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On blowing up algebraic cocycles

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§1. Blowing up algebraic cocycles

Let $f: \tilde{X} \to X$ be the blowing up of a non-singular algebraic variety X with non-singular center Y of codimension r. Then as for cohomology groups of $\tilde{X}$ the following proposition is known.

Proposition 1. We have an isomorphism

$$\Phi : H^s(\tilde{X}, Z) \simeq H^s(X, Z) \oplus H^{s-2}(Y, Z) \oplus \cdots \oplus H^{\nu}(Y, Z),$$

where Z is the ring of integers and

$$\nu = \max\left(s-2r+2,\ \tfrac{1}{2}(1+(-1)^{s-1})\right).$$

Here the isomorphism is given as follows: Letting $\tilde{Y} = f^{-1}(Y)$ be the inverse image of Y, $j: \tilde{Y} \hookrightarrow \tilde{X}$ and $g: \tilde{Y} \to Y$ the injection map and the restriction of f to $\tilde{Y}$ respectively, and $\hat{y}$ the cohomology class dual to the cycle $\tilde{Y}$, we define the homomorphism Φ^{-1} from $H^s(X, Z) \oplus \sum_{\alpha \geq 1} H^{s-2\alpha}(Y, Z)$ to $H^s(\tilde{X}, Z)$ by

$$\Phi^{-1}\left(x + \sum_{\alpha \geq 1} u_{s-2\alpha}\right) = f^*(x) + \sum j_*(g^*(u_{s-2\alpha}))\cdot \tilde{y}^{\alpha-1},$$

where j_* is the Gysin homomorphism induced from the injection j.

Now we consider an algebraic cocycle z dual to an irreducible subvariety Z of X of codimension s which contains Y properly. Let $\tilde{Z}$ be the strict transform of Z by the blowing up f and $\tilde{z}$ the cohomology class dual to $\tilde{Z}$. Then we have

Proposition 2. Every component of $\Phi(\tilde{z})$ is an algebraic cocycle.

A proof of this proposition is given as follows. Let $\nu(Y, X)$ be the normal bundle of Y in X. Then $g : \tilde{Y} \to Y$ is a fibre bundle associated to $\nu(Y, X)$ with the projective space $\mathbf{P}^{r-1}$ as fibre and hence there is a canonically defined complex line bundle L on $\tilde{Y}$ such that its restriction to a fibre is associated to a hyperplane of $\mathbb{P}^{r-1}$! We denote by $h = c_1(L)$ the first Chern class of L. Putting

$$\Phi(\tilde{z}) = f^*(z) + \sum j_*(g^*(u_{2(s-\alpha)}))\cdot \tilde{y}^{\alpha-1},$$

we apply the homomorphism j^* on both hands. Since $j^*(\tilde{z})$ is a linear combination of duals of irreducible components of $\tilde{Z}\cap\tilde{Y}$ and
$j^*(j_*(g^*(u_{2(s-\alpha)}))\cdot \tilde{y}^{\alpha-1}) = g^*(u_{2(s-\alpha)})\cdot(-h)^{\alpha}$, Grothendieck [1] yields analycity of $u_{2(s-\alpha)}$. It is easily seen that $x = z$, which ends the proof.

When Z is non-singular, we have the following

Proposition 3.

$$\tilde{z} \equiv f^*(z) + \sum_{\alpha\geq 1}(-1)^{\alpha} j_*(g^*(c_{s-\alpha}(\nu(Z,X))))\cdot\tilde{y}^{\alpha-1} \bmod p,$$

where $\nu(Z,X)$ is the restriction to Y of the normal bundle of Z in X. and p is a prime.

Proof is omitted.

§2 Analycity of Steenrod powers of algebraic cocycles

Letting $P^k : H^s(X, Z_p) \to H^{s+2k(p-1)}(X, Z_p)$ be the Steenrod power operation, we consider the Steenrod power $P^k(z)$ of an algebraic cocycle z dual to a subvariety Z. When Z is non-singular, we obtain from Thom [3, 4] the following proposition.

Proposition 4. Let $\nu(Z)$ be the normal bundle of Z in X and λ_* the Gysin homomorphism induced by the injection $\lambda : Z \hookrightarrow X$. Then we have

$$P^k(z) = \lambda_*(T^k(\nu(Z))),$$

where the cohomology class $T^k(\nu(Z))$ is a polynomial of Chern class $c_\ell(\nu(Z))$ of $\nu(Z)$ defined as follows: Let $\sum c_\ell(\nu(Z))\, x^\ell = \prod (1+\gamma_\ell x)$ be formal factorization, and

$$T^k(\nu(Z)) = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq s} (\gamma_{\alpha_1}\gamma_{\alpha_2} \cdots \gamma_{\alpha_k})^{p-1}.$$

If Z has singularities, by a theorem of Hironaka [2] we can resolve the singularities by successive applications of blowings-up. Now we consider the blowing-up of X with non-singular center Y contained in Z and we use the same notations as in §1. Since $P^k(z) = f_* f^*(P^k(z)) = f_* P^k(f^*(z))$, f_* being the Gysin homomorphism induced by f, analycity of $P^k(f^*(z))$ induces analycity of $P^k(z)$. By means of Thom isomorphism we obtain

Lemma. $P^k j_*(w) = j_*(P^k w) + j_*(P^{k-1} w)\cdot \tilde{y}^{p-1}$ for $w \in H^*(\tilde{Y}, Z_p)$.

From this Lemma and Proposition 2 we infer readily analycity of $P^k(f^*(z))$ by induction on singularities and codimension of subvariety. Thus we obtain

Theorem. Steerod powers of analytic cocycles are also analytic.

References

[1] A. Grothendieck, Sur quelques propriétés fondamentales en théorie des intersections, Seminaire Chevalley (1958), Anneaux de

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[2] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math., 79 (1964), 109 – 326.

[3] R. Thom, Espaces fibrés en sphères et carrés de Steenrod, Ann. Sci. Ecole Norm. Sup., 69 (1952), 109 – 182.

[4] ——, Quelques propriétés globales des variétés différentiables, Comment. Math. Helv., 28 (1954), 17 – 86.