(\frac{\zeta(2)}{2}x^2 - \frac{1}{2}x\log x + \frac{1}{2}(1-\gamma-\log 2\pi)x - \frac{1}{24}\right),$$

and

$$R(x,0) = \sum_{n\le x} \left(\log\frac{x}{n}\right) d(n) - \left(x\log x - 2(\gamma-1)x + \frac{1}{4}\log x + \frac{1}{2}\log 2\pi\right).$$

(Since, if $r < 0$, we have $R(x,r) = -x^r R(x,-r)$, $-r > 0$, we may restrict ourselves to the case $r \geq 0$).

Now we may state

Theorem 1. Let $-\frac{1}{2} \leq r < 3$, $r \neq 0, 1$. Then

$$R(x,r) = -\frac{r}{2}x^{r-1}G_{1-r,2}(x) + O\left(x^{(2r-1)/4}\right).$$

As a simple consequence of (1.9), (1.10) and Theorem 1, we have

Corollary 1.

$$R(x,r) = \begin{cases} O\left(x^{(2r-1)/4}\right) & \text{if} \quad 0 < r < \frac{1}{2} \\ O(\log x) & \text{if} \quad r = \frac{1}{2} \\ O\left(x^{(7r-5)/7}\log x\right) & \text{if} \quad \frac{1}{2} < r \leq 1 . \end{cases}$$

Incidentally, the first two improve upon the so far known best estimates due to Wilson [55], and the third improves upon the result of Landau [29]. Chowla's conjecture (1.2) is equivalent to the estimation $R(x,1) = O(x^{1/4+\varepsilon})$ (cf. also §2.3).

2.2. Before stating our main theorems let us introduce the definition of a general functional equation, essentially due to (Bochner and) Chandrasekharan and Narasimhan [7]:

Let $\{a_n\}$, $\{b_n\}$ be two sequences of complex numbers not all the terms of which are zero. Let $\{\lambda_n\}$, $\{\mu_n\}$ be strictly increasing sequences of positive numbers. Let

$$\Delta(s) := \prod_{\nu=1}^{N} \Gamma(\alpha_\nu s + \beta_\nu),$$

where $N \in \mathbb{N}$, β_ν are arbitrary complex numbers, $\alpha_\nu > 0$, and $A := \sum_{\nu=1}^{N} \alpha_\nu > \frac{1}{2}$. Suppose that

$$\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} b_n \mu_n^{-s},$$

each of which converges in some half-plane with finite abscissa of absolute convergence σ_a^* and σ_b^*, respectively. Then $\phi(s)$ and $\psi(s)$ are said to satisfy the <u>functional equation</u> ($\delta \in \mathbb{R}$)

$$\phi(s)\Delta(s) = \psi(\delta - s)\,\Delta(\delta - s)$$

if there exists in the s-plane a domain $\mathcal{D}$, which is the exterior of a compact set $\mathcal{S}$, in which there exists a holomorphic function χ with the property

$$\lim_{|t|\to\infty} \chi(\sigma + it) = 0,$$

uniformly in any finite interval $-\infty < \sigma_1 \le \sigma \le \sigma_2 < \infty$, and

$$\chi(s) = \phi(s)\,\Delta(s) \quad \text{for} \quad \sigma > \sigma_a^*$$

$$\chi(s) = \psi(\delta - s)\,\Delta(\delta - s) \qquad \text{for} \qquad \sigma > \sigma_b^*$$

For $\rho \geq 0$ we form the Riesz sum of a_n of order ρ:

$$A^\rho(x) := \frac{1}{\Gamma(\rho+1)} \sum_{\lambda_n \leq x}{}' (x - \lambda_n)^\rho a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho}\, ds,$$

where $c > 0$, $c > \sigma_a^*$, and define the residual function

$$S^\rho(x) := \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho}\, ds,$$

where $\mathscr{C}$ is the rectangle with vertices at $c_1 - iR$, $c_1 + iR$, $c_2 + iR$, $c_2 - iR$. Here $c_1 > 0$, $c_1 > \sigma_a^*$; R is so large that the integrand is regular for $|t| \geq R$; $c_2 = -(m_0 + \frac{1}{2})$, $0 \leq m_0 \in \mathbb{Z}$; m_0 is chosen so large that $\mathscr{C}$ encloses all the singularities of the integrand to the right of $\sigma = -\rho - 1 - k$, where k is such that $k > |\delta/2 - 1/4A|$, and all the singularities of $\phi(s)$ lie in $\sigma > -k$ (so that if ρ is integral then the integral is simply $\int_{\mathscr{C}} \frac{\phi(s)\, x^{s+\rho}}{s(s+1)\ldots(s+\rho)}\, ds$, and $\mathscr{C}$ encloses all the singularities of $\phi(s)$ and the poles $0, -1, \ldots, -\rho$ of Γ).
Suppose the following inequalities hold:

$$\begin{cases} \delta + \frac{1}{2} + m_0 > \sigma_b^*, & \delta + \frac{1}{2} + m_0 > \mathrm{Re}\left(-\frac{\beta_\nu}{\alpha_\nu}\right), \\ \frac{\delta}{2} + \frac{1}{2} + m_0 > \max\left(\frac{3}{2A}, \frac{\rho + 1/2}{2A}\right), & \frac{1}{2} + m_0 > \mathrm{Re}\left(\frac{\beta_\nu - 1}{\alpha_\nu}\right), \quad 1 \leq \nu \leq N. \end{cases}$$

Then we consider the error term $P^\rho(x)$ defined by

$$P^\rho(x) := A^\rho(x) - S^\rho(x) = \frac{1}{2\pi i} \int_{\mathscr{C}_1} \frac{\Gamma(s)\phi(s)}{\Gamma(s+\rho+1)} x^{s+\rho}\, ds,$$

where $\mathscr{C}_1$ is a broken line made up of $c_1 - i\infty$, $c_1 - iR$, $c_2 - iR$,

$c_2 + iR$, $c_1 + iR$, $c_1 + i\infty$.

Theorem 2. Suppose that for any $x \geq 1$, the set $\{\mu_n \leq x\}$ contains a subset $Q = Q_x := \{\mu_{n_k} \leq x \mid k = 1, \ldots, N(x)\}$ such that no number $\mu_n^{1/2A}$ is expressible as a linear combination of the numbers $\mu_{n_k}^{1/2A}$ with coefficients ± 1, unless $\mu_n^{1/2A} = \mu_{n_r}^{1/2A}$ for some r, in which case $\mu_n^{1/2A}$ has no other representation, and that

$$\sum_{\mu_{n_k} \leq x} \left| \operatorname{Re} b_{n_k} \right| \mu_{n_k}^{-\frac{A\delta + \rho + 1/2}{2A}} \geq c_3 L(x), \tag{2.2.1}$$

for all $x \geq 1$, where $L(x)$ is an increasing, slowly varying function (for the details of the theory of slowly varying functions, see Seneta [46]).

Let

$$S_x := \left\{ \eta = \left| \mu_m'^{\frac{1}{2A}} + \sum_{k=1}^{N} r_k \mu_{n_k}^{\frac{1}{2A}} \right| \;\middle|\; \mu_m' = 0, \mu_1, \mu_2, \ldots;\ r_k = 0, \pm 1,\ \sum_{n=1}^{N} r_k^2 \geq 1 \right\}.$$

Suppose there exists an $\tilde{\eta} \in S_x$ such that $\eta < 1$. Then

$$\tilde{\eta}(x) := \inf S_x = \min S_x .$$

Suppose there exist constants c_4, $c_5 > 0$ such that

$$c_4 x \leq q(x) \leq \exp\left(c_5 \frac{x}{\log x} \right),$$

where

$$q(x) := -\log \tilde{\eta}(x) \quad (> 0) .$$

Suppose that

$$N(x) \leq \exp\left(B_2' \frac{x}{\log x} \right), \qquad B_2' > 0 ,$$

$$\max \mu_{n_k} = o\left(\exp\exp\left(c\frac{x}{\log x}\right)\right),$$

$$c_6 := \max\{c_5, 2B_2'\} > 0$$

$$Q(x) := \exp\left(c_6\frac{x}{\log x}\right),$$

and that there exists an n such that

$$\text{(2.2.2)} \qquad \operatorname{Re} b_n \neq 0 .$$

Then there exists c_7 such that

$$\operatorname{Re} P^{\rho}(x) = \Omega_{\pm}\left(x^{\delta/2-1/4A+(1-1/2A)\rho} L(c_7 \log\log x \cdot \log\log\log x)\right).$$

If in (2.2.1) and (2.2.2) we have Im in place of Re, then in the conclusion, we should have $\operatorname{Im} P^{\rho}(x)$ in place of $\operatorname{Re} P^{\rho}(x)$.

Theorem 3. Under the suppositions of Theorem 2, let $\mu_n := A_1 n^b$, $A_1 > 0$, $b > 0$, $\frac{2A}{b} \in \mathbb{N}$. Take Q as follows: $Q = Q_x = \{q^b \mid q \in Q'\}$, where $Q' = Q_x' = \{$square-free integers with prime factors in $P\}$, where $P = P_x$ is the set of prime numbers for which there exist constants B_1, $B_2 > 0$ such that

$$B_1\frac{x}{\log x} \leq \sum_{p\in P_x} 1 \leq B_2\frac{x}{\log x},$$

and take $L(x)$ as

$$L(x) := \exp\left(c_{12}\frac{x^{\lambda}}{\log x}\right), \qquad 0 < \lambda < 1 .$$

Then there exists c_{13} such that

$$\operatorname{Re} P^{\rho}(x) = \Omega_{\pm}\left(x^{\delta/2-1/4A+(1-1/2A)\rho} \exp\left(c_{13}\frac{(\log\log x)^{\lambda}}{(\log\log\log x)^{1-\lambda}}\right)\right)$$

If in (2.2.1) and (2.2.2) we have Im in place of Re, so should do we in the conclusion.

Corollary 1. If, in addition to the conditions of the theorem, we suppose that $\mathrm{Re}\, b_n = \mathrm{Re}\, b(n)$ is a multiplicative function of n satisfying the inequality

$$|\mathrm{Re}\, b(p)| \geq c_{14}\, p^a$$

for all $p \in P_x$ and some a such that

$$a > \frac{\delta}{2} + \frac{1}{4A} + \frac{\rho}{2A} - \frac{1}{b},$$

then there exists c_{15} such that

$$\mathrm{Re}\, P^\rho(x) = \Omega_\pm\left(x^{\frac{\delta}{2}-\frac{1}{4A}+(1-\frac{1}{2A})\rho} \exp\left[c_{15}\frac{(\log\log x)^{1+b(a-\delta/2-1/4A-\rho/2A)}}{(\log\log\log x)^{b(\delta/2+1/4A+\rho/2A-a)}}\right]\right).$$

Here the same proviso as that of Theorem 3 holds also.

Corollary 2. Under the notation of §2.1, if

$$r \geq 0,\ \rho < r + \frac{1}{2},\ m_0 + \frac{1}{2} > \max\left(\frac{r}{b},\ \frac{r+2}{2b},\ \frac{r+\rho-1/2}{2b}\right),$$

then we have

$$P^\rho(x,r,b) = \Omega_\pm\left(x^{-\frac{r}{2}+\frac{1}{4}+(b-\frac{1}{2})\rho} \exp\left[c_{16}\frac{(\log\log x)^{(1/2+r-\rho)/2}}{(\log\log\log x)^{(-r+3/2+\rho)/2}}\right]\right),$$

where, more explicitly than in §2.1,

$$P^\rho(x,r,b) = \frac{1}{\Gamma(\rho+1)} \sum_{n\leq x}{}' (x^b - n^b)^\rho \sigma_{-r}(n) - \left(\frac{\Gamma\left(\frac{1-r}{b}\right)\zeta(1-r)x^{1-r+b\rho}}{b\Gamma\left(\frac{1-r}{b}+\rho+1\right)} + \frac{\zeta(1+r)\Gamma(\frac{1}{b})x^{1+b\rho}}{b\Gamma(\frac{1}{b}+\rho+1)} + \sum_{n=0}^{m_0} \frac{(-1)^n}{n!}\frac{\zeta(-bn)\zeta(r-bn)}{\Gamma(-n+\rho+1)} x^{b(\rho-n)}\right)$$

Incorporating the results deducible from the theorems of Berndt [4], Chandrasekharan and Narasimhan [7], Hafner [16], Steinig [49] and ours, we now state

Corollary 3. There exist c_{17}, c_{18} such that

$$R(x,r) = \begin{cases} \Omega_{\pm}\left\{x^{r/2-1/4} \exp\left(c_{17} \dfrac{(\log\log x)^{r/2-1/4}}{(\log\log\log x)^{5/2-r/2}}\right)\right\} & \text{(Corollary 2)} \qquad r \geq \frac{3}{2} \\ O(x^{r/2}) & \text{(trivial)} \end{cases}$$

$$R(x,r) = \begin{cases} \Omega_{\pm}\left\{x(\log x)^{r/2-1/4}\right\} & \text{(Berndt, Hafner, Steinig)} \\ O(x^{r/2}) & \text{(trivial)} \qquad 1 < r < \frac{3}{2} \end{cases}$$

$$R(x,r) = \begin{cases} \Omega_{-}\left\{(x\log x)^{r/2-1/4}\right\} & \text{(Berndt, Hafner, Steinig)} \\ \Omega_{+}\left\{x^{r/2-1/4} \exp\left[c_{18} \dfrac{(\log\log x)^{r/2-1/4}}{(\log\log\log x)^{5/2-r/2}}\right]\right\} & \text{(Corollary 2)} \\ O\left\{x^{(7r-5)/7}\log x\right\} & \text{(Nowak)} \qquad \frac{1}{2} < r \leq 1 \end{cases}$$

$$R(x,\tfrac{1}{2}) = \begin{cases} \Omega_{-}(\log\log x) & \text{(Hafner)} \\ O(\log x) \end{cases}$$

$$R(x,r) = \begin{cases} \Omega_{\pm}\left\{x^{r/2-1/4}\right\} & \text{(Chandrasekharan and Narasimhan)} \\ O\left\{x^{r/2-1/4}\right\} & |r| < \frac{1}{2}\,. \end{cases}$$

2.3. Once one notices that the function $\zeta(bs)\zeta(bs+r)\pi^{-bs}$ satisfies the functional equation, one could obtain a series representation for $P^{\rho}(x,r,b)$ appealing to a theorem of Hafner [15]; in particular, a series representation for $R(x,r)$ would follow, which, then, yields a series representation for $G_{1-r,2}(x)$ in view of Theorem 1. However, our original proof in [23] of the series representation for $G_{1-r,2}(x)$ is not only elementary and self-contained but can be used e.g. in the investigation of the logarithmic Riesz sum of $\sigma_r(n)$, which is relevant to Chowla's and Walum's conjecture, as will be seen below.

We write

$$L_r(x) := \sum_{n\leq x} \sigma_r(n) \log\frac{x}{n} = \Psi_r(x) + H_r(x)\,,$$

and consider the error term $H_r(x)$, where $\Psi_r(x)$ is the sum of the residues of the function $x^s\zeta(s)\zeta(s-r)/s^2$ at the points $s = 0,\ r+1,\ 1$.

Proposition 1. We have

$$L_{-1}(x) = \frac{\pi^2}{6}x - \frac{1}{4}\log^2 x - a\log x + b - \frac{1}{2x}G_{0,2}(x) + O(x^{-3/4})\,,$$

where $a = \frac{1}{2}(\gamma + \log 2\pi)$, $b = -\frac{\gamma}{2}\log 2\pi - \frac{A_1}{2} + \frac{\zeta''(0)}{2}$, and A_1 is one of generalized Euler constants defined by

$$\zeta(s) = \frac{1}{s-1} + \gamma + A_1(s-1) + \ldots\,,$$

so that Chowla's conjecture is equivalent to the estimation

$$H_{-1}(x) = O(x^{-3/4+\varepsilon})\,.$$

Since $H_{-1}(x) = x^{-1}R(x,1) + \int_1^x R(y,1)y^{-2}dy + O(1)$, one should refer to §2.1 for the known upper estimates of $H_{-1}(x)$.

Corollary 1. We have

$$H_{-1}(x) = \frac{x^{-3/4}}{2^{3/2}\pi^2} f_0(4\pi\sqrt{x}) + O(x^{-3/4}),$$

$$H_{-1}(x) = \Omega_-\left\{x^{-3/4}(\log x)^{1/4}\right\},$$

$$\limsup_{x\to\infty} x^{3/4}H_{-1}(x) = +\infty .$$

Proposition 2. We have

$$\int_1^\infty \frac{R(y,1)}{y^2}\,dy = b + a - \frac{1}{2} + \frac{\pi^2}{12},$$

$$\int_1^\infty \frac{G_{0,2}(y)}{y^2}\,dy = \frac{2}{3}\sum_{n=1}^\infty \int_n^\infty \frac{P_3(u)}{u^3}\,du,$$

$$\int_1^\infty \frac{G_{2,2}(y)}{y^3}\,dy = \sum_{n=1}^\infty \int_n^\infty \frac{P_3(u)}{u^3}\,du .$$

Theorem 5. We have

$$L_0(x) = x\log x + 2(\gamma - 1)x + \frac{1}{4}\log x + \frac{1}{2}\log 2\pi$$
$$- \frac{G_{1,2}(x)}{x} - \frac{P_1(\sqrt{x}) + 8P_3(\sqrt{x})}{\sqrt{x}} + O(x^{-3/4}) .$$

This should be compared with the results of Oppenheim |37| and MacLeod |33|.

Proposition 3. We have the identities

$$\int_1^x \frac{\Delta_1(t)}{t^2}\,dt = \frac{1}{2}\log 2\pi + \gamma - \frac{3}{2},$$

where $\Delta_1(x) = \sum_{n\le x} (x - n)d(n) - \left\{\frac{1}{2} x^2\log x + (\gamma - \frac{3}{4})x^2 + \frac{x}{4}\right\};$

$$\zeta(3) = 180\left(\int_1^\infty \frac{G_{1,2}(t)}{t^2}\,dt - \frac{1}{2}\sum_{n=1}^{\infty}\int_n^\infty \frac{P_4(u)}{u^4}\,du\right);$$

$$\int_1^\infty \frac{P_1(\sqrt{t}) + 8P_3(\sqrt{t})}{t^{3/4}}\,dt = 2\left(\int_1^\infty \frac{P_2(u) - B_2}{u^3}\,du + 8\int_1^\infty \frac{P_4(u) - B_4}{u^4}\,du\right).$$

When $0 \neq |r| < 1$, we can obtain a similar asymptotic formula for $L_r(x)$.

§3. Possible further developements.

In MacLeod |32,33| various asymptotic formulas can be found, e.g. defining $E^a_{-t}(x)$ $(0 \leq t \leq a)$ by

$$E^a_{-t}(x) = \begin{cases} \sum\limits_{n\leq x} \dfrac{\sigma(n)}{n} - \zeta(2)x + \dfrac{1}{2}\log x & a = t = 1, \\[2ex] \sum\limits_{n\leq x} \dfrac{\sigma_a(n)}{n^t} - \dfrac{1}{a-t+1}\zeta(a+1)x^{a-t+1} & \text{otherwise,} \end{cases}$$

Theorem 8,(iv) asserts that

$$\int_1^x E^1_{-1}(u)\,du = -\frac{1}{2}(\log 2\pi + \gamma)x - \frac{1}{2x}G_{2,2}(x) - \frac{1}{2}G_{0,2}(x) + O(1),$$

which can be simplified, on using (1.9)(of which he was unaware), as:

$$\int_1^x E^1_{-1}(u)\,du = -\frac{1}{2}(\log 2\pi + \gamma)x - \frac{1}{2}G_{0,2}(x) + O(x^{1/4}); \tag{3.1}$$

on the other hand, Theorem 4,(b) gives

$$\begin{aligned} \int_1^x E^1_{-1}(u)\,du &= \sum_{n\leq x} (x-n)\frac{\sigma(n)}{n} - \int_1^x \left(\zeta(2)u - \frac{1}{2}\log u\right)du \\ &= \sum_{n\leq x} (x-n)\frac{\sigma(n)}{n} - \frac{\zeta(2)}{2}x^2 + \frac{1}{2}(x\log x - x) + O(1) \end{aligned} \tag{3.2}$$

$$= -\frac{1}{2}(\log 2\pi + \gamma)x + R(x,1) + O(1),$$

so that in studying the error term of $\int_1^x E_{-1}^1(u)du$, we may use the method of trigonometrical sums to estimate $G_{0,2}(x)$ from above, and general omega theorems in §2.2 to estimate $R(x,1)$ from below. A similar situation reveals itself also in the study of

$$\int_1^x E_{-2}^2(u)du = -\frac{1}{2}\zeta(2)x + \frac{1}{12}\log x + G_{-1,2}(x) + \left[A_1 + \frac{1}{12}\gamma + \frac{1}{2}\zeta(3)\right] + O(x^{-1/4})$$

$$= \sum_{n \leq x} (x-n)\sigma_{-2}(n) - \frac{1}{2}\zeta(3)x^2 + \frac{1}{2}\zeta(3).$$

Thus the estimation of the error term for $\int_1^x E_{-2}^2(u)du$ reduces, on the one hand, to that of $G_{-1,2}(x)$ $(= O(\log x))$ from above, and, on the other hand, to that of $\sum_{n \leq x} (x-n)\sigma_{-2}(n) - \left(\frac{1}{2}\zeta(3)x^2 - \frac{1}{2}\zeta(2)x + \frac{1}{12}\log x + (A_1 + \frac{1}{12}\gamma)\right)$ from below. Actually, the key hindges on Lemma 20 in [33], which gives, with the aid of (1.9), an asymptotic relation between $G_{a,2}(x)$ and the error term of the Riesz sum of order 1 of some divisor functions of the form $\sigma_a(n)n^{-t}$. Once one notices this relation and applies the upper estimation for G and lower estimation for R obtainable from general omega theorems, one easily gets a precise asymptotic formula for $\int_1^x E_{-t}^a(u)du$, which, in general, readily yields an asymptotic formula for $\sum_{n \leq x} E_{-t}^a(n)$. It seems to me that MacLeod's theorems 6-12 should be examined in the above mentioned way. Moreover, MacLeod treats only those sums of the form $\sum_{n \leq x} \sigma_a(n)n^{-t}$ with $a, t \in \mathbb{Z}$, but we need to consider them with $a, t \in \mathbb{R}$ to investigate $G_{a,k}(x)$ for $0 \leq a \in \mathbb{R}$.

Nowak [34] considers an analogue to the function $G_{a,k}(x)$, in the circle problem, i.e. the function $H_{a,k}(x) := \sum_{n \le \sqrt{x/2}} (\sqrt{x-n^2})^a P_k(\sqrt{x-n^2})$, and provides a precise series representation for $H_{1,2}(x)$. Considering that the divisor problem and the circle problem have been discussed in rather a parallel way, it seems plausible that the consideration of $H_{a,k}$ for various a, k might cut open a new phase in the circle problem, and at least it does not seem absurd to expect that the sum $\sum_{n \le x} r(n)\log(x/n)$ appearing in [1] could be treated by a method similar to that used in the proof of results in §2.3, using Nowak's estimate of $H_{a,k}$.

In [36] Nowak obtains another interesting analogue to Chowla's and Walum's conjecture (1.6) in the case of the Piltz divisor problem in three dimension, and sets forth a conjecture relating to the n-dimensional case ($n > 4$). In the three-dimensional, the analogous function to $G_{a,k}(x)$ above is $G_{a,b;k}(x) := \sum_{D(x)} n^a m^b P_k(\frac{x}{mn})$, where $D(x)$ stands for the domain $\frac{1}{2} < n \le x^{1/3}$, $n < m \le (x/n)^{1/2}$, and his Theorem estimates $G_{a,b;k}(x)$ very precisely in the case $k \ge 2$, $2a - 1 \ge b \ge 1$. The investigation of this function as well as the function $H_{a,k}(x)$ may enrich the theory.

Smith [47] considers the seemingly relevant sum $\sum_{n \le x} (x-n)(\sigma_{-2}(n))^2$ to ours, but this seems to belong to another field.

Finally, we note that the most difficult case of (1.6) for $k = 1$ has not been studied enough, the obtained estimate [21] is still very weak. It should be possible to obtain as good an estimate as (1.4).

References

1. R. Ayoub and S. Chowla, On a theorem of Müller and Carlitz, J. Number Theory 2 (1970), 342-344.
2. U. Balakrishnan and S. Srinivasan, A footnote to a conjecture of S. Chowla and H. Walum. II., to appear.
3. R. Bellman, Analytic number theory: An introduction, Benjamin, London-Amsterdam-Don Mills, Ontario-Sydney-Tokyo 1980.
4. B. C. Berndt, On the average order of some arithmetical functions, Bull. Amer. Math. Soc. 76 (1970) 856-859.
5. B. C. Berndt, On the average order of a class of arithmetical functions, I, J. Number Theory 3 (1971) 184-203.
6. B. C. Berndt, On the average order of a class of arithmetical functions, II, J. Number Theory 3 (1971), 288-305.
7. K. Chandrasekharan and Raghavan Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Ann. of Math. 76 (1962), 93-136.
8. K. Chandrasekharan and Raghavan Narasimhan, On the mean value of the error term for a class of arithmetical functions, Acta Math. 112 (1964) 41-67.
9. S. Chowla and H. Walum, On the divisor problem, Norske Vid. Selsk. Forh. (Trondheim) 36 (1963), 127-134; Proc. Sympos. Pure Math. Vol. 8, Amer. Math. Soc., Providence, R. I. 138-143.
10. S. Chowla, The Riemann hypothesis and Hilbert's tenth problem, Math. and its applications, Vol. 4, Gordon and Breach, New York 1965, 89-91.

11. K. Corrádi and I. Katái, A comment on K. S. Gangadharan's paper entitled "Two classical lattice point problems." (Hungarian. English summary). Magyar. Tud. Akad. Mat. Fiz. Oszt. Közl. 17 (1967), 89-97.

12. H. Cramér, Über zwei Sätze des Herrn G. H. Hardy, Math. Z. 15 (1922), 201-210.

13. K. S. Gangadharan, Two classical lattice point problems, Proc. Cambridge Philos. Soc. 57 (1961), 699-721.

14. J. L. Hafner, New omega theorems for two classical lattice point problems, Invent. Math. 63 (1981), 181-186.

15. J. L. Hafner, On the representation of the summatory functions of a class of arithmetical functions, Lect. Notes in Math. No. 899, Springer Verlag, Berlin - Heidelberg - New York 1981, 148-165.

16. J. L. Hafner, On the average order of a class of arithmetical functions, J. Number Theory 15 (1982), 36-76.

17. E. Hlawka, Theorie der Gleichverteilung, Mannheim - Wien - Zürich, Bibl. Inst. 1979.

18. A. E. Ingham, On two classical lattice point problems, Proc. Cambridge Philos. Soc. 36 (1940), 131-138.

19. H. Joris, Ω-Sätze für zwei arithmetische Funktionen, Comment. Math. Helv. 47 (1972), 220-248.

20. H. Joris, Ω-Sätze für gewiße multiplikative arithmetische Funktionen, Comment. Math. Helv. 48 (1973), 409-435.

21. S. Kanemitsu, On a kind of divisor problem, Mem. Fac. Sci. Kyushu Univ. Ser. A 32 (1978), 81-89.

22. S. Kanemitsu and R. Sita Rama Chandra Rao, On a conjecture of S. Chowla and of S. Chowla and H. Walum, to appear in J. Number Theory.

23. S. Kanemitsu and R. Sita Rama Chandra Rao, On a conjecture of S. Chowla and of S. Chowla and H. Walum, II, to appear in J. Number Theory.

24. S. Kanemitsu and R. Sita Rama Chandra Rao, On a conjecture of S. Chowla and of S. Chowla and H. Walum, III, Proc. Japan Acad. Ser. A Math. Sci. 56 (1980), 185-187.

25. S. Kanemitsu, Some results in the divisor problems, to appear.

26. G. A. Kolesnik, On the order of $\zeta(1/2+it)$ and $\Delta(R)$, Pacific J. Math. 98 (1982), 107-122.

27. E. Landau, Besprechung von S. Wigert: Sur quelqeus fonctions arithmétiques,, Gött. gel. Anz. 7 & 8 (1915), 377-414.

28. E. Landau, Über Dirichlets Teilerproblem, Nachr. Akad. Wiss. Göttingen 1920, 13-32.

29. E. Landau, Ueber einige zahlentheoretische Funktionen, Nachr. Akad. Wiss. Göttingen 1920 , 116-134.

30. E. Landau, Ueber das Konvergenzgebiet einer mit der Riemannschen Zetafunktion zusammenhängenden Reihe, Math. Ann. 97 (1926), 251-259.

31. E. Landau, Ausgewählte Abhandlungen zur Gitterpunktlehre, herausgegangen von A. Walfisz, VEB Deutscher Verlag der Wissenschaften, Berlin 1962.

32. R. A. MacLeod, "Fractional Part Sums and Divisor Functions", Univ. of Victoria Int. Report DM-119-IR, January 1978, modified February 1979.

33. R. A. MacLeod, Fractional part sums and divisor functions, J. Number Theory 14 (1982), 185-227.

34. W. G. Nowak, On a certain sum connected with the circle problem, to appear.

35. W. G. Nowak, On a problem of S. Chowla and H. Walum, to appear.

36. W. G. Nowak, An analogue to a conjecture of S. Chowla and H. Walum, to appear.

37. A. Oppenheim, Some identities in the theory of numbers, Proc. London Math. Soc. (2) 26 (1926), 295-350.

38. Pan Cheng-Tung, On $\sigma(n)$ and $\phi(n)$, Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 637-638.

39. Pan Cheng-Tung, On $\sigma(n)$ and $\phi(n)$, Acta Natur. Univ. Pekinensis, (1956), 303-322.

40. H. Y. Peng, A result in the divisor problem, Acad. Sinica Science Record 1 (1942), 69-72.

41. S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, Trans. Cambridge Philos. Soc. 22 (1918), 259-276; Collected papers of Srinivasa Ramaujan, Cambridge Univ. Press, Cambridge 1927, 179-199.

42. D. Redmond, Mean value theorems for a class of Dirichlet series, Pacific J. Math. 78 (1978), 191-231 .

43. D. Redmond, Omega theorems for a class of Dirichlet series, Rockey Mountain J. Math. 9 (1979), 733-748.

44. S. L. Segal, A note on the average order of number-theoretic error terms, Duke Math. J. 32(1965), 279-284; erratum,

ibid. 32 (1965), 765.

45. S. L. Segal, Errata: "A note on the average order of number -theoretic error terms". Duke Math. J. 33 (1966), 821.

46. E. Seneta, Regularly varying functions, Lect. Notes in Math. No. 508, Springer Verlag, Berlin-Heidelberg-New York 1976.

47. R. A. Smith, An error term of Ramanujan, J. Number Theory 2 (1970), 91-96.

48. S. Srinivasan, A footnote to a conjecture of S. Chowla and H. Walum, to appear.

49. J. Steinig, On an integral connected with the average order of a class of arithmetical functions, J. Number Theory 4 (1972), 463-468.

50. D. Suryanarayana, On a conjecture of S. Chowla, Proc. Amer. Math. Soc. 56 (1976), 27-33.

51. E. C. Titchmarsh, The theory of the Riemann zeta-function, Clarendon Press, Oxford 1951.

52. A. Walfisz, Teilerprobleme (vierte Abhandlung), Ann. Scuola Norm. Sup. Pisa (2) 5 (1936), 289-298.

53. A. Walfisz , Gitterpunkte in mehrdimensionalem Kugeln, PWN, Warszawa 1957.

54. S. Wigert, Sur quelques fonctions arithmétiques, Acta Math. 37 (1914), 113-140.

55. B. M. Wilson, An asymptotic relation between the arithmetical sums $\sum_{n \leq x} \sigma_r(n)$ and $x^r \sum_{n \leq x} \sigma_{-r}(n)$, Proc. Cambridge Philos. Soc. 21 (1923), 140-149.