144
数理解析研究所講究録
501巻 1983年 144-159

Actions of symplectic groups on a product of projective spaces

Aiko NAKANISHI and Fuichi UCHIDA

国際基督教大 中西あい子
山形大・理 内田伏一

0. Introduction

In previous papers [6],[7], smooth actions of special unitary (resp. symplectic) groups on a product of complex (resp. quarternion) projective spaces have been studied. Here we shall study smooth actions of symplectic group $Sp(n)$ on certain product manifolds and we shall prove the following.

Theorem. Let X be a closed orientable manifold on which $Sp(n)$ acts smoothly and non-trivially. Suppose $n \gtrsim 7$.

(i) Suppose $X \sim P_a(C) \times P_b(C)$, $1 \leqq b \leqq a < 2n$ and $a+b \leqq 4n-3$. Then $a = 2n-1$ and X is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_0$, where Y_0 is a closed orientable manifold such that $Y_0 \sim P_b(C)$ and $Sp(n)$ acts naturally on $P_{2n-1}(C)$ and trivially on Y_0.

(ii) Suppose $X \sim P_a(H) \times P_b(C)$, $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, and $2a+b \leqq 4n-4$. Then there are three cases :

(a) $a = n-1$ and X is equivariantly diffeomorphic to $P_{n-1}(H) \times Y_1$, where Y_1 is a closed orientable manifold such that $Y_1 \sim P_b(C)$ and $Sp(n)$ acts naturally on $P_{n-1}(H)$ and trivially on Y_1,

(b) $b = 2n-1$ and X is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_2$, where Y_2 is a closed orientable manifold such that $Y_2 \sim P_a(H)$ and $Sp(n)$ acts naturally on $P_{2n-1}(C)$ and trivially on Y_2,

(c) $b = 2n-1$ and X is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/Sp(1)$, where Y_3 is a closed orientable $Sp(1)$ manifold such that $Y_3 \sim S^2 \times P_a(H)$, $Sp(1)$ acts as right scalar multiplication on S^{4n-1}, the unit sphere of H^n, and $Sp(n)$ acts naturally on S^{4n-1} and trivially on Y_3. In addition, $F \sim S^0 \times P_a(C)$ and the induced homomorphism $i^* : H^2(Y_3) \to H^2(F)$ is trivial, where F denotes the fixed point set of the restricted $U(1)$ action on Y_3. Conversely, if Y_3 satisfies the above conditions, then $(S^{4n-1} \times Y_3)/Sp(1) \sim P_{2n-1}(C) \times P_a(H)$ for $1 \leqq a \leqq n-2$.

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients. By $X_1 \sim X_2$ we mean $H^*(X_1) \cong H^*(X_2)$ as graded algebras. Denote by $P_n(C)$ and $P_n(H)$ the complex and quarternion projective n-spaces, respectively.

1. Preliminary results

First we prepare the following two lemmas which are proved by a standard method (cf. [2],[3],[5]).

Lemma 1.1. Suppose $n \geqq 7$. Let G be a closed connected proper subgroup of $Sp(n)$ such that $\dim Sp(n)/G < 8n$. Then G coincides with $Sp(n-i) \times K$ $(i = 1,2,3)$ up to an inner automorphism of $Sp(n)$, where K is a closed connected subgroup of $Sp(i)$.

Lemma 1.2. Suppose $r \geqq 5$ and $k < 8r$. Then an orthogonal non-trivial representation of $Sp(r)$ of degree k is equivalent to $(\nu_r)_R \oplus \theta^{k-4r}$. Here $(\nu_r)_R : Sp(r) \to O(4r)$ is the canonical inclusion, and θ^t is the trivial representation of degree t.

In the following, let X be a closed connected orientable manifold with a non-trivial smooth $Sp(n)$ action, and suppose $n \geqq 7$ and $\dim X < 8n$. Put

$$F_{(i)} = \{x \in X : Sp(n-i) \subset Sp(n)_x \subset Sp(n-i) \times Sp(i)\},$$
$$X_{(i)} = Sp(n)F_{(i)} = \{gx : g \in Sp(n),\ x \in F_{(i)}\}.$$

Here $Sp(n)_x$ denotes the isotropy group at x. Then, by Lemma 1.1, we obtain $X = X_{(0)} \cup X_{(1)} \cup X_{(2)} \cup X_{(3)}$. Moreover, from Lemma 1.2, we can show the following Propositions. The proofs are omitted.

Proposition 1.3. If $X_{(k)}$ is non-empty, then $X_{(i)}$ is empty for each $i \geqq k+2$.

Proposition 1.4. Suppose $X = X_{(k)} \cup X_{(k+1)}$. If $X_{(k)}$ and $X_{(k+1)}$ are non-empty, then the codimension of each connected component of $F_{(k)}$ in X is equal to $4(k+1)(n-k)$.

Corollary 1.5. Suppose $X = X_{(2)} \cup X_{(3)}$. Then either $X_{(2)}$ or $X_{(3)}$ is empty.

Remark. $\dim Sp(n)/Sp(n-k)\times Sp(k) = 4k(n-k)$ and $\chi(Sp(n)/Sp(n-k)\times Sp(k)) = {}_nC_k$, where $\chi(\ )$ denotes the Euler characteristic, and ${}_nC_k$ denotes the binomial coefficient.

Remark. If $\dim X < 4n$, then we see $X = X_{(1)}$. In addition, if $H^{odd}(X) = 0$, then X is equivariantly diffeomorphic to $P_{n-1}(H)$, $P_{n-1}(H) \times S^2$ or $P_{2n-1}(C)$, where $Sp(n)$ acts naturally on $P_{n-1}(H)$, $P_{2n-1}(C)$ and trivially on S^2. So we assume $\dim X \geqq 4n$, in the following sections.

2. Cohomological aspects

Throughout this section, suppose that X is a closed orientable manifold with a non-trivial smooth $Sp(n)$ action, $n \geqq 7$ and $X = X_{(0)} \cup X_{(1)}$.

Proposition 2.1. Suppose either $X \sim P_a(C) \times P_b(C)$, $1 \leqq b \leqq a < 2n \leqq a+b \leqq 4n-3$, or $X \sim P_a(H) \times P_b(C)$, $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a+b \leqq 4n-4$. Then $X_{(0)}$ is empty.

(Proof) Suppose that $X_{(0)}$ is non-empty. Let U be an invariant closed tubular neighbourhood of $X_{(0)}$ in X, and put $E = X - \operatorname{int} U$. Let $i : E \to X$ be the inclusion. Then $i^* : H^t(X) \to H^t(E)$ is an isomorphism for each $t \leqq 4n-2$, because the codimension of each connected component of $X_{(0)}$ is $4n$ by Lemma 1.2. Put $Y = E \cap F_{(1)}$. Then Y is a connected compact orientable manifold with non-empty boundary ∂Y, and $Sp(1)$ acts naturally on Y. There is a natural diffeomorphism $E = (S^{4n-1} \times Y)/Sp(1)$. By the Gysin sequence of the principal $Sp(1)$ bundle $p : S^{4n-1} \times Y \to E$, we obtain an exact sequence :

$$0 \to H^{2k-1}(S^{4n-1} \times Y) \to H^{2k-4}(E) \to H^{2k}(E) \to H^{2k}(S^{4n-1} \times Y) \to 0,$$

where $2k = \dim Y = \dim X - (4n-4)$. Hence we obtain $\operatorname{rank} H^{2k}(Y)$

- rank $H^{2k-1}(Y) \geqq 1$, by the cohomology ring strucure of X. Considering the homology exact sequence of the pair $(Y, \partial Y)$ and the Poincare-Lefschetz duality, we obtain

$$\operatorname{rank} H_0(\partial Y) \leqq \operatorname{rank} H_0(Y) + \operatorname{rank} H^{2k-1}(Y) - \operatorname{rank} H^{2k}(Y) \leqq 0.$$

Therefore ∂Y is empty; this is a contradiction. q.e.d.

In the remaing of this section, we assume $X = X_{(1)} = (S^{4n-1} \times F_{(1)})/Sp(1)$, where $F_{(1)}$ is a closed connected orientable manifold with a natural $Sp(1)$ action.

Here we describe certain situations which appear in the proofs of the following Propositions. Let Y be a closed orientable $Sp(1)$ manifold such that $H^{odd}(Y) = 0$. Put $M = S^{4n-1} \times Y$, where $Sp(1)$ acts as right scalar multiplication on S^{4n-1}. Let T be a closed toral subgroup of $Sp(1)$. Consider the following commutative diagram :

$$(D\text{-}1) \qquad \begin{array}{ccc} M/T & \xrightarrow{p_1} & M/Sp(1) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ P_{2n-1}(C) & \xrightarrow{q} & P_{n-1}(H) \end{array}$$

where π_1, π_2 are projections of fibre bundles with Y as the fibre, and p_1, q are projections of 2-sphere bundles. Since $H^{odd}(Y) = 0$, we can apply the Leray-Hirsh theorem to the fibrations π_1, π_2. In particular, we see $H^{odd}(M/Sp(1)) = 0$. By the Gysin sequence of the principal $Sp(1)$ bundle $p : M \to M/Sp(1)$, we obtain an exact sequence :

$$(A_i) \quad 0 \to H^{2i-1}(M) \to H^{2i-4}(M/Sp(1)) \xrightarrow{\mu} H^{2i}(M/Sp(1)) \xrightarrow{p^*} H^{2i}(M) \to 0$$

for each i, where μ is the multiplication by $e(p)$, the Euler class.

We regard S^∞ as the inductive limit of S^{4N-1} on which T acts naturally. Let F denote the fixed point set of the restricted T action on Y. Consider the following commutative diagram :

$$\text{(D-2)} \qquad \begin{array}{ccc} H^r((S^\infty \times Y)/T) & \xrightarrow{j^*} & H^r(M/T) \\ \downarrow{\scriptstyle i_\infty^*} & & \downarrow{\scriptstyle i_1^*} \\ H^r((S^\infty/T) \times F) & \xrightarrow{j_F^*} & H^r(P_{2n-1}(C) \times F), \end{array}$$

where i_1, i_∞, j, j_F are natural inclusions. Since $H^{odd}(Y) = 0$, we see that (cf. [/])

(1) i_∞^* is injective, j^* is surjective and i_∞^* is surjective for $r > \dim Y$.

On the other hand, j_F^* is an isomorphism for $r \leqq 4n-2$, and hence

(2) i_1^* is injective for $r \leqq 4n-2$.

Also we prepare the following for later use. The proof is omitted.

Lemma 2.2. Let S be a closed connected smooth $Sp(1)$ manifold. Let F be the fixed point set of the restricted T action on S, where T is a closed toral subgroup of $Sp(1)$. Suppose that $\operatorname{codim} F = 2$ and F is not connected. Then there is an equivariant diffemorphism : $S = Sp(1)/T \times F_1$, where F_1 is a connected component of F.

2-A. Now we consider the case $X \sim P_a(H) \times P_b(C)$.

Proposition 2.3. Suppose $X \sim P_a(H) \times P_b(C)$, $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a+b \leqq 4n-4$. Then either $a = n-1$ and $F_{(1)} \sim P_b(C)$, or $b = 2n-1$ and $F_{(1)} \sim S^2 \times P_a(H)$.

(Proof) The cohomology ring is as follows.

$H^*(X) = \mathbb{Q}[u,v]/(u^{a+1}, v^{b+1})$; deg $u = 4$, deg $v = 2$.

We can express $e(p) = \alpha u + \beta v^2$; $\alpha, \beta \in \mathbb{Q}$, where $p : S^{4n-1} \times F_{(1)} \to X$ is the principal $Sp(1)$ bundle. By definition, the $Sp(1)$ bundle p is a pull-back of the canonical principal $Sp(1)$ bundle over $P_{n-1}(H)$, and hence $e(p)^n = 0$. Thus we obtain $\alpha\beta = 0$, by considering the term $u^a v^{2n-2a}$ in the expression of $e(p)^n$. On the other hand, we can prove $e(p) \neq 0$ by making use of the exact sequence (A_i). Moreover we see, from (A_i), that if $\beta = 0$ then $a = n-1$ and $F_{(1)} \sim P_b(C)$; if $\alpha = 0$ then $b = 2n-1$ and $F_{(1)} \sim S^2 \times P_a(H)$. q.e.d.

Now we consider the $Sp(1)$ action on $F_{(1)}$. Let T be a toral subgroup of $Sp(1)$. Denote by F the fixed point set of the restricted T action on $F_{(1)}$. Since $\chi(F_{(1)}) \neq 0$, we see that F is non-empty. We shall show the following.

Proposition 2.4. If $a = n-1$ and $F_{(1)} \sim P_b(C)$, then the $Sp(1)$ action on $F_{(1)}$ is trivial. If $b = 2n-1$ and $F_{(1)} \sim S^2 \times P_a(H)$, then $F \sim S^0 \times P_a(H)$ or $F \sim S^0 \times P_a(C)$. Moreover the induced homomorphism $i^* : H^2(F_{(1)}) \to H^2(F)$ is trivial.

(Proof) Put $Y = F_{(1)}$ in the diagram (D-1). Let $t \in H^2(P_{2n-1}(C))$ and $w \in H^4(P_{n-1}(H))$ be the canonical generators. Then $\pi_2^*(w) = e(p)$ by definition. We see that $e(p) = \alpha u$, $\alpha \neq 0$ or $e(p) \neq \beta v^2$, $\beta \neq 0$ in Proposition 2.3.

Suppose first $e(p) = \alpha u$. Then $a = n-1$ and $F_{(1)} \sim P_b(C)$. We can prove $M/T \sim P_{2n-1}(C) \times P_b(C)$, $b \leqq 2n-2$ by making use of the Leray-Hirsch theorem, and hence the T action on $F_{(1)} \sim P_b(C)$ is trivial (cf. [6], Proposition 3.3). Therefore the Sp(1) action on $F_{(1)}$ is trivial.

Suppose next $e(p) = \beta v^2$. Then $b = 2n-1$ and $F_{(1)} \sim S^2 \times P_a(H)$. Put $u_1 = p_1^*(u)$, $v_1 = p_1^*(v)$ and $t_1 = \pi_1^*(t)$. We can apply the Leray-Hirsch theorem to the bundles π_1, π_2 in the diagram (D-1), and we obtain

$$H^*(M/T) = Q\,[t_1, u_1, v_1]/(u_1^{a+1}, v_1^{2n}, t_1^2 - \beta v_1^2), \quad \beta \neq 0.$$

Consider the diagram (D-2) for $Y = F_{(1)}$. Let u_2, v_2 be homogeneous elements of $H^*((S^\infty \times F_{(1)})/T)$ such that $j^*(u_2) = u_1$ and $j^*(v_2) = v_1$. Let t be the canonical generator of $H^2(S^\infty/T) = H^2(P_{2n-1}(C))$. Then we can express $i_\infty^*(u_2) = t^2 \times f_0 + t \times f_1 + 1 \times f_2$, $i_\infty^*(v_2) = t \times g_0 + 1 \times g_1$, where f_k, g_k are elements of $H^{2k}(F)$. Since $j_F^* i_\infty^*(\beta v_2^2) = i_1^*(\beta v_1^2) = i_1^*(t_1^2) = j_F^*(t^2 \times 1)$, we obtain $g_0^2 = \beta^{-1}$ and $g_1 = 0$. Moreover we see that g_0 is not constant, and hence F is not connected. Since $j_F^* i_\infty^*(u_2^{a+1}) = 0$ and $a+1 \leqq n-1$, we obtain $f_0 = 0$ and hence $i_\infty^*(u_2) = t \times f_1 + 1 \times f_2$. Let F_1 (resp. F_2) be the union of connected components F_σ of F on which $g_0 | F_\sigma$ is positive (resp. negative). Then each element of $H^k((S^\infty \times F_s)/T)$ for $k > 4a+2$ is expressed as a polynomial of $t \times 1$ and $t \times (f_1 | F_s) + 1 \times (f_2 | F_s)$

with rational coefficients for $s = 1,2$, because $H^*((S^\infty \times F_{(1)})/T$ is generated by two elements u_2, v_2 as graded $H^*(S^\infty/T)$-algebra and i_∞^* is surjective for $k > 4a+2$. In particular, if $f_1|F_s \neq 0$, then we can express $t^{4a-1} \times (f_1|F_s) = \sum_j (c_j(t \times (f_1|F_s) + 1 \times (f_2|F_s))^j (t \times 1)^{4a-2j})$, for $c_j \in Q$. Then we obtain $c_0 = 0$, $c_1 = 1$ and $f_2|F_s = -c_2(f_1|F_s)^2$. Therefore

$$H^*(F_s) = Q[x_s]/(x_s^{a+1}) \ ; \ \deg x_s = 2 \ \text{ or } \ 4,$$

because $f_k^{a+1} = 0$ $(k = 1,2)$ and $\chi(F_1) + \chi(F_2) = \chi(F_{(1)}) = 2a$. If $F_s \sim P_a(H)$ for some s, then $F \sim S^0 \times P_a(H)$ by Lemma 2.2. Thus we obtain $F \sim S^0 \times P_a(H)$ or $F \sim S^0 \times P_a(C)$. Finally we shall show that $i^* : H^2(F_{(1)}) \to H^2(F)$ is trivial for the case $F \sim S^0 \times P_a(C)$. Consider the following commutative diagram :

$$\begin{array}{ccc} H^2(M/T) & \xrightarrow{k_1^*} & H^2(F_{(1)}) \\ \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i^*} \\ H^2(P_{2n-1}(C) \times F) & \xrightarrow{k_0^*} & H^2(F), \end{array} \qquad i, i_1, k_0, k_1 \ : \text{natural inclusions.}$$

We see that $k_1^*(v_1)$ generates $H^2(F_{(1)})$ and $i_1^*(v_1) = t \times g_0$, and hence $i^* k_1^*(v_1) = k_0^*(t \times g_0) = 0$. Thus $i^* : H^2(F_{(1)}) \to H^2(F)$ is trivial. q.e.d.

Suppose $F \sim S^0 \times P_a(H)$. Then by Lemma 2.2, there is an equivariant diffeomorphism : $F_{(1)} = Sp(1)/T \times Y_2$, where Y_2 is a connected component of F. Thus we obtain an equivariant diffeomorphism :

$$X = X_{(1)} = (S^{4n-1} \times F_{(1)})/Sp(1) = P_{2n-1}(C) \times Y_2.$$

Consequently we obtain the following.

Theorem 2.5. Let X be a closed orientable manifold with a non-trivial smooth $Sp(n)$ action. Suppose $n \geqq 7$, $X = X_{(0)} \cup X_{(1)}$ and $X \sim P_a(H) \times P_b(c)$; $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a+b \leqq 4n-4$. Then there are three cases :

(a) $a = n-1$ and X is equivariantly diffeomorphic to $P_{n-1}(H) \times Y_1$, where Y_1 is a closed orientable manifold such that $Y_1 \sim P_b(C)$,

(b) $b = 2n-1$ and X is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_2$, where Y_2 is a closed orientable manifold such that $Y_2 \sim P_a(H)$,

(c) $b = 2n-1$ and X is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/Sp(1)$, where Y_3 is a closed orientable $Sp(1)$ manifold such that $Y_3 \sim S^2 \times P_a(H)$, $F \sim S^0 \times P_a(C)$ and $i^* : H^2(Y_3) \to H^2(F)$ is trivial, where F denotes the fixed point set of the restricted T action on Y_3. Conversely, if Y_3 satisfies the above conditions, then $(S^{4n-1} \times Y_3)/Sp(1) \sim P_{2n-1}(C) \times P_a(H)$ for $a \leqq n-2$.

Remark. In the above theorem 2.5, it remains to prove the final statement in the case (c). But the proof is omitted here. To prove this, the condition that $i^* : H^2(Y_3) \to H^2(F)$ is trivial is not necessary for $a > 1$, but it can not be omitted for $a = 1$.

2-B. Next we consider the case $X \sim P_a(C) \times P_b(C)$. By the same way as in the case 2-A, we have the following Propositions. The proofs are omitted.

Proposition 2.6. Suppose $X \sim P_a(C) \times P_b(C)$, $1 \leqq b \leqq a < 2n \leqq a+b \leqq 4n-3$. Then $a = 2n-1$ and $F_{(1)} \sim S^2 \times P_b(C)$.

Proposition 2.7. $F \sim S^0 \times P_b(C)$.

Consequently we obtain the following.

Theorem 2.8. Let X be a closed orientable manifold with a non-trivial smooth $Sp(n)$ action. Suppose $n \geqq 7$, $X = X_{(0)} \cup X_{(1)}$ and $X \sim P_a(C) \times P_b(C)$, $1 \leqq b \leqq a < 2n \leqq a+b \leqq 4n-3$. Then $a = 2n-1$ and X is equivariantly diffeomorphic to $P_{2n-1}(C) \times Y_0$, where Y_0 is a closed orientable manifold such that $Y_0 \sim P_b(C)$.

3. Cohomologies of certain homogeneous spaces

In this section, we give the cohomologies of $V_{n,2}/G = Sp(n)/Sp(n-2) \times G$ for certain closed connected subgroups G of $Sp(n)$. The results are as follows. The actual proofs are omitted here (see [8]).

Lemma 3.1. $H^*(V_{n,2}/Sp(1) \times Sp(1)) = Q[u,v]/(u^n, \sum_i u^i v^{n-1-i})$, $\deg u = \deg v = 4$.

Lemma 3.2. $H^*(V_{n,2}/T^2) = Q[x,y]/(x^{2n}, \sum_i x^{2i} y^{2n-2-2i})$, $\deg x = \deg y = 2$.

Lemma 3.3. The graded algebra $H^*(V_{n,2}/Sp(2))$ is isomorphic to the subalgebra of $Q[u,v]/(u^n, \sum_i u^i v^{n-1-i})$, consisting of symmetric polynomials, where $\deg u = \deg v = 4$.

Lemma 3.4. The graded algebra $H^*(V_{n,2}/U(2))$ is isomorphic to the subalgebra of $Q[x,y]/(x^{2n}, \sum_i x^{2i} y^{2n-2-2i})$, consisting of symmetric polynomials, where $\deg x = \deg y = 2$.

Lemma 3.5. The graded algebra $H^*(V_{n,2}/U(1) \times Sp(1))$ is isomorphic to the subalgebra of $Q[x,y]/(x^{2n}, \sum_i x^{2i} y^{2n-2-2i})$ generated by x^2, y.

From these lemmas, we have

Propositon 3.6. Let G be one of T^2, $U(2)$ and $U(1) \times Sp(1)$. Let w_1, w_2 be any non-zero homogeneous elements of $H^*(V_{n,2}/G)$ such that $\deg w_k = 2k$. Then w_1^{2n-1} and w_2^{n-1} are non-zero elements.

4. Finish of the proof

Throughout this section, suppose that $n \geqq 7$ and X is a closed orientable manifold with a non-trivial smooth $Sp(n)$ action, and $X \sim P_a(C) \times P_b(C)$ for some a, b such that

$$1 \leqq b \leqq a < 2n \leqq a+b \leqq 4n-3,$$

or $X \sim P_c(H) \times P_d(C)$ for some c, d such that

$$1 \leqq c \leqq n-1,\ 1 \leqq d \leqq 2n-1 \text{ and } 2n \leqq 2c+d \leqq 4n-4.$$

Then, from the results in the section 3, we can show that $X_{(2)}$ and $X_{(3)}$ are empty sets. That is, we can prove the follwoing Propositions. We shall give here the outline of the proofs of those Propositions (see [8] for the details).

Proposition 4.1. $X \neq X_{(k)}$; $k = 2,3$.

(Outline of the proof) Suppose $X = X_{(k)}$. Then $X = (Sp(n)/Sp(n-k) \times F_{(k)})/Sp(k)$. In particular, we obtain $\chi(X) = {}_nC_k\chi(F_{(k)})$. From this fact, we see that $k \neq 3$ and the possibilities remain only in the following cases :

(a) $\dim F_{(2)} = 8$, $\chi(F_{(2)}) = 8$; $(a,b) = (2n-1,2n-3)$,

(b) $\dim F_{(2)} = 6$, $\chi(F_{(2)}) = 4$; $(c,d) = (n-1,2n-3)$, $(n-2,2n-1)$,

(c) $\dim F_{(2)} \leqq 4$.

If $\dim F_{(2)} \leqq 4$, then $X = V_{n,2}/\ Sp(1) \times Sp(1)$ or $V_{n,2}/Sp(2) \times F_{(2)}$, and hence (c) does not happen by Lemmas 3.1 and 3.3.

In the cases (a), (b), if the $Sp(2)$ action on $F_{(2)}$ is transitive, then $X = V_{n,2}/T^2$, $V_{n,2}/U(2)$ or $V_{n,2}/U(1) \times Sp(1)$, and hence such cases do not happen by Proposition 3.6. So the remainder of the possibilities is as follows : in each case below, $Sp(2)$ action on $F_{(2)}$ is not transitive,

(a)' the case (a) and the restricted G action on $F_{(2)}$ has a fixed point, where $G = U(2)$ or $U(1) \times Sp(1)$,

(b)' the case (b) and the $Sp(2)$ action on $F_{(2)}$ is trivial,

(c)' the case (b) and the $Sp(2)$ action on $F_{(2)}$ has no fixed point,

(d)' the case (b) and the $Sp(2)$ action on $F_{(2)}$ has a fixed point but this action is not trivial.

Consider the case (a)'. Then the natural projection π_1 : $(V_{n,2} \times F_{(2)})/G \to V_{n,2}/G$ has a cross section s, and we have the following commutative diagram :

$$\begin{array}{ccc} (V_{n,2} \times F_{(2)})/G & \underset{s}{\overset{\pi_1}{\rightleftarrows}} & V_{n,2}/G \\ \Big\downarrow q & & \Big\downarrow p \\ X = (V_{n,2} \times F_{(2)})/Sp(2) & \xrightarrow{\pi_2} & V_{n,2}/Sp(2), \end{array}$$

π_1, π_2, p, q : natural projections.

From this diagram and the cohomologies of $V_{n,2}/Sp(2)$ and $V_{n,2}/G$ (see Lemmas 3.1,3.4 and 3.5), we can see that if $X \sim P_a(C) \times P_b(C)$, then $p^*|H^4(V_{n,2}/Sp(2))$ is not injective.

This is a contradiction.

If $F_{(2)}$ is such as in the case (b)', then $X = V_{n,2}/Sp(2) \times F_{(2)}$, and hence such a case does not happen by Lemma 3.3.

If $F_{(2)}$ is such as in the case (c)', then we can see that the identity component of an isotropy subgroup is conjugate to $Sp(1) \times Sp(1)$ and that the fixed point set F of the restricted $Sp(1) \times Sp(1)$ action on $F_{(2)}$ is a closed orientable surface with $\chi(F) = 4$ and F has at most two components. Therefore $X = (V_{n,2}/Sp(1) \times Sp(1)) \times S^2$, and hence such a case does not happen by Lemma 3.1.

Finally consider the case (d)'. Then we see that the fixed point set F' of the $Sp(2)$ action is 1-dimensional, and the identity component of the other isotropy subgroup is conjugate to $Sp(1) \times Sp(1)$. Let U be a closed tubular neighbourhood of F', and let F'' be the fixed point set of the restricted $Sp(1) \times Sp(1)$ action on $F_{(2)}$ - int U. Then we see that F'' is a compact orientable surface with $\chi(F'') = 4$, F'' has at most two components and each component of F'' has a non-empty boundary. Such a case does not happen, because $\chi \leqq 1$ for each compact connected orientable surface with non-empty boundary.

Proposition 4.2. If $X_{(1)}$ is non-empty, then $X_{(2)}$ is empty.

(Outline of the proof) Suppose that both of $X_{(1)}, X_{(2)}$ are non-empty. Then $X = X_{(1)} \cup X_{(2)}$ and codim $F_{(1)} = 8n-8$

by Propositions 1.3, 1.4. Since $\dim X \leqq 8n-6$, we obtain $\dim F_{(1)} = 0$ or 2. Then we have the following possibilities :

(a) the $Sp(1)$ action on $F_{(1)}$ is non-trivial,

(b) the $Sp(1)$ action on $F_{(1)}$ is trivial, and

(b.1) $\dim F_{(1)} = 0$; $(a,b) = (2n-1,2n-3)$ or $(2n-2,2n-2)$, $(c,d) = (n-1,2n-2)$,

or (b.2) $\dim F_{(1)} = 2$; $(a,b) = (2n-1,2n-2)$.

For each case above, we first investigate the possibilities of the orbit types. And, from the results (in the section 3) about the cohomologies of such orbits, we deduce that those cases do not happen. For example, consider the case (a). Then $\dim F_{(1)} = 2$, and $X \sim P_{2n-1}(C) \times P_{2n-2}(C)$. Considering the slice representation at a point of $F_{(1)}$, we see that the $Sp(n)$ action on X has a codimension one orbit, and hence X is a union of closed invariant tubular neighbourhoods of just two non-principal orbits (cf. [4]). Calculating the Euler characteristics, we see that two non-principal orbits are $P_{2n-1}(C)$ and $V_{n,2}/T^2$. Since $\operatorname{codim} P_{2n-1}(C) = 4n-4$ in X, the inclusion $i : V_{n,2}/T^2 \to X$ induces an isomorphism $i^* : H^2(X) \to H^2(V_{n,2}/T^2)$, and hence $x^{2n-1} \neq 0$ for each non-zero element $x \in H^2(X)$ by Proposition 3.6. This is a contradiction.

Similarly, we can deduce a contradiction for the case (b.1). The proof for the case (b.2) is a little more complicated than that for the above two cases, but we omit it here.

Under the consideration of the section 1, we obtain the main theorem stated in Introduction, by combining Theorems 2.5, 2.8 and Propositions 4.1, 4.2. The full proofs of the results in this paper will appear in [8].

References

[1] G.E.Bredon : The cohomology ring structure of a fixed point set, Ann. of Math. 80(1964), 524-537.

[2] E.B.Dynkin : The maximal subgroups of the classical groups, A.M.S. Translations 6(1957), 245-378.

[3] W.C.Hsiang and W.Y.Hsiang : Differentiable actions of compact connected classical groups I, Amer.J.Math. 89(1967), 705-786.

[4] F.Uchida : Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits, Japan.J.Math. 3(1977), 141-189.

[5] F.Uchida : Real analytic SL(n,R) actions on spheres, Tôhoku Math.J. 33(1981), 145-175.

[6] F.Uchida : Actions of special unitary groups on a product of complex projective spaces, Osaka J.Math. to appear

[7] F.Uchida : Actions of symplectic groups on a product of quarternion projective spaces, to appear.

[8] F.Uchida and A.Nakanishi : Actions of symplectic groups on certain manifolds, Tôhoku Math.J. to appear.