数理解析研究所講究録
501巻 1983年 99-117

Finite coverings of punctured torus bundles and the first Betti number

by

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1. Introduction

A compact orientable 3-manifold is called a Haken manifold if it is irreducible and it contains a two-sided incompressible surface. Through the works of Haken[3], Waldhausen[10], Thurston [7] and many others, this property has proved to be one of the most powerful hypotheses in 3-manifold theory. However there are many 3-manifolds which do not satisfy this hypothesis. In fact there are certain Seifert fibered spaces which are irreducible but not Haken (cf. Waldhausen [9] ; these had been the only known examples of such manifolds before [7]) and recently Thurston [7] has shown that most 3-manifolds obtained by performing Dehn surgeries on S^3 along the figure eight knot have the same property. Moreover he has shown that they admit hyperbolic structures so that they cannot be Seifert fibered spaces. Thus they are essentially new examples of non-Haken manifolds. There are generalizations of this result to 2-bridge knots by Hatcher-Thurston [5], 2-bridge links by Floyd-Hatcher [3] and to punctured torus bundles by Floyd-Hatcher [2], Culler-Jaco-Rubinstein [1]. On the other hand Waldhausen [9] conjectured that any orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken, namely some finite covering space of it is Haken and Thurston has proposed a related problem to determine whether every aspherical or hyperbolic 3-manifold is virtually Haken or more strongly whether some finite covering space of such a manifold has positive first Betti number (Hempel [6] has expressed this property as the fundamental group of the manifold is virtually representable to $\mathbb{Z}$). Thurston has even asked if every hyperbolic 3-manifold has a finite covering space which fibers over the circle (cf. Problems 16-18 of [8]). The purpose of this paper is to examine the above questions for the 3-manifolds which are obtained by performing Dehn surgeries along a section in torus bundles over the circle with hyperbolic monodromy. More precisely, assuming a certain condition on the monodromy, we shall construct explicit finite coverings of such manifolds with positive first Betti number. In particular we prove that the fundamental group of the 3-manifold $M(\mu,\lambda)$ which is obtained by performing (μ,λ) Dehn surgery on S^3 along the figure eight knot (cf. [7]) is virtually representable to $\mathbb{Z}$ if $\mu \equiv 0 \pmod 4$ and $\lambda + \frac{1}{4}\mu \equiv 0 \pmod 3$. In fact it has a 240-fold covering with the first Betti number at least 2. Our method does not apply to the general problem as it stands. However we hope that our result might serve as supporting evidence towards an affirmative solution of the problem.

2. Finite coverings of punctured torus bundles

Let f be a homeomorphism of the punctured torus $T_0 = T^2 - \mathring{D}^2$ which is the identity near the boundary and let E_f be the mapping torus of f. f extends naturally to a homeomorphism of the torus T^2 (which we denote by the same letter) by using the identity mapping on D^2. The corresponding mapping torus M_f

is a closed orientable 3-manifold fibering over the circle with characteristic homeomorphism (= monodromy) f. Let $[f] \in SL_2\mathbb{Z}$ be the matrix representing the action of f on the first homology group $H_1(T^2)$ (homology is always assumed to be with integer coefficients) with respect to the natural generators of it. We call it the monodromy matrix of E_f and also of M_f. The oriented homeomorphism class of E_f or M_f depends only on the conjugacy class of the monodromy matrix $[f]$. Since we have assumed that f fixes the disk $D^2 \subset T^2$, a tubular neighborhood of the "zero-section" of M_f is identified with $D^2 \times S^1$. We denote $E_f(\mu,\lambda)$ for the closed 3-manifold obtained by performing (μ,λ) Dehn surgery on M_f along the zero-section. Thus we have $E_f(1,0) = M_f$ in particular. Hereafter we assume that the monodromy matrix $[f]$ is hyperbolic, namely $[f]$ is assumed to have two distinct real eigenvalues. In this case it is easy to show that $E_f(\mu,\lambda)$ is a rational homology 3-sphere except for the trivial case $E_f(1,0)$. Thurston [7] has proved that $E_f(\mu,\lambda)$ admits a hyperbolic structure if we exclude a finite set of choices for (μ,λ). Floyd-Hatcher [2] and Culler-Jaco-Rubinstein [1] have classified all incompressible surfaces in E_f and concluded that $E_f(\mu,\lambda)$ is an irreducible non-Haken 3-manifold except for finite set of (μ,λ).

Now let $\tilde{T}_0$ be a regular finite covering space of T_0 determined by a surjective homomorphism $\pi_1(T_0) \longrightarrow G$, where G is a finite group. $\pi_1(T_0)$ is a free group generated by two elements α,β. Here we suppose α, β to be the standard generators so that the commutator $[\alpha,\beta]$ corresponds to the boundary circle, (see Figure 1, α and β are represented by the paths which consist of the simple closed curves x and y together with the curves which connect them to the base point b_0).

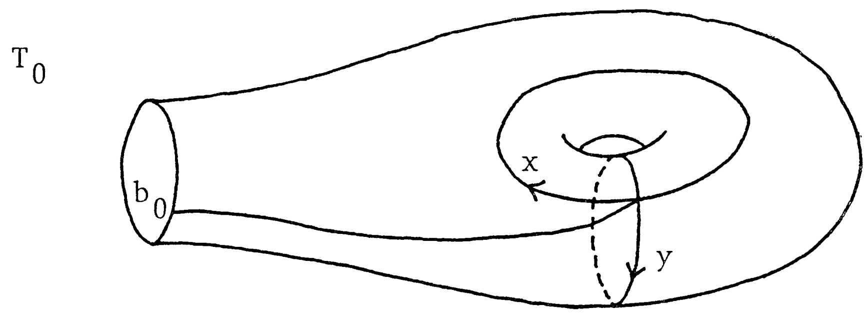

Figure 1. The generators of $\pi_1(T_0)$

Now let a, b be the images of α, β in G and let n and m be the orders of G and $[a,b]$ respectively.

2.1. Lemma. Let g, c, b_1 be the genus of the surface $\tilde{T}_0$, the number of the connected components of $\partial\tilde{T}_0$ and the first Betti number of $\tilde{T}_0$, respectively. Then we have

$$g = \frac{1}{2}\left(n-\frac{n}{m}\right) + 1$$

$$c = \frac{n}{m}$$

$$b_1 = n+1.$$

The proof is easy and we omit it.

Now it is easy to see that some power of the homeomorphism $f\colon T_0 \longrightarrow T_0$ leaves $\pi_1(\tilde{T}_0)$ invariant. Hence there is a natural number ℓ and a homeomorphism $\tilde{f}\colon \tilde{T}_0 \longrightarrow \tilde{T}_0$ so that the following diagram commutes:

$$\begin{array}{ccc} \tilde{T}_0 & \xrightarrow{\tilde{f}} & \tilde{T}_0 \\ \downarrow & & \downarrow \\ T_0 & \xrightarrow[f^{\ell}]{} & T_0 \end{array} .$$

Here we always assume that $\tilde{f}$ is the identity near the boundary of $\tilde{T}_0$. This is possible if we change ℓ if necessary. Let $E_{\tilde{f}}$ be the mapping torus of $\tilde{f}$. Then the natural map $E_{\tilde{f}} \longrightarrow E_{f^\ell}$ defines an n-fold covering projection. If we compose another natural map $E_{f^\ell} \longrightarrow E_f$ to this projection, we obtain an $n\ell$-fold covering projection $E_{\tilde{f}} \longrightarrow E_f$. The boundary tori of $E_{\tilde{f}}$ are "(m,ℓ)- fold covering spaces" of the boundary torus of E_f.

2.2. Lemma. The covering space $E_{\tilde{f}} \longrightarrow E_f$ extends to a covering space over $E_f(\mu,\lambda)$ if and only if $\mu \equiv 0 \pmod{m}$ and $\lambda \equiv 0 \pmod{\ell}$.

Proof. As mentioned before the Lemma, the restriction of the covering projection $E_{\tilde{f}} \longrightarrow E_f$ to any boundary component of $E_{\tilde{f}}$ is canonically isomorphic to the "(m,ℓ)-fold covering projection" $p_{m,\ell}: T^2 \longrightarrow T^2$, where $p_{m,\ell}(s,t) = (ms,\ell t)$ $((s,t) \in T^2)$. In $E_f(\mu,\lambda)$ a simple closed curve on $\partial E_f = T^2$ representing the homology class $(\mu,\lambda) \in H_1(T^2)$ bounds a disk. Therefore if the covering projection $E_{\tilde{f}} \longrightarrow E_f$ extends to $E_f(\mu,\lambda)$, this simple closed curve must lift to a closed curve in the upper T^2 under the projection $p_{m,\ell}$. It follows that $\mu \equiv 0 \pmod{m}$ and $\lambda \equiv 0 \pmod{\ell}$. Conversely if this condition is satisfied, then a simple calculation shows that the (μ,λ) Dehn surgery on the lower T^2 lifts to the $(\frac{\mu}{m},\frac{\lambda}{\ell})$ Dehn surgery on the upper T^2. This implies that the projection $E_{\tilde{f}} \longrightarrow E_f$ extends to a covering projection $E_{\tilde{f}}(\frac{\mu}{m},\frac{\lambda}{\ell}) \longrightarrow E_f(\mu,\lambda)$, where $E_{\tilde{f}}(\frac{\mu}{m},\frac{\lambda}{\ell})$ is the closed 3-manifold obtained by performing "$(\frac{\mu}{m},\frac{\lambda}{\ell})$ Dehn surgery" on each boundary component of $E_{\tilde{f}}$ simultaneously.

Recall that $\tilde{f}$ is a homeomorphism of $\tilde{T}_0$. Let

$$k = \text{rank Ker}(\tilde{f}_*\text{-Id}): H_1(\tilde{T}_0) \longrightarrow H_1(\tilde{T}_0).$$

2.3. Lemma. The first Betti number b_1 of $E_{\tilde{f}}(\frac{\mu}{m},\frac{\lambda}{\ell})$ is not less than $k\text{-}c\text{+}1$, where c is the number of connected components of $\partial\tilde{T}_0$ (see Lemma 2.1).

Proof. Consider the Wang exact sequence for the fibration $\tilde{T}_0 \to E_{\tilde{f}} \to S^1$:

$$0 \to H_2(E_{\tilde{f}}) \to H_1(\tilde{T}_0) \xrightarrow{\tilde{f}_*\text{-Id}} H_1(\tilde{T}_0) \to H_1(E_{\tilde{f}}) \to H_0(\tilde{T}_0) \to 0.$$

From this we conclude

$$\text{rank of } H_2(E_{\tilde{f}}) = k.$$

The Mayer-Vietoris exact sequence for the triple $(E_{\tilde{f}}(\frac{\mu}{m},\frac{\lambda}{\ell}); E_{\tilde{f}}, \bigcup_{i=1}^{c} (D^2\times S^1)_i)$ is:

$$0 \to H_3(E_{\tilde{f}}(\tfrac{\mu}{m},\tfrac{\lambda}{\ell})) \to H_2(\bigcup_{i=1}^{c}(S^1\times S^1)_i) \longrightarrow$$
$$H_2(E_{\tilde{f}}) \oplus H_2(\bigcup_{i=1}^{c}(D^2\times S^1)_i) \to H_2(E_{\tilde{f}}(\tfrac{\mu}{m},\tfrac{\lambda}{\ell})) \to \cdots .$$

From this we obtain

$$b_1 = \text{rank } H_2(E_{\tilde{f}}(\tfrac{\mu}{m},\tfrac{\lambda}{\ell})) \geq k\text{-}c\text{+}1.$$

3. A special case

In this section we consider a special case which should clarify the key point of our construction. Let D_x and D_y be the left handed Dehn twists about the curves x and y on the punctured torus T_0, respectively. The associated monodromy matrices are

$$[D_x] = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$[D_y] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

3.1. Proposition. Let $f = D_x^2 \circ D_y^3$ (hence $[f] = \begin{pmatrix} -5 & -2 \\ 3 & 1 \end{pmatrix}$). Then $E_f(\mu,\lambda)$ has a 6-fold covering space with the first Betti number at least 2 provided $\mu \equiv 0 \pmod 3$.

Proof. Let

$$\rho\colon \pi_1(T_0) \longrightarrow S_3$$

be the homomorphism defined by

$$\rho(\alpha) = (12)$$

$$\rho(\beta) = (123)$$

where as before α (resp. β) is the element in $\pi_1(T_0)$ corresponding to the closed curve x (resp. y) and S_3 is the symmetric group of degree 3. Then $[\rho(\alpha),\rho(\beta)] = (123)$ has order 3 so that the corresponding 6-fold covering space $\tilde{T}_0$ has genus 3 and its boundary has 2 connected components (cf.

Lemma 2.1). On $\tilde{T}_0$, let x_1, x_2, x_3 and y_1, y_2 be the lifts of x and y, respectively (see Figure 2)

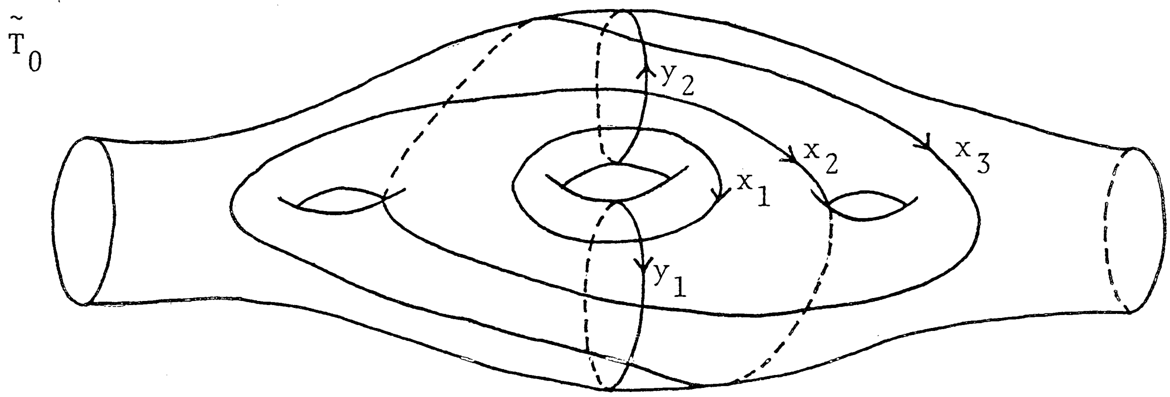

Figure 2

It is easy to see that D_x^2 lifts to a homeomorphism $\tilde{D}_x$ of $\tilde{T}_0$, which is the simultaneous left handed Dehn twists about the three curves x_1, x_2, x_3. Similarly D_y^3 lifts to a homeomorphism $\tilde{D}_y$ which is the left handed Dehn twists about the curves y_1, y_2. Put

$$\tilde{f} = \tilde{D}_x \circ \tilde{D}_y .$$

Then clearly $\tilde{f}$ is a lift of f and it is the identity near $\partial\tilde{T}_0$. Consider the following three homology classes

$$[x_1] - [x_2]$$
$$[x_2] - [x_3]$$
$$[y_1] - [y_2]$$

in $H_1(\tilde{T}_0)$. It is easy to see that these homology classes are left invariant under the action of $\tilde{D}_x$ and $\tilde{D}_y$. Hence they are invariant cycles under the homeomorphism $\tilde{f}$. In fact simple calculations show that the number $k = \text{rank Ker}(\tilde{f}_* - \text{Id}) : H_1(\tilde{T}_0) \longrightarrow H_1(\tilde{T}_0)$ is equal to 3. Therefore by Lemma 2.3 we conclude that the first Betti number of $E_{\tilde{f}}(\frac{\mu}{3}, \lambda)$, which is a 6-fold covering space of $E_f(\tilde{\mu}, \lambda)$, is not less than 2. In fact further calculations show that it is equal to 2 unless $(\mu, \lambda) = (0,1)$ in which case it is equal to 3.

4. The case of the figure eight knot complement

In this section we prove

4.1. Theorem. Let $M(\mu, \lambda)$ be the closed 3-manifold obtained by performing (μ, λ) Dehn surgery on S^3 along the figure eight knot (see [7]). Then $\pi_1(M(\mu, \lambda))$ is virtually representable to $\mathbb{Z}$ if $\mu \equiv 0 \pmod 4$ and $\lambda + \frac{1}{4}\mu \equiv 0 \pmod 3$. In fact there is a 240-fold covering space of $M(\mu, \lambda)$ with first Betti number at least 2.

4.2. Remark. Thurston [7] has shown that the manifold $M(4,1) = M(4,-1)$ is Haken and Culler-Jaco-Rubinstein [1] proved that $M(16,1) = M(16,-1)$ is virtually Haken. By the above theorem we know that some finite covering spaces of the manifolds $M(4,-1)$ and $M(16,-1)$ have positive first Betti number.

As is well known, the figure eight knot complement can be considered as a punctured torus bundle over the circle with monodromy matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. More precisely let

$$f = D_x^{-1} \circ D_y \quad \left(\text{hence} \quad [f] = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

be a homeomorphism of T_0. Then it is easy to prove

4.3. Lemma. $M(\mu,\lambda)$ is homeomorphic to $E_f(\lambda,\mu)$ for all (μ,λ).

The matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is conjugate to $\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

and we have

$$\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

We make use of this expression to prove Theorem 4.1. To do this we use the following model for T_0. Let

$$r\colon \pi_1(T_0) \longrightarrow PSL_2\mathbb{R}$$

be the homomorphism defined by

$$r(\alpha) = \begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$r(\beta) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}.$$

Then r is an isomorphism of $\pi_1(T_0)$ onto a discrete subgroup of $PSL_2\mathbb{R}$. Hence we can identify $\operatorname{Int} T_0$ with $D/\operatorname{Im} r$, where $D = \{z \in \mathbb{C} \; ; \; |z| < 1\}$ is the Poincaré disc on which $PSL_2\mathbb{R}$ acts by isometries. A fundamental domain for the action of $\operatorname{Im} r$ on D and the loops x and y, which are now closed geodesics, are expressed in Figure 3.

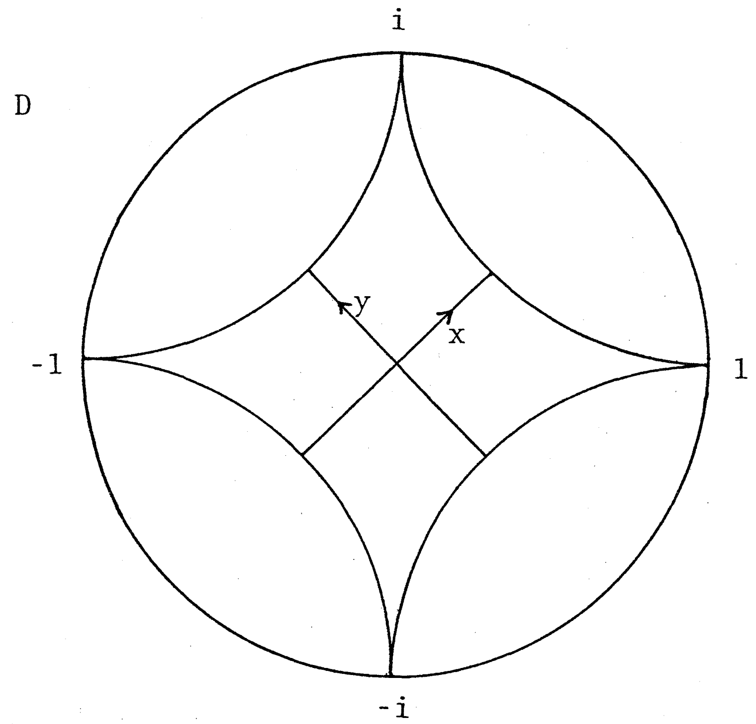

Figure 3.

Now let $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \in PSL_2\mathbb{R}$. It acts on D as a rotation by $90°$. We have

$$R\, r(\alpha)\, R^{-1} = r(\beta)$$

$$R\, r(\beta)\, R^{-1} = r(\alpha)^{-1}.$$

Hence R acts on $\mathrm{Int}\, T_0$ as an isometry of order 4. We can consider R as a homeomorphism of T_0 of order 4 such that its restriction to the boundary ∂T_0 is a rotation by $90°$ with respect to the orientation on ∂T_0 determined by the closed curve $[\alpha,\beta]$. We have

$$[R] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence if we put

$$g = R \circ D_y^{-3},$$

then $[g] = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$. Now g^4 is a homeomorphism of T_0 which is the identity on ∂T_0, because D_y is the identity on ∂T_0 and R has order 4. Since $[g^4]$ is conjugate to $[f]^4$, E_{g^4} is a 4-fold covering space of E_f. We prove

4.4. Lemma. $E_{g^4}(\mu,\lambda)$ is a 4-fold covering space of $E_f(\mu-\lambda, 4\lambda)$.

Proof. It is easy to see that $\pi_1(E_f)$ has the following presentation:

$$\pi_1(E_f) = \langle \alpha,\beta,\gamma;\ \gamma\alpha\gamma^{-1} = \alpha\beta\alpha,\ \gamma\beta\gamma^{-1} = \beta\alpha \rangle$$

where γ is represented by the closed curve $\{\text{base point}\} \times S^1 \subset E_f$. $\pi_1(\partial E_f) = \pi_1(T^2)$ is generated by $[\alpha,\beta]$ and γ. On the other hand $\pi_1(E_g)$ has the presentation:

$$\pi_1(E_g) = \langle \bar\alpha,\bar\beta,\bar\gamma;\ \bar\gamma\bar\alpha\bar\gamma^{-1} = \bar\alpha\bar\beta\bar\alpha^{2},\ \bar\gamma\bar\beta\bar\gamma^{-1} = \bar\alpha^{-1} \rangle$$

where $\bar\gamma$ is represented by the curve $\{\text{base point}\}\times[0,1]$ followed by rotation by $90°$ on ∂T_0. $\pi_1(\partial E_g)$ is generated by $[\bar\alpha,\bar\beta]$ and $\bar\gamma$. Now two homeomorphisms $f \circ D_x$ and $D_x \circ g$ of T_0 are isotopic. Hence we can define a homeomorphism of E_g to E_f which induces an isomorphism $\theta\colon \pi_1(E_g) \to \pi_1(E_f)$ such that

$$\theta(\bar\alpha) = \alpha$$
$$\theta(\bar\beta) = \beta\alpha^{-1} \qquad \text{and}$$
$$\theta(\bar\gamma) = \gamma.$$

It is easy to check that $\theta([\bar\alpha,\bar\beta]) = [\alpha,\beta]$ and θ induces a canonical isomorphism $\theta\colon \pi_1(\partial E_g) \cong \pi_1(\partial E_f)$. Now consider E_{f^4}

and E_g4. θ restricts to an isomorphism $\theta\colon \pi_1(\partial E_g4) \cong \pi_1(\partial E_f4)$ where $\pi_1(\partial E_g4)$ (resp. $\pi_1(\partial E_f4)$) is a free abelian group generated by $[\bar{\alpha},\bar{\beta}]$ and $\bar{\gamma}^4$ (resp. $[\alpha,\beta]$ and γ^4). Let $\bar{u}$, $\bar{v}$ (resp. u, v) be the elements of $H_1(\partial E_g4)$ (resp. $H_1(\partial E_f4)$) representing the "meridian" and "longitude" homology classes. Then clearly u (resp. v) is represented by $[\alpha,\beta]$ (resp. γ^4) and $\bar{u}$ is represented by $[\bar{\alpha},\bar{\beta}]$. However $\bar{v}$ is represented by $\bar{\gamma}^4\ [\bar{\alpha},\bar{\beta}]^{-1}$. Hence $\theta_*(\bar{u}) = u$ and $\theta_*(\bar{v}) = v-u$. It follows that $E_g4(\mu,\lambda)$ is homeomorphic to $E_f4(\mu-\lambda,\lambda)$, which is a 4-fold covering space of $E_f(\mu-\lambda,4\lambda)$. This completes the proof.

Proof of Theorem 4.1. First we wish to construct a regular finite covering space $\tilde{T}_0$ of T_0 such that the homeomorphisms D_y^{-3} and R of T_0 lift to $\tilde{T}_0$. In view of the equations

$$R\ r(\alpha)\ R^{-1} = r(\beta)$$
$$R\ r(\beta)\ R^{-1} = r(\alpha)^{-1}$$

it suffices to construct a surjective homomorphism

$$\rho\colon \pi_1(T_0) \longrightarrow G$$

where G is a finite group generated by two elements a and b such that (i) b has order 3 and (ii) $a \to b$, $b \to a^{-1}$ defines an automorphism of G. We define G as a subgroup of S_5 generated by the following two elements

$$a = (123)$$
$$b = (145).$$

It is easy to see that G is actually equal to A_5 (the alternating group of degree 5) and it satisfies the above conditions (i) and (ii). In fact (i) is clear and the inner automorphism of S_5 defined by (4253) restricts to the required automorphism of $G = A_5$. Let $\tilde{T}_0$ be the corresponding 60-fold covering space of T_0. Since $[a,b] = (153)$ has order 3, the genus of $\tilde{T}_0$ is equal to 21, the boundary $\partial\tilde{T}_0$ has 20 connected components and b_1 of $\tilde{T}_0$ is equal to 61 (cf. Lemma 2.1). Now let x_i (resp. y_j) $(i,j = 1,\ldots,20)$ be the connected components of the inverse images of x (resp. y) in $\tilde{T}_0$. Each projection $y_j \to y$ is a 3-fold covering space. Hence the homeomorphism D_y^{-3} of T_0 lifts to a homeomorphism $\tilde{D}_y^{-1}$ of $\tilde{T}_0$ which is the simultaneous right handed Dehn twists about the closed curves y_j $(j=1,\ldots,20)$. By the condition (ii) R lifts to a homeomorphism $\tilde{R}$ on $\tilde{T}_0$, which is an isometry of order 4 on Int $\tilde{T}_0$. Moreover $\tilde{R}$ sends the closed curve x_i (resp. y_j) to some $y_{i'}$ (resp. $-x_{j'}$) Now put

$$\tilde{g} = \tilde{R}\circ\tilde{D}_y^{-1}.$$

Clearly $\tilde{g}$ is a lift of g. Hence $\tilde{g}^4$ is a lift of g^4. Since $\tilde{D}_y^{-1}$ is the identity on $\partial\tilde{T}_0$ and $\tilde{R}$ has order 4, $\tilde{g}^4$ is the identity on $\partial\tilde{T}_0$. For any homology class $u \in H_1(\tilde{T}_0)$, we have

$$(\tilde{D}_y^{-1})_*(u) = u + \text{linear combination of } [y_j]\text{'s}$$

where $[y_j] \in H_1(\tilde{T}_0)$ is the homology class of y_j. On the other hand we have

$$\tilde{R}_*([x_i]) = [y_{i'}]$$
$$\tilde{R}_*([y_j]) = -[x_{j'}].$$

Hence we conclude

$$\tilde{g}_*^4(u) - u = \text{linear combination of } [x_i] \text{ and } [y_j]\text{'s}.$$

It follows that the rank of $\tilde{g}_*^4\text{-Id}\colon H_1(\tilde{T}_0) \longrightarrow H_1(\tilde{T}_0)$ cannot exceed 40. Hence $k = \text{rank Ker}(\tilde{g}_*^4\text{-Id})$ is not less than 21. Therefore b_1 of $E_{\tilde{g}}4(\mu,\lambda)$ is not less than 2 by Lemma 2.3. Now $E_{\tilde{g}}4(\mu,\lambda)$ is a 60-fold covering space of $E_g4(3\mu,\lambda)$, which in turn is a 4-fold covering space of $E_f(3\mu-\lambda,4\lambda)$ by Lemma 4.4. In view of Lemma 4.3, we can conclude that $E_{\tilde{g}}4(\mu,\lambda)$ is a 240-fold covering space of $M(4\lambda,3\mu-\lambda)$. This completes the proof.

5. Generalizations

It is clear that the arguments of the preceding two sections can be generalized in various ways. In this section we present some of them.

Let G be a finite group generated by two elements a and b. Let n = order of G, p = order of a, q = order of b and m = order of $[a,b]$. Consider the following condition on (G,a,b):

$$(*) \quad e(G,a,b) = n+2-\left(\frac{n}{p}+\frac{n}{q}\right)-\frac{n}{m} > 0.$$

5.1. Proposition. Suppose that there is a finite group G satisfying the condition $(*)$. Then for any homeomorphism f of T_0 which can be expressed as

$$f = D_x^{p_1} \circ D_y^{q_1} \circ \cdots \circ D_x^{p_d} \circ D_y^{q_d}$$

so that

$$\text{g.c.d.}(p_1,\dots,p_d) \equiv 0 \pmod p \quad \text{and}$$
$$\text{g.c.d.}(q_1,\dots,q_d) \equiv 0 \pmod q,$$

$\pi_1(E_f(\mu,\lambda))$ is virtually representable to $\mathbb{Z}$ if $\mu \equiv 0 \pmod{m}$.

Proof. Let $\pi\colon \tilde{T}_0 \longrightarrow T_0$ be the n-fold covering space of T_0 defined by the surjective homomorphism

$$\rho\colon \pi_1(T_0) \longrightarrow G$$

where $\rho(\alpha) = a$ and $\rho(\beta) = b$. By Lemma 2.1, the genus of $\tilde{T}_0$ is $\frac{1}{2}(n-\frac{n}{m})+1$, the number of connected components of $\partial\tilde{T}_0$ is $\frac{n}{m}$ and b_1 of $\tilde{T}_0$ is $n+1$. By the assumption, the homeomorphisms D_x^p and D_y^q lift to homeomorphisms $\hat{D}_x$ and $\hat{D}_y$ of $\tilde{T}_0$ which are simultaneous left handed Dehn twists about each connected components of $\pi^{-1}(x)$ and $\pi^{-1}(y)$, respectively. Let $\tilde{f}$ be the corresponding lift of f. Then as in the proof of Theorem 4.1, we have

$$k = \operatorname{rank} \operatorname{Ker}(\tilde{f}_*-\mathrm{Id})\colon H_1(\tilde{T}_0) \longrightarrow H_1(\tilde{T}_0) > n+1-(\frac{n}{p}+\frac{n}{q}).$$

Therefore if (*) is satisfied, then $\pi_1(E_f(\mu,\lambda))$ is virtually representable to $\mathbb{Z}$ if $\mu \equiv 0 \pmod{m}$ by Lemma 2.3.

There are many finite groups G with two generators a, b satisfying the condition (*). Here are several examples.

5.2. Examples. (I) The dihedral group $D_p = \langle a,b;\ a^p=b^2=1,\ bab^{-1} = a^{-1}\rangle$. $e(D_p,a,b) = \begin{cases} p-2 & (p \equiv 1 \pmod 2) \\ p-4 & (p \equiv 0 \pmod 2) \end{cases}$ so that (D_p,a,b) satisfies (*) if $p \neq 2, 4$.

(II) G = the subgroup of S_{p+1} generated by $a = (12...p)$ and $b = (23...p+1)$. $[a,b]$ has order 2 so that $e(G,a,b) = n(1-\frac{2}{p}-\frac{1}{2})+2 > 0$ if $p > 3$.

(III) G is the subgroup of S_p generated by $a = (12...p)$

and $b = (12\ldots q)$ $(q<p)$. $[a,b]$ has order 3 so that $e(G,a,b) = n(1-\frac{1}{p}-\frac{1}{q}-\frac{1}{3})+2 > 0$ if $q>2$.

(IV) $G = A_5$ generated by $a = (123)$ and $b = (145)$. In this case $p = q = m = 3$ so that $e(G,a,b) = 2$ (cf. Proof of Theorem 4.1).

Combining Proposition 5.1 and Examples 5.2, we can deduce

5.3. Theorem. Let $A \in SL_2\mathbb{Z}$ be expressed as

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{p_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{q_1} \cdots \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{p_d} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{q_d}$$

and let $p = \text{g.c.d.}(p_1,\ldots,p_d)$, $q = \text{g.c.d.}(q_1,\ldots,q_d)$. Assume that $p>1$, $q>1$ and p or $q>2$. Then among those 3-manifolds which are obtained by performing Dehn surgeries on the mapping torus M_A of A along the zero-section, there are infinitely many such manifolds whose fundamental groups are virtually representable to $\mathbb{Z}$.

Finally we generalize Theorem 4.1.

5.4. Theorem. Let $A \in SL_2\mathbb{Z}$ be expressed as

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{r_1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{p_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{q_1} \cdots \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{r_d} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{p_d} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{q_d}$$

and let $p = \text{g.c.d.}(p_1,\ldots,p_d)$, $q = \text{g.c.d.}(q_1,\ldots,q_d)$. Assume that $\text{g.c.d.}(p,q) > 2$. Then among those 3-manifolds which are obtained by performing Dehn surgeries on M_A along the zero-section, there are infinitely many such manifolds whose fundamental groups are virtually representable to $\mathbb{Z}$.

Proof can be given almost parallel to that of Theorem 4.1. It is only necessary to replace the alternating group A_5 by the following finite group. Consider a regular polygon K with h (=g.c.d.(p,q)) vertices which is placed on the zx-plane so that at least one vertex is on the z-axis and it is symmetric with respect to the z-axis. Let K' be the image of K under the rotation around the z-axis by $90°$. Let V be the set of vertices of K and K'. Now define G to be the subgroup of the group of permutations of V generated by the following two elements a and b. a permutes the vertices of K cyclically and it fixes the vertices outside K. b is defined to be the element which is conjugate to a by the rotation around the z-axis by $90°$. Then it is easy to check that both of a and b have order h, $[a,b]$ has order 3 so that $e(G,a,b) = n(1-\frac{2}{h}-\frac{1}{3})+2>0$. Moreover the correspondences $a \longrightarrow b$ and $b \longrightarrow a^{-1}$ define an automorphism of G.

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