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Introduction to the work of Sullivan on the splitting of various K-theories

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In this note, we will briefly summarize the results of Sullivan [7], [8].

<u>The starting point was</u>

To establish a uniqueness theorem of Novikov-Browder theory in PL and Diff category, i.e., given a homotopy equivalence

$$f : N \longrightarrow M$$

of closed PL or Diff manifolds of dimension ≥ 5, to decide when f is homotopic to PL or Diff isomorphism.

This is a question in manifold theory. However the t-regularity theorem and the surgery technique make it possible to convert this problem into that in homotopy theory, at least partially. (The Alexander duality!)

To put it concretely, let $g : M \longrightarrow N$ be a homotopy inverse to f. Then, since g is a homotopy equivalence, the stable bundle $\tau(M) \oplus g^* \nu(N)$ (where τ and ν are the stable tangent and normal bundle respectively) is fibre homotopy trivial. Sullivan showed that this bundle has a <u>canonical</u> G/H-framing (H = PL or 0 according as the category is PL or Diff) [6]. From homotopy theoretical point of view, this means that there is a canonical map

$$N(f) : M \longrightarrow G/H$$

which is called the classifying map for the associated G/H-bundle with f.

Now assume that f is a homeomorphism, then $N(f) \simeq 0$ as a G/Top bundle and by definition, if $f \simeq$ PL or Diff isomorphism, then

$$N(f) \simeq 0.$$

Moreover Sullivan showed that if $\pi_1(M) = 0$ and H = PL, then

$$N(f) \simeq 0 \Longleftrightarrow f \simeq \text{PL isomorphism.}$$

Thus the "Alexander duality" is complete in this case. Sullivan solved the Hauptvermutung under the hypothesis that $H^4(M; \mathbb{Z})$ has no 2-torsion.

In case $\pi_1 \neq 0$, the "Alexander duality" is not complete as the manifold $S^3 \times S^1 \times S^1$ presents an interesting counter example. (Shaneson [5], also cf. the work of Fukuhara).

Thus we meet with the

Problem (unstable obstruction).

If $\pi_1 \neq 0$, H = PL or Diff, then what is the additional obstruction for $f \simeq$ PL or Diff isomorphism besides $N(f) \simeq 0$.

Comments $N(f) \simeq 0$ implies that $f \simeq$ PL or Diff isomorphism on (n - 3)-skeleton (n = dim M).

(1) The (n - 2)-obstruction seems to be very difficult to analyse. Perhaps some theory of codim 2 manifolds pair will serve to this problem (cf. the work of Y. Matsumoto).

(2) The (n - 1)-obstruction is zero for PL and Diff cases.

(3) The n-obstruction is: H = PL $\Longrightarrow$ zero (by the generalized Poincaré conjecture). H = Diff $\Longrightarrow$ the obstruction group is θ_n, the group of homotopy n-spheres.

Henceforth we will be concerned in only the stable obstruction $=$ normal invariant $N(f) : M \longrightarrow G/H$.

Thus we need a homotopy theory of G/PL and G/O.

<u>Classifying spaces</u>, <u>homotopy groups</u>

Let us consider the following diagram of various universal spaces which are important in geometric topology.

$$\begin{array}{ccccccccccccc}
 & & PL/O & & & & PL/O & = & PL/O & & & & \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
\cdots & \longrightarrow & SO & \xrightarrow{J} & SG & \longrightarrow & G/O & \longrightarrow & BSO & \longrightarrow & BSG & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \| & & \\
\cdots & \longrightarrow & SPL & \longrightarrow & SG & \longrightarrow & G/PL & \longrightarrow & BSPL & \longrightarrow & BSG & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
 & & \vdots & & & & \vdots & & \vdots & & & &
\end{array}$$

1. $\pi_*(SG) \cong \pi_*(S)$

$\cong \mathrm{im}\, J \oplus \mathrm{cok}\, J$ (J. F. Adams [1]).

2. $\pi_*(PL/O) \cong \Gamma_*$

$\cong \Gamma(\partial\pi) \oplus \mathrm{cok}\, J$.

The second equality is valid at odd primes, at 2 we have the Kervaire invariant one problem. (Conjectured by Novikov, proved by Brumfiel [2]).

3. $\pi_*(G/PL) \cong L_*(1)$

$$\cong \begin{cases} 0 & * \equiv 1, 3 \pmod 4 \\ \mathbb{Z}_2 & * \equiv 2 \pmod 4 \\ \mathbb{Z} & * \equiv 0 \pmod 4 \end{cases}$$

(Kervaire-Milnor [4], Sullivan).

4. $\pi_*(G/O) \cong \begin{cases} \mathrm{cok}\, J & * \not\equiv 0 \pmod 4 \\ \mathbb{Z} \oplus \mathrm{cok}\, J & * \equiv 0 \pmod 4 \end{cases}$ (Sullivan [6]).

These theorems are all "group level" theorems, and it will be very convenient to geometric topology if we have more deep theorems, i.e., "space-level theorems" or at least "functor-level theorems". Sullivan has done this.

Homotopy type of G/PL

Theorem (Sullivan and Kirby-Siebenmann)

$$G/PL(2) = K(\mathbb{Z}_2, 2) \underset{\delta Sq^2}{\times} K(\mathbb{Z}, 4) \times \prod_{i \geq 1} K(\mathbb{Z}_2, 4i+2) \times \prod_{i \geq 2} K(\mathbb{Z}, 4i)$$

localized at 2

$$G/Top(2) = \prod_{i \geq 0} K(\mathbb{Z}_2, 4i+2) \times \prod_{i \geq 2} K(\mathbb{Z}, 4i) \quad \text{localized at 2}$$

$$G/PL(odd) = G/Top(odd)$$
$$= 1 \times BSO(odd)$$

(where $1 \times BSO$ is the classifying space for $KSO^* \equiv 1 + \widetilde{KSO}$).

PL-bundle theory at odd primes

Theorem. Oriented PL microbundle is KO_{odd}-orientable.

Proof. cf. [7], [9].

The point is as follows. Following the idea of Conner-Floyd [3], Sullivan characterizes geometrically the KO_{odd}-theory using bordism. Then the Thom isomorphism theorem for bordism of PL-microbundles (this follows from the t-regularity theorem) implies the existence of the KO_{odd}-Thom class.

Now consider the fibration

$$G/PL \longrightarrow BSPL \longrightarrow BSG.$$

In the last section, we saw that $G/PL_{odd} = 1 \times BSO_{odd}$. Thus we have

Corollary. Stable PL-microbundle theory = KO_{odd} oriented stable spherical fibration theory at odd primes.

The Minkowski-Hasse principle in geometric topology.

Let X be a sufficiently nice space (e.g. $\pi_1(X) = 0$ and $\pi_i(X)$: finitely generated for all $i > 0$). Then we have the "geometric square".

$$\begin{array}{ccc} X & \xrightarrow{c} & \hat{X} \\ {\scriptstyle \ell_0}\downarrow & & \downarrow{\scriptstyle \ell_0} \\ X_0 & \xrightarrow[\text{f.c.}]{} & X_A \end{array}$$

where ℓ_0 is the localization at 0, c is the profinite completion, f.c. is the formal completion and X_A is the finite Adele type of X. The point is that the above square is a fibre square. Now we need a homotopy theory of various classifying spaces which appear in geometric topology, e.g. BSO, BSPL, BSTop, G/O, G/PL, G/Top, etc. These spaces are all sufficiently nice. Hence we can construct the "geometric square" for these spaces. Since the "geometric square" is a fibre product, to study the homotopy type of these spaces, we have only to investigate the following three theories:

(i) The rational theory.

(ii) The profinite theory.

(iii) The compatibility condition of the above two theories on the Adele theory.

Now in case X is a homotopy associative H-space (which is all we need), the rational theory of X is equivalent to the Q-cohomology theory $H^*(X; \mathbb{Q})$.

For example the rational theories of BSO, BSPL, BSTop, G/O. G/PL, G/Top equal the theory of Pontrjagin classes and that of BSG is trivial.

Next we consider the profinite theory. One of the most convenient properties of the profinite theory is that it factors as the direct product of p-adic profinite theories for all prime p. Algebraically this can be seen by observing the fact that

$$\hat{\mathbb{Z}} = \prod_{p:\text{prime}} \hat{\mathbb{Z}}_p .$$

The Adele theory equals the A-cohomology theory, where $A = \widetilde{\prod_p} Q_p$ is the set of finite Adèles. (Here we also assume that X is a homotopy associative H-space).

The compatibility condition is the problem to decide whether a cohomology class with A-coefficient is rational or not.

<u>Étale homotopy type of algebraic variety.</u>

Let V be a normal algebraic variety of finite type over $\mathbb{C}$. Then, a "Čech-like nerve construction" using the Zariski opens and étale covers provides the complete étale homotopy type of V, V_{et}. (Artin-Mazur, Lubkin).

From our point of view, the point is that the finite cohomology of V is known by an algebraic method:

$$H^*(V;\ \text{finite}) \cong H^*(V_{et};\ \text{finite}).$$

By the construction of V_{et}, the homotopy groups of V_{et} are profinite groups. Thus if $\pi_1(V) = 0$, then we have

$$V_{et} \simeq \hat{V}.$$

Now the algebraic nature of the construction of the complete étale homotopy type induces a certain algebraic symmetry. More precisely, if V is defined over $\mathbb{Q}$, then the Galois group $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$) acts on V_{et} by isomorphisms. Thus we obtain many self-homotopy equivalences of $\hat{V}$.

The Galois group in geometric topology

Let $BSO = \varinjlim_{n,k} G_{n,k}(\mathbb{R})$ be the direct limit of the finite real Grassmannians. $G_{n,k}(\mathbb{R}) = O(n+k)/O(n) \times O(k)$. To use the étale theory, we replace $O(n)$ by $O(n, \mathbb{C})$ and $G_{n,k}(\mathbb{R})$ by

$$"G_{n,k}(\mathbb{R})" = O(n+k, \mathbb{C})/O(n,\mathbb{C}) \times O(k,\mathbb{C}).$$

By the theorem of Chevalley [3a], these replacements do not alter the homotopy types.

Now $"G_{n,k}(\mathbb{R})"$ is an algebraic variety defined over $\mathbb{Q}$, thus $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on

$$BSO\hat{} \cong \varinjlim_{n,k} "G_{n,k}(\mathbb{R})"_{et}$$

by self homotopy equivalences.

This action is abelian, i.e., it reduces to the action of

$$G/[G, G] \cong \mathrm{Gal}(\text{maximal abelian extension of } \mathbb{Q}/\mathbb{Q}) \cong \hat{Z}^* .$$

This follows from a cohomology calculation and the fact that

$$[BSO\hat{}, BSO\hat{}] \underset{ph\otimes\hat{\mathbb{Z}}}{\subset} H^*(BSO\hat{}; \hat{\mathbb{Z}}) \otimes Q \cong \hat{Z}[p_1, p_2, \cdots] \otimes Q.$$

Now let $\psi k\hat{}: BSO\hat{} \longrightarrow BSO\hat{}$ be the profinitely completed Adams operation and let

$$a(k) \in \hat{\mathbb{Z}}^*$$

be defined by

$$a(k) = \prod_{p \nmid k} (k) \prod_{p | k} 1 .$$

Then it is easy to verify that

$$a(k) \simeq \psi^{k\hat{}}$$

on $\prod_{p \nmid k} BSO_p^{\hat{}}$.

On the other hand, the Galois action exists in the unstable range:

$$\sigma \in G \Rightarrow \exists \sigma : BSO(k)^{\hat{}} \longrightarrow BSO^{\hat{}}(k) \quad \text{for all } k.$$

From this fact, we can deduce that the following diagram is homotopy commutative,

$$\begin{array}{ccc} BSO^{\hat{}} & \xrightarrow{\sigma} & BSO^{\hat{}} \\ & \searrow \quad \swarrow & \\ & BSG^{\hat{}} & \\ & \| & \\ & BSG & \end{array}$$

i.e. the Galois action preserves the underlying fibre homotopy type.

From the above facts, the Adams conjecture follows.

<u>Splitting of various K-theories</u>.

Consider the following situation:

$$K : \hat{\mathbb{Z}}_p\text{-module}$$

and $\hat{\mathbb{Z}}_p^*$ acts on K by $\hat{\mathbb{Z}}_p$-endomorphism (p : odd prime). (For example, $KSO_p^{\hat{}}$, $KSPL_p^{\hat{}}$ etc. satisfies the above condition.)

Now $\hat{\mathbb{Z}}_p^* \cong \mathbb{Z}/p-1 \oplus \hat{\mathbb{Z}}_p$ as topological groups. Let $\xi \in \hat{\mathbb{Z}}_p^*$ be a generator for $\mathbb{Z}/p-1$ and let $T : K \rightarrow K$ be the associated action on K with ξ . Consider the following operations

$$\pi_{\xi^i} = \prod_{j \neq i} \frac{T - \xi^j}{\xi^i - \xi^j} \ : K \longrightarrow K .$$

It is easy to verify the following properties of π_{ξ^i}.

$$\pi_{\xi^i} \circ \pi_{\xi^j} = \begin{cases} 0 & (i \neq j) \\ \pi_{\xi^i} & (i = j) \end{cases}$$

$$\sum_{i=0}^{p-2} \pi_{\xi^i} = 1 \qquad \text{etc.}$$

From these facts, we can deduce that K splits as a direct sum of various eigenspaces:

$$K \cong K_1 \oplus K_{\xi^1} \oplus \cdots \oplus K_{\xi^{p-2}}$$

$$K_1 = \{x \in K; \ Tx = x\}$$

$$K_{\xi^i} = \{x \in K; \ Tx = \xi^i x\}.$$

Since K_{ξ^i}'s depend on the choice of ξ, we group these factors and obtain

$$K \cong K_1 \oplus K_\xi$$

$$(K_\xi = K_{\xi^1} + \cdots + K_{\xi^{p-2}}).$$

We apply this splitting to various K-theories which admit a Galois action. Thus we obtain, for example,

$$\widetilde{KSO}^\wedge_p = (\widetilde{KSO}^\wedge_p)_1 \oplus (\widetilde{KSO}^\wedge_p)_\xi$$

$$\widetilde{KSPL}^\wedge_p = (\widetilde{KSPL}^\wedge_p)_1 \oplus (\widetilde{KSPL}^\wedge_p)_\xi .$$

The action of $\hat{\mathbb{Z}}^*_p$ on $\widetilde{KSPL}^\wedge_p$-theory is defined via the characterization theorem of PL-microbundles at odd primes. (See p.5)

Remark. $\hat{\mathbb{Z}}^*_2 \cong \mathbb{Z}/2 \oplus \hat{\mathbb{Z}}_2$.

Let ξ be the generator of $\mathbb{Z}/2$. Then, ξ acts on $BSO^\wedge$ trivially (A cohomology calculation).

The main theorem.

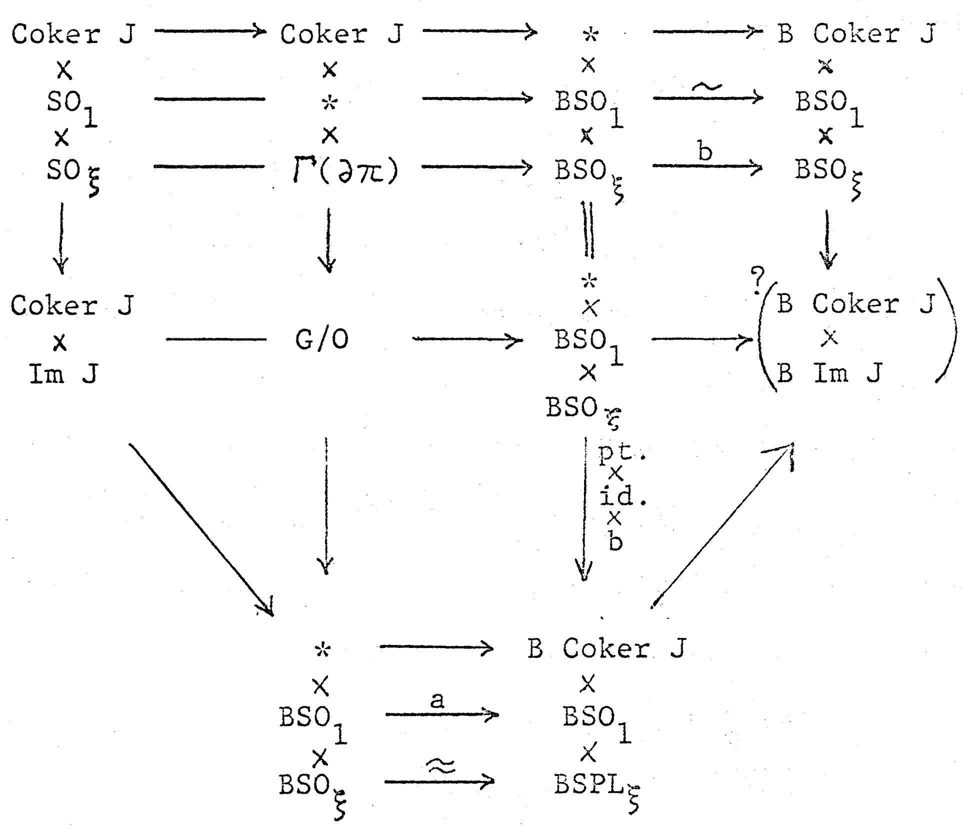
In this section, X will stand for $\hat{X}$ odd.

(X = SPL, SG, BSO etc.)

Theorem.

$$\begin{array}{ccccccc}
SPL & \longrightarrow & PL/O & \longrightarrow & BSO & \longrightarrow & BSPL \\
\downarrow & & \downarrow & & \| & & \downarrow \\
SG & \longrightarrow & G/O & \longrightarrow & BSO & \longrightarrow & BSG \\
\| & & \downarrow & & \downarrow & & \| \\
SG & \longrightarrow & G/PL & \longrightarrow & BSPL & \longrightarrow & BSG
\end{array}$$

$$\|$$



Remark. The question mark ? at BSG corresponds to the question: To what extent is the splitting SG = Coker J × Im J additive?

G/O-bundle theory

Consider the following fibration

$$SG \xrightarrow{i} G/O \xrightarrow{j} BSO \xrightarrow{k} BSG.$$

Let ξ be an element in $\widetilde{K}_{G/O}(X)$ and let $\eta = j_*(\xi) \in \widetilde{KSO}(X)$. Here X is a finite CW-complex.

Since G/O = fibre (B Spin $\longrightarrow$ B Spin G), η has a canonical spin structure. Let $\Delta\eta \in \widetilde{KO}(T(\eta))$ be a Thom class (which is not unique, but we choose one so that the necessary conditions for the following argument will be satisfied). Since η comes from ξ, we have a fibre homotopy trivialization:

$$\begin{array}{ccc} E(\eta) & \xrightarrow{\quad f \quad} & X \times \mathbb{R}^* \\ \downarrow & & \downarrow P_1 \\ X & = & X \end{array}$$

This induces,

$$\begin{array}{ccc} \widetilde{KO}(T(\eta)) & \underset{\sim}{\xleftarrow{\quad f^* \quad}} & \widetilde{KO}(S^*X_+) \\ \wr\uparrow \Phi & & \wr\uparrow \Phi \\ K(X) & = & K(X) \end{array}$$

The above diagram is not commutative in general and we set

$$\theta(\xi) \equiv \Phi^{-1} f^{*-1}(\Delta\eta) \in 1 + \widetilde{KO}(X).$$

Since $\pi_1(G/O) = 0$, we have

$$\theta(\xi) \in 1 + \widetilde{KSO}(X).$$

Thus we have a map

$$\theta : G/O \longrightarrow BSO.$$

Since $\Delta(\eta_1 \hat{\oplus} \eta_2) = \Delta(\eta_1) \times \Delta(\eta_1)$, the map θ is additive.

Now we define a map $\lambda : G/O^\wedge \longrightarrow BSO^\wedge$ as follows:

$$\lambda = \prod_p \lambda_p ,$$

$$\lambda_p : G/O^\wedge_p \longrightarrow BSO^\wedge_p$$

(i) $\lambda_2 = \hat{\theta}_2$

(ii) $\lambda_{odd\ 1} : G/O^{\wedge}_{odd} \xrightarrow{\text{natural}} G/PL^{\wedge}_{odd} \xrightarrow{\simeq} BSO^{\wedge}_{odd} \xrightarrow{\text{projection}} BSO^{\wedge}_{odd\ 1}$

$\lambda_{odd\ \xi} : G/O^{\wedge}_{odd} \xrightarrow{\text{natural}} BSO^{\wedge}_{odd} \xrightarrow{\text{projection}} BSO^{\wedge}_{odd\ \xi}$.

Let $\alpha \in \hat{\mathbb{Z}}^*$ be such that α_p is a topological generator for each odd prime p and α_2 topologically generates $\hat{\mathbb{Z}}_2 \subset \hat{\mathbb{Z}}_2^*$. Now since η is fibre homotopy trivial, there is a (canonical) vector bundle $\zeta \in \widetilde{KO}(X)$ such that

$$\eta \cong \zeta^{\alpha} - \zeta .$$

Consider the following diagram

$$0 \xleftarrow{f} \eta \cong \zeta^{\alpha} - \zeta \xrightarrow[\text{canonical fibre homotopy trivialization}]{} 0 .$$

This construction induces a map

$$\alpha : G/O^{\wedge} \longrightarrow SG .$$

Now consider the map

$$(\lambda, \alpha) : G/O^{\wedge} \longrightarrow BSO^{\wedge} \times SG .$$

Sullivan claims that there is a sub-theory $\mathcal{L} \subset SG$ such that (λ, α) is an isomorphism onto the theory $BSO^{\wedge} \times \mathcal{L}$.
Moreover $\mathcal{L}_{odd} = \text{Coker } J_{odd}$.

For the details, see [8].

In this note, we have been concerned only with odd primes. However, Sullivan seems to have finished a similar analysis at 2.

References

[1] Adams, J. F. On the groups J(X), IV, Topology 5 (1966), 21-71.

[2] Brumfiel, G. On the homotopy groups of BPL and PL/O, I, Ann. of Math. 88 (1968), 291-311.

[3a] Chevalley, C. "Theory of Lie groups I", Princeton, 1946.

[3] Conner, P. E. and Floyd, E. E. "The Relation of Cobordism to K-theories", Springer Lecture Note 28, 1966.

[4] Kervaire, M. and Milnor, J. Groups of homotopy spheres, I, Ann. of Math. 77 (1963), 504-537.

[5] Shaneson, J. "On non-simply-connected manifolds", Lecture Notes prepared for th3 1970 Madison Conference on Algebraic Topology.

[6] Sullivan, D. "Triangulating and Smoothing Homotopy Equivalences and Homeomorphisms", Geometric Topology Seminar Notes, Princeton University, 1967.

[7] ———— . "Geometric Topology, Part I. Localization, Periodicity and Galois Symmetry", Note, M.I.T., 1970.

[8] ———— . "Geometric Periodicity and Invariants of Manifolds", Manifolds — Amsterdam 1970, Springer Lecture Note 197, 1971.

[9] Tsuchiya, A. "Characteristic classes for PL microbundles", Lecture Notes.

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