

The generalized least-square estimate of autoregressive
coefficients

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The purpose of this note is to derive the asymptotic distribution of the generalized least-square estimate of autoregressive coefficients. Let $\{\varepsilon_t : t = 0, \pm 1, \pm 2, \dots\}$ be a real-valued stationary Gaussian process with mean 0 and suppose that the spectrum of the process is absolutely continuous with respect to the Lebesgue measure and has the spectral density $f_\varepsilon(\omega)$ ($-\pi \leq \omega \leq \pi$). Let $\{X_t\}$ be a stationary Gaussian process generated by the equation $X_t - \sum_{j=1}^p \alpha_j X_{t-j} = \varepsilon_t$, where the coefficients α_j are such that the zeroes of $z^p - \sum_{j=1}^p \alpha_j z^{p-j}$ are all inside the unit circle. Denote by f_X the spectral density of the process $\{X_t\}$ and denote by $f_{X\varepsilon}$ the cospectral density of the bivariate process $\{(X_t, \varepsilon_t)\}$; then it evidently holds that $f_X(\omega) = f_\varepsilon(\omega) / |1 - \sum \alpha_j e^{i\omega j}|^2$ a.e., and $f_{X\varepsilon}(\omega) = (1 - \sum \alpha_j e^{i\omega j}) f_X(\omega)$. For later use, write the covariances $E(X_t X_{t-s})$, $E(X_t \varepsilon_{t-s})$ and $E(\varepsilon_t \varepsilon_{t-s})$ respect

-ively as $\gamma_x(s)$, $\gamma_{x\epsilon}(s)$ and $\gamma_\epsilon(s)$.

If the spectral density f_ϵ is known, an estimate of the α_j can be obtained, based on observations $X_{t-p}, \dots, X_t, \dots, X_N$, as the value which minimizes the weighted square integral given as

$$\int_{-\pi}^{\pi} \left| \sum_{t=1}^N (X_t - \sum_{j=1}^p \hat{\alpha}_j X_{t-j}) e^{i\omega t} \right|^2 / f_\epsilon(\omega) d\omega.$$

Call the estimate $\hat{\alpha}_{j,N}$ thus obtained the generalized least-square estimate of the α_j . Suppose that the inverse of $f_\epsilon(\omega)$ is integrable and let $D(k)$ be the k -th Fourier coefficient of $1/f_\epsilon(\omega)$; namely, $D(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1/f_\epsilon(\omega) d\omega$. Let Γ_N be the p by p matrix whose (k,j) element is $\sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s)$ and let γ_N be the p -vector whose k -th component is $\sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_s D(t-s)$. Denote by α without suffix the p -vector whose k -th component is α_k . Define $\hat{\alpha}_N$ in the same way; that is, the j -th element of $\hat{\alpha}_N$ is $\hat{\alpha}_{j,N}$. Then it holds that $\Gamma_N \hat{\alpha}_N = \gamma_N$ and, if α^0 is the true value of α , $\Gamma_N (\hat{\alpha}_N - \alpha^0) = \gamma_N$ where γ_N is the p -vector with $\sum \sum X_{t-j} \epsilon_s D(t-s)$ in the j th element. Let Ω be the p by p matrix whose (k,j) element is $\frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - \sum \alpha_k e^{i\omega k}|^{-2} e^{i\omega(k-j)} d\omega$; then the following theorem establishes that the generalized least-square estimate $\hat{\alpha}_N$ is asymptotically normally distributed with covariance matrix Ω^{-1} ; namely, the asymptotic covariance of the generalized least-square estimate

is the same with that of the least-square estimate applied to the ordinary autoregressive process (that is, $\{\varepsilon_t\}$ is an independent process).

Theorem. Assume that

- i) f_ε and $1/f_\varepsilon$ are square-integrable and $1/f_\varepsilon$ has a bounded derivative with respect to w ,
- ii) $\sum_{j=0}^{\infty} |Y_\varepsilon(j)| < \infty$, and $\sum_{j=0}^{\infty} |D(j)| < \infty$
- iii) The process $\{\varepsilon_t\}$ is uniform mixing; namely, if $B(t \leq p)$ and $B(t \geq q)$ are the Borel fields determined by $\{\varepsilon_t; t \leq p\}$ and $\{\varepsilon_t; t \geq q\}$ respectively, there exists a sequence of positive numbers g_n such that $g_n \rightarrow 0$ as $|n| \rightarrow \infty$ and $|\Pr(E \cap F) - \Pr(E)\Pr(F)| < g_{p-q}$, where $E \in B(t \leq p)$ and $F \in B(t \geq q)$. Then under assumptions i), ii), iii), $\sqrt{N}(\hat{\alpha}_N - \alpha^0)$ is asymptotically normally distributed with zero mean vector and with covariance matrix Ω^{-1} .

Proof.

Define $F_{k,N}(l)$ as $F_{k,N}(l) = \sum_{m=1}^{N-l} X_{m+l-k} \varepsilon_m$ for $l \geq 0$;
 $F_{k,N}(l) = \sum_{m=|l|+1}^N X_{m+l-k} \varepsilon_m$ for $l < 0$. Moreover let
 $\xi_N(k) = \sum_{l=-N+1}^{N-1} \{F_{k,N}(l) - E(F_{k,N}(l))\} D(l) / \sqrt{N}$,
 $\xi_{N,L}(k) = \sum_{l=-L}^L \{F_{k,N}(l) - E(F_{k,N}(l))\} D(l) / \sqrt{N}$ and $\xi_N^*(k) = \xi_N(k) - \xi_{N,L}(k)$. Observe first of all for a fixed positive integer that by assumptions i) and iii) the statistics $\{F_{k,N}(l) - E(F_{k,N}(l))\} / \sqrt{N}$

$(l = 0, \pm 1, \dots, \pm L; k = 1, \dots, p)$ are asymptotically jointly normally distributed with covariances

$$\begin{aligned} C_{l,m}(k,j) &= \lim_{N \rightarrow \infty} E \left\{ (F_{k,N}(l) - E(F_{k,N}(l)))(F_{j,N}(m) - E(F_{j,N}(m))) \right\} / N \\ &= \sum_{u=-\infty}^{\infty} \left\{ r_X(u) r_{\varepsilon}(u+l+m-k-j) + r_{X\varepsilon}(u+l-k) r_{\varepsilon X}(u-m+j) \right\} \end{aligned}$$

where the last expression above is finite by assumption ii [cf. Hannan (1970), p 209 and 228]. Accordingly, $\xi_{N,L}(k)$, $k = 1, 2, \dots, p$, are asymptotically jointly normally distributed with mean 0 and with covariance matrix whose (k,j) element $C_L(k,j)$ is given as $C_L(k,j) = \sum_{l=-L}^L \sum_{m=-L}^L D(l) D(m) C_{l,m}(k,j)$. Now,

$$\begin{aligned} \lim_{L \rightarrow \infty} C_L(k,j) &= \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} D(l) D(m) \left\{ \sum_{u=-\infty}^{\infty} r_X(u) r_{\varepsilon}(u+l-m-k+j) \right. \\ &\quad \left. + r_{X\varepsilon}(u+m-j) r_{\varepsilon X}(u-l+k) \right\} \end{aligned}$$

where the right-hand side converges absolutely. Repeated applications of the Parseval equality lead to the equation

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} D(l) D(m) \sum_{u=-\infty}^{\infty} r_X(u) r_{\varepsilon}(u+l-m-k+j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_X(\omega)}{f_{\varepsilon}(\omega)} e^{i(k-j)\omega} d\omega,$$

whereas, since $\int_{-\pi}^{\pi} e^{il\omega} f_{X\varepsilon} f_{\varepsilon}^{-1} d\omega = 0$ for $l < 0$,

$$\begin{aligned} &\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} D(l) D(m) r_{X\varepsilon}(u+m-j) r_{\varepsilon X}(u-l+k) \\ &= \sum_{u=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i(u+k)} \frac{f_{X\varepsilon}(\omega)}{f_{\varepsilon}(\omega)} d\omega \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(u-j)} \frac{f_{X\varepsilon}(\omega)}{f_{\varepsilon}(\omega)} d\omega = 0. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\lim_{L \rightarrow \infty} C_L(k, j) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\omega) f_{\varepsilon}^{-1}(\omega) e^{(k-j)\omega} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \sum_{\ell=1}^p \alpha_\ell e^{i\omega\ell} \right|^{-2} e^{(k-j)\omega} d\omega.\end{aligned}$$

Secondly, an upper bound of the absolute mean of $\xi_{N,L}^*(k)$ is evaluated as follows :

$$E|\xi_{N,L}^*(k)| \leq \sum_{L+1 \leq |\ell| \leq N-1} |D(\ell)| \left[E\{F_{k,N}(\ell) - E(F_{k,N}(\ell))\}^2/N \right]^{\frac{1}{2}},$$

whereas, for a certain positive number S, it holds uniformly that

$$E\{F_{k,N}(\ell) - E(F_{k,N}(\ell))\}^2/N < S.$$

Therefore $E|\xi_{N,L}^*(k)| \leq 2S \sum_{\ell=L+1}^{\infty} |D(\ell)|$ uniformly in N and k.

By use of Chebychev's inequality for the first-order absolute moment, it follows from Assumption ii) that there exists a L_0 such that for $L > L_0$ $Pr\{|\xi_{L,N}^*| > \delta\} < \varepsilon$ for all $N (\geq L)$. Then the limit theorem given by T.W. Anderson (1971) says that the asymptotic distribution of $\tilde{\xi}_N = \xi_{L,N} + \xi_{L,N}^*$ is multivariate normal distribution with zero mean vector and with covariance matrix Ω , where

$\tilde{\xi}_N$, $\xi_{L,N}$ and $\xi_{L,N}^*$ are p-vectors whose k-th elements are $\tilde{\xi}_N(k)$, $\xi_{L,N}(k)$ and $\xi_{L,N}^*(k)$ respectively.

The convergence of $\frac{1}{N}\Gamma_N$ to Ω is shown as this. Observe that $E \frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s) = \sum_{\ell=-N+1}^{N-1} \left(1 - \frac{|k|}{N}\right) \gamma_{\ell+k-j} D(\ell).$

Then,

$$\lim_{N \rightarrow \infty} E \frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N X_{t-k} X_{s-j} D(t-s) = \sum_{\ell=-\infty}^{\infty} \gamma_{\ell+k-j} D(\ell)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_X(\omega) f_{\Sigma}(\omega)^{-1} e^{i(k-j)\omega} d\omega.$$

On the other hands, it is straightforward to see that

$$\frac{1}{N} \sum_{t=1}^N \sum_{s=1}^N \{ X_{t-k} X_{s-j} - E(X_{t-k} X_{s-j}) \} D(t-s)$$

converges in mean-square to 0. Thus $\frac{1}{N} \Gamma_N$ converges in probability to Ω .

Finally the convergence of $\frac{1}{\sqrt{N}} E \left\{ \sum_{\ell=-N+1}^{N-1} F_{k,N}(\ell) D(\ell) \right\}$ to 0 is demonstrated as follows. By the application of the Grenander-Rosenblatt theorem (1953, pp 543-544) after its slight modification

$$\frac{1}{\sqrt{N}} \left| E \left(\sum F_{k,N}(\ell) D(\ell) \right) - \sum_{-\infty}^{\infty} Y_{X,\epsilon}(k-\ell) D(\ell) \right| = O\left(\frac{\log N}{\sqrt{N}}\right).$$

On the other hand,

$$\sum_{-\infty}^{\infty} Y_{X,\epsilon}(k-\ell) D(\ell) = 0$$

$$\text{since } \sum_{-\infty}^{\infty} Y_{X,\epsilon}(\ell-k) D(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\omega} \frac{1 - \sum \alpha_j e^{ij\omega}}{|1 - \sum \alpha_j e^{ij\omega}|^2} d\omega = 0.$$

Consequently, $\frac{1}{\sqrt{N}} \sum (F_{k,N}(\ell) - E(F_{k,N}(\ell))) D(\ell)$ is asymptotically distributed in the same way as $\frac{1}{\sqrt{N}} \sum F_{k,N}(\ell) D(\ell)$. To summarize, $\sqrt{N} (\hat{\alpha}_N - \alpha_0)$ is asymptotically distributed as $(\frac{1}{N} \Gamma_N)^{-1} \xi_N$, while $\frac{1}{N} \Gamma_N \rightarrow \Omega$ in probability and ξ_N is asymptotically normal with zero mean vector and with covariance Ω . Thus the proof is complete.

References

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