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ON THE CIRCULAR COUÉTTE FLOW

BY

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1. Introduction.

In the present paper we shall study properties of special solutions of the 3-dimensional stationary and non-stationary Navier-Stokes equations from several points of view. The main solution considered in this paper is the classical Couétte flow between two rotating concentric cyliners.

There are mainly two reasons why we wish to treat this flow. First it is important to study precise properties of special solutions of the Navier-Stokes equations(1) in special cases along with the mathematical analysis for the equation in the general formulation, since we still not have complete theory for the problem of existence or non-existence of global in time regular solutions of the 3-dimensional N-S equations. Secondly, the Couétte flow itself has many properties which are quite interesting mathematically as well as physically. For example, as the celebrated experiment by G. I. Taylor in 1923 revealed and as was rigorously proved mathematically by W. Velte [6] in 1966, the Couétte flow is not necessarily the unique solution. And moreover, what is more interesting and seems even a peculiar phenomenum to mathematicians

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(1) For simplicity, we call it an N-S equation.

is the experimental fact that in experiments the Couétte flow is actually observed in one circumstance and in another it is not, although mathematically it is a solution for both cases. In the latter case another flow different from the Couétte flow is observed. The explanations tried by physicists for this phenomenum are done from the stand-point of the stability theory (See, for example, C. C. Lin [3]). But at the present state of the mathematical treatment of the N-S equations where we do not know whether unique and regular global in time solution of the 3-dimensional N-S equation really exists or not, the stability theory is confronted with theoretical difficulties.

In this paper we shall treat problems related to the Couétte flows in general cases. First we shall study the problem whether or not for any given $T(> 0)$ there exists a regular solution in the interval $[0,T]$ of the corresponding non-stationary N-S equation for every initial data given near the Couétte flow. After establishing an affirmative answer to this question, we next prove the differentiability in the sense of Fréchet of the evolution operator which gives a mathematical foundation to the linear stability theory. Thirdly, we shall discuss the eigenvalue problem for the Fréchet derivative of the evolution operator at the Couétte flow and show that the Couétte flow is unstable for infinitesimal perturbations under certain circumstances. Finally we shall prove that the Couétte flow is an isolated solution under almost all circumstances.

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2. Formulation of the problem and the results.

We consider the non-stationary and stationary N-S equations in a domain G between two concentric cylinders of radii R_1 and $R_2 (> R_1)$. More precisely G is defined by $G = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \; ; \; R_1^2 < x_1^2 + x_2^2 < R_2^2\}$. The two cylinders rotates with constant angular velocities ; the inner with Ω_1 and the outer with Ω_2 counter clockwise.

In G the N-S equation is expressed for the non-stationary motion

(NSE)
$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - \nabla_v v - \operatorname{grad} q, & t > 0,\ x \in G \\ \operatorname{div} v(x) = 0, & x \in G \\ v(0,x) = a(x) & \\ \text{and the boundary condition of adherence at the boundary that} & \\ \text{the fluid on the boundary move with the boundary.} & \end{cases}$$

and for the stationary motion

(SE)
$$\begin{cases} \Delta v - \nabla_v v - \operatorname{grad} q = 0 \\ \operatorname{div} v = 0 \\ \text{the boundary condition of adherence at the boundary.} \end{cases}$$

The boundary condition will be given explicitly later. In these equations v is the velocity vector field of the fluid in question and q is a scalar function which is the pressure in the fluid and the unknowns are v and q. ∇ is the canonical affine connection in $\mathbb{R}^3$ and Δ is the

Laplacian. In the sequel to treat the equations (NSE) and (SE) effectively, we adopt two coordinate systems, the cartesian coordinate system (x_1, x_2, x_3) and the cylindrical coordinate system (r, ϕ, z). In these coordinate systems a vector field v is expressed ; in the first, $v = (v_1, v_2, v_3) = \sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i}$ and in the latter, $v = (v_r, v_\phi, v_z) = v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r} \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}$. The well-known Couétte flow is thus expressed by $W = (A + \frac{B}{r^2})\frac{\partial}{\partial \phi}$ and $q_0 = \int^r \frac{1}{\rho} (A\rho + \frac{B}{\rho})^2 d\rho$ where $A = (R_2^2\Omega_2 - R_1^2\Omega_1)/(R_2^2 - R_1^2)$ and $B = R_1^2R_2^2(\Omega_1 - \Omega_2)/(R_2^2 - R_1^2)$. This is a solution of (SE) for all R_1, R_2, Ω_1, Ω_2. We preserve the letter w to denote the Couétte flow exclusively in this paper. It is the aim of this paper to study the properties of the Couétte flow and those of the solutions of the equations (NSE) and (SE) near the Couétte flow. To that end, we consider a portion G_h of G and treat the equations in G_h. G_h is defined by $G_h = \{x \in G ; 0 < x_3 < h\}$ and the union of the lower and the upper bottom is denoted by δG_h. In G_h we consider the following initial value problem and boundary value problem.

$$
\text{(IVP)} \quad \begin{cases}
\frac{\partial v}{\partial t} = \Delta v - \nabla_v v - \text{grad } q \\
\text{div } v = 0 \\
v(x_1, x_2, 0) = v(x_1, x_2, h), \ \frac{\partial v}{\partial x_3}(x_1, x_2, 0) = \frac{\partial v}{\partial x_3}(x_1, x_2, h) \\
v = (A + \frac{B}{R_i^2})\frac{\partial}{\partial \phi} \quad \text{for } r^2 = x_1^2 + x_2^2 = R_i^2, \ i = 1,2.
\end{cases}
$$

and

$$\text{(BVP)} \quad \begin{cases} \Delta v - \Delta_V v - \operatorname{grad} q = 0 \\ \operatorname{div} v = 0 \\ v(x_1, x_2, 0) = v(x_1, x_2, h), \ \dfrac{\partial v}{\partial x_3}(x_1, x_2, 0) = \dfrac{\partial v}{\partial x_3}(x_1, x_2, h). \\ v = (A + \dfrac{B}{R_i^2})\dfrac{\partial}{\partial \phi} \quad \text{for } r = R_i, \ i = 1,2. \end{cases}$$

In order to treat the problem in a functional analysis setting, we introduce some function spaces and operators. $L^2 = L^2(G_h)$ is a Hilbert space of all $\mathbf{R}^3$-valued functions $v = (v_1(x), v_2(x), v_3(x))$ defined in G_h for which the norm $\| v \| = \left\{ \int_{G_h} \sum_{i=1}^{3} v_i^2(x)dx \right\}^{1/2}$ is finite.

$C_{0,\sigma}^{\infty} \equiv C_{0,\sigma}^{\infty}(G_h)$ is a space of all R^3-valued functions $\varphi = (\varphi_1(x), \varphi_2(x), \varphi_3(x))$ such that (i) every component $\varphi_j \in C^{\infty}(\overline{G}_h)$ (ii) $\varphi_j \equiv 0$ near $G_h \equiv \{x \ ; \ x_1^2 + x_2^2 = R_i^2, \ i = 1,2. \ \ 0 \leqq x_3 \leqq h\}$ (iii) $\operatorname{div} \varphi = 0$. (iv) $\varphi(x_1, x_2, 0) = \varphi(x_1, x_2, h)$, $\dfrac{\partial}{\partial x_3}(x_1, x_2, 0) = \dfrac{\partial}{\partial x_3}(x_1, x_2, h)$. $L_\sigma^2 \equiv L_\sigma^2(G_h)$ is the completion of $C_{0,\sigma}^{\infty}$ with respect to the norm of $L^2(G_h)$. By P we denote the orthogonal projection of L^2 onto L_σ^2. For $\varphi \in C_{0,\sigma}^{\infty}$ we define an operator A by $A\varphi = -P\Delta\varphi$. It is easy to verify that A is a strictly positive symmetric operator in the Hilbert space L_σ^2. The positivity is verified by the Poincaré inequality. We take the Friedrichs extension of A which we denote also by the same letter A. Then A is a strictly positive self-adjoint operator with compact inverse A^{-1}. For real γ we denote A^{γ} the fractional power of A and by $\mathcal{D}(A^{\gamma})$ the domain of definition of A^{γ} endowed with its graph norm $\|\varphi\|_{\gamma} = \|A^{\gamma}\varphi\|$. Transforming the unknowns from (v,q) to (u,p) by the identities $v = u + w$, $q = p + q_0$, and making use of the above notations, the equations (IVP) and (BVP) are transformed (formally) into the following abstract evolution equation (EE) and an operator equation (E) in L^2_σ, respectively.

$$\text{(EE)} \qquad \begin{cases} \dfrac{du}{dt} = -Au - P(\nabla_u u + \nabla_w u + \nabla_u w) \\ u(0) = a \end{cases}$$

and

$$\text{(E)} \qquad Au + P(\Delta u + \nabla_w u + \nabla_u w) = 0.$$

In (EE) $u = u(t)$ is regarded as an L^2_σ-valued function defined on $\{t \geqq 0\}$ In order to investgate the integrability of the equation (EE), we introduce the following integral equation (IE),

$$\text{(IE)} \qquad u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P(\nabla_{u(s)}u(s) + \nabla_w u(s) + \nabla_{u(s)}w)ds$$

where by e^{-tA} we denote the semi-group of operators generated by $-A$. If we can prove the existence of solution $u(t)$ of (IE) with certain regularity property, it is easy to verify that it is a regular solution of (EE). Hence we shall be engaged exclusively in (IE).

Now we can state our theorems which we are going to prove in this paper.

<u>Theorem 1</u> (an existence theorem)

(1) For every $r > 0$ there exists $T > 0$ such that there exists uniquely a solution of (IE) on the interval $[0,T]$ for every $a \in (A^{1/2})$ with $\|A^{1/2}a\| < r$.

(2) For every $T > 0$ there exists $r > 0$ such that the statement in (1) holds.

<u>Theorem 2</u> (differentiability of the evolution operator)

The evolution operator $S_t : \mathcal{D}(A^{1/2}) \to \mathcal{D}(A^{1/2})$ is Fréchet differentiable at $0 \in \mathcal{D}(A^{1/2})$ for any $t > 0$.

<u>Theorem 3</u> (eigenvalue problem for the Fréchet derivative of the evolution operator)

For any $\omega = (\omega_1, \omega_2) \in S^1$ (the 1-sphere) such that $(R_2^2\omega_2^2 - R_1^2\omega_1^2)(\omega_2 - \omega_1) > 0$ there exists $\rho = \rho_\omega > 0$ such that the Fréchet derivative of the evolution operator S_t at 0 has real positive eigenvalue greater than 1 for every $t > 0$.

<u>Theorem 4</u> 0 is an isolated solution of (E) in $\mathcal{D}(A^\gamma)$ with $\gamma > \frac{3}{4}$ for almost all $\Omega = (\Omega_1, \Omega_2) \in R^2$.

3. Existence theorems.

First we state two lemmas concerning the operators A and e^{-tA}. For the statement of Lemma 1, we introduce the operator B defined as follows. The domain of definition of B is $\mathcal{D}(B) = W_2^2(G_h) \cap H_2^1(G_h)$ where $W_2^m(G_h)$ is a L^2-Sobolev space of order m, and $H_2^2(G_h) =$ $\{u \in W_2^1(G_h)\ ;\ u\big|_{G_h} = 0,\ u(x_1, x_2, 0) = u(x_1, x_2, h),\ \frac{\partial u}{\partial x_3}(x_1, x_2, 0) =$ $\frac{\partial u}{\partial x_3}(x_1, x_2, h)\}$. And for $u \in \mathcal{D}(B)$, $Bu = -\Delta u$.

Lemma 1. For $0 < \gamma < 1$, $\mathcal{D}(A^\gamma) = \mathcal{D}(B^\gamma) \cap L^2_\sigma$. And therefore $\mathcal{D}(A^\gamma) \subset C(\overline{G}_h)$ for $\gamma > \frac{3}{4}$ and $\sup_{x \in G_h} |u(x)| \leq C_\gamma \| A^\gamma u \|$ for $u \in \mathcal{D}(A^\gamma)$.

We can prove the lemma by the interpolation theory of Lions and a certain fact concerning $\mathcal{D}(A)$. For details, see H. Fujita-H. Morimoto [2] where analogical result is proved.

Lemma 2. For $0 < \gamma < e$, $\| A^\gamma e^{-tA} \| \leq t^{-\gamma}$.

Proof. The proof is easy if we use the spectral representation of A and so we omit the proof.

We note here that by virtue of Lemma 1, the nonlinear operator $P\nabla_u v$ is well-defined for every $u \in \mathcal{D}(A^\gamma)$ if $\gamma > \frac{3}{4}$, and $v \in (A^{1/2})$ and we have $\| P\nabla_u v \| \leq C_\gamma \|A^\gamma u\| \; \| A^{1/2} v \|$. We note also that in the cartesian coordinate system, $\nabla_u v = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} u_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \equiv (u \cdot \nabla) v$ and so the above estimate is an immediate consequence.

Now we are ready to study the integral equation (IE). For that purpose we introduce a function space Ψ_T^γ, for $T\ (>0)$ and $(\frac{3}{4}<)\ \gamma\ (<1)$. Ψ_T^γ is a space of all $\mathcal{D}(A^{1/2})$-valued functions $u(t)$ defined on the interval $[0,T]$ such that (i) $u(t) \in C([0,T]\ ;\ \mathcal{D}(A^{1/2})) \cap C((0,T]\ ;\ \mathcal{D}(A^\gamma))$, (ii) and the norm

$$|||u||| = \sup_{0\leq t\leq T} \|A^{1/2}u(t)\| + \sup_{0<t\leq T}\ \sup_{0<s\leq t} s^{\gamma-\frac{1}{2}} \|A^\gamma u(s)\|$$

is finite.

We are going to obtain solutions of the integral equation (IE) in the class Ψ_T^γ by the iteration method. We use the following iteration scheme.

$$\text{(3.1)}\quad \begin{cases} u_0(t) = 0, \\ u_{n+1}(t) = e^{-tA}a - \displaystyle\int_0^t e^{-(t-s)A} P[(u_n(s)\cdot\nabla)u_n(s) + (w\cdot\nabla)u_n(s) \\ \qquad\qquad + (u_n(s)\cdot\nabla)w]ds. \end{cases}$$

First we must verify that the iteration is possible in Ψ_T^γ. To that end we introduce functions $K_u(t)$, $M_u(t)$ for functions $u(t)$ in Ψ_T^γ. They are defined as follows.

$$K_u(t) = \sup_{0<s\leq t} s^{\gamma-\frac{1}{2}} \|A^\gamma u(s)\|, \qquad M_u(t) = \max_{0\leq s\leq t} \|A^{1/2}u(s)\|.$$

And in addition we define operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ by

$$\mathcal{A}(u)(t) = \int_0^t e^{-(t-s)A} P(u(s)\cdot\nabla)u(s)ds$$

$$\mathcal{B}(u)(t) = \int_0^t e^{-(t-s)A} P(w\cdot\nabla)u(s)ds$$

$$\mathcal{C}(u)(t) = \int_0^t e^{-(t-s)A} P(u(s)\cdot\nabla)wds$$

Then we obtain the following estimates which justify the iteration.

Lemma 3.

$$\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \|A^{\gamma}\mathcal{A}(u)(s)\| \leqq t^{1-\gamma} C_{\gamma} B_1 K_u(t) M_u(t)$$

$$\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \|A^{\gamma}\mathcal{B}(u)(s)\| \leqq t^{1/2} \frac{C_{\gamma}}{1-\gamma} \|A^{\gamma}\omega\| M_u(t)$$

$$\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \|A^{\gamma}\mathcal{C}(u)(s)\| \leqq t^{1-\gamma} C_{\gamma} B_1 \|A^{1/2}\omega\| K_u(t)$$

$$\sup_{0<s\leqq t} \|A^{1/2}\mathcal{A}(u)(s)\| \leqq t^{1-\gamma} C_{\gamma} B_1 K_u(t) M_u(t)$$

$$\sup_{0<s\leqq t} \|A^{1/2}\mathcal{B}(u)(t)\| \leqq t^{1/2}\, 2C_{\gamma} \|A^{\gamma}\omega\| M_u(t)$$

$$\sup_{0<s\leqq t} \|A^{1/2}\mathcal{C}(u)(t)\| \leqq t^{1-\gamma} C_{\gamma} B_1 \|A^{1/2}\omega\| K_u(t)$$

Here we write $B_1 = B(1-\gamma, \frac{3}{2}-\gamma)$ where $B(\cdot,\cdot)$ is the beta function.

Proof. We prove the first estimate only. The others are proved similarly.

$$\int_0^s \|A^{\gamma} e^{-(s-\sigma)A}\cdot P(u(\sigma)\cdot\nabla)u(\sigma)\|\, d\sigma \leqq \int_0^s (s-\sigma)^{-\gamma} C_{\gamma} \|A^{\gamma}u(\sigma)\| \ \|A^{1/2}u(\sigma)\|\, d\sigma$$

$$\leqq C_{\gamma} \int_0^s (s-\sigma)^{-\gamma}\sigma^{\frac{1}{2}-\gamma} K_u(s)M_u(s)d\sigma = C_{\gamma}K_u(s)M_u(s) \int_0^1 (1-\rho)^{-\gamma}\rho^{\frac{1}{2}-\gamma} s^{\frac{3}{2}-2\gamma} d\rho$$

$$= s^{\frac{3}{2}-2\gamma} C_{\gamma} B(1-\gamma, \tfrac{3}{2}-\gamma)K_u(s)M_u(s).$$

Hence we have

$$\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \|A^{\gamma}A(u)(s)\| \leqq C_{\gamma}B_1 \sup_{0<s\leqq t} s^{1-\gamma}K_u(s)M_u(s) = C_{\gamma}B_1 t^{1-\gamma}K_u(t)M_u.$$

In the last equality we use the fact that $K_u(t)$ and $M_u(t)$ are increasing functions and the assumption that $\frac{3}{4} < \gamma < 1$. Q. E. D.

By Lemma 3, we have

$$\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \| A\, [\mathcal{A}(u(s)) + \mathcal{B}(u)(s) + \mathcal{C}(u)(s)] \| \leqq t^{1-\gamma}[C_1K_u(t)M_u(t) + C_2K_u(t) + C_3t^{\gamma-\frac{1}{2}} M_u(t)]$$

and

$$\sup_{0<s\leqq t} \| A^{1/2}[\mathcal{A}(u)(s) + \mathcal{B}(u)(s) + \mathcal{C}(u)(s)] \| \leqq t^{1-\gamma}[C_1K_u(t)M_u(t) + C_2K_u(t) + C_3M_u(t)]$$

Therefore, defining $N_u(t) = \max\{K_u(t), M_u(t)\}$ and $\mathcal{D}(u) = \mathcal{A}(u) + \mathcal{B}(u) + \mathcal{D}(u)$, we have

$$(3.2) \qquad N_{\{u\}}(t) \leqq t^{1-\gamma}[C_1N_u^2(t) + (c_2 + c_3t^{\gamma-\frac{1}{2}})N_u(t)]$$

where we used c_1, c_2, c_3 to denote positive constants depending only on γ and the Couette flow w.

We now return to the iteration scheme (3.1). By the estimate (3.2) we have the following reccurence inequality

$$(3.3) \qquad N_{u_{n+1}}(t) \leqq \| A^{1/2}a\| + t^{1-\gamma}[c_1N_{u_n}^2(t) + (c_2 + c_3t^{\gamma-\frac{1}{2}})N_{u_n}(t)].$$

By a simple consideration we have for every u_n

$$(3.4) \qquad N_{u_n}(t) \leqq \chi(t)$$

if $c_2t^{1-\gamma} + c_3t^{1/2} < 1$ and $\Delta(t) \geqq 0$

where we define

$$(3.5) \qquad \Delta(t) = (c_2t^{1-\gamma} + c_3t^{1/2} - 1)^2 - 4c_1t^{1-\gamma}\| A^{1/2}a\|$$

and

$$\chi(t) = [1 - (c_2 t^{1-\gamma} + c_3 t^{1/2}) - \Delta(t)^{1/2}]/2c_1 t^{1-\gamma}. \tag{3.6}$$

We can now study the convergence of the iteration. Setting $v_n(t) = u_{n+1}(t) - u_n(t)$, we have

$$v_0(t) = e^{-tA}a \quad \text{and}$$

$$v_n(t) = -\int_0^t e^{-(t-s)A} P[(v_{n-1}(s)\cdot\nabla)u_n(s) + (u_{n-1}(s)\cdot\nabla)v_{n-1}(s)$$
$$+ (w\cdot\nabla)v_n(s) + (v_{n-1}(s)\cdot\nabla)w]ds.$$

In order to estimate $v_n(t)$ we define

$$D_n(t) = \sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \|A^\gamma v_n(s)\|, \quad E_n(t) = \sup_{0<s\leqq t} \|A^{1/2} v_n(s)\|,$$

$$F_n(t) = \max\{D_n(t), E_n(t)\}, \qquad M_n(t) = K_{u_n}(t),\ M_n(t) = M_{u_n}(t),\ N_n(t) = N_{u_n}(t).$$

Then we have

$$F_n(t) \leqq t^{1-\gamma}(d_1\chi(t) + d_2 \|A^\gamma w\| t^{\gamma-\frac{1}{2}} + d_3 \|A^{1/2} w\|)F_{n-1}(t) \equiv \rho(t,w)F_n(t)$$

and

$$F_0(t) \leqq \|A^{1/2}a\|$$

where d_1, d_2, d_3 are positive constants depending only on γ. If we note that for every fixed w, $\chi(t)$ tends to 0 as t tends to 0, we immediately see that there exists a positive T depending only on $\|A^\gamma w\|$ such that $\sum_{n=0}^{\infty} F_n(t)$ converges uniformly in $t \in [0,T]$. Hence we see, noting that A is a closed operator and has a continuous inverse, that $u(t) = \lim_{n\to\infty} u_n(t)$ exists in L^2_σ and $\mathcal{D}(A^{1/2})$ for

$t \in [0,T]$ and in $\mathcal{D}(A^{\gamma})$ for every $t \in (0,T]$ and that the former convergence is uniform on $[0,T]$ and the latter locally uniform in $(0,T]$. The fact that the limit function belongs to Ψ_T^{γ} is evident. The estimates above show that for a given $r > 0$ we can choose $T > Q$ such that the limit function exists on the interval $[0,T]$ for every initial data $a \in \mathcal{D}(A^{1/2})$ such that $\| A^{1/2}a \| \leqq r$.

For the proof of assertion (2) of Theorem 1, we note that (3.5) and (3.6) show that $X(t) \to 0$ when $\| A^{1/2}a \| \to 0$. This and the fact that in the inequality $c_2T^{1-\gamma} + c_3T < 1$ c_2 and c_3 depend only on γ indicate that for arbitrary given $T' > 0$ we can choose $r > 0$ such that we can construct a solution $u(t) \in \Psi_{T'}^{\gamma}$ for every $u(0) = a \in \mathcal{D}(A^{1/2})$ such that $\| A^{1/2}a \| \leqq r$.

Thus we have proved the following

Theorem 1.

(1) For every given $r > 0$ we can choose $T > 0$ such that for every $a \in \mathcal{D}(A^{1/2})$ with $\| A^{1/2}a \| \leqq r$, there exists a solution $u(t)$ of the integral equation (IE) in the interval $[0,T]$ which belongs to the class Ψ_T^{γ}.

(2) For every given $T > 0$ we can choose $r > 0$ such that there exists a solution of (IE) on $[0,T]$ for every $a \in \mathcal{D}(A^{1/2})$ with $\| A^{1/2}a \| \leqq r$.

Remark. We have not mentioned the uniqueness of the solution. We have omitted the proof since it is not difficult to prove the uniqueness of the solution in the class Ψ_T^{γ}.

4. Differentiability of the evolution operator S_t.

First we remember the definition of the Fréchet derivative.

Definition. Let X, Y be Banach spaces and Φ be a continuous mapping defined in a neighbourhood U of an element $a \in X$. A bounded linear operator A is said to be the Fréchet derivative of Φ at a if

$$\Phi(a + h) - \Phi(a) = Aa + o(\|h\|) \quad \text{as } a + h \text{ tends to } a \text{ in } U.$$

And when this is the case, Φ is said to be Fréchet differentiable at a.

We now return to the integral equation (IE) and define the evolution operator $S_t : \mathcal{D}(A^{1/2}) \to \mathcal{D}(A^{1/2})$ by $S_t a = u(t)$ where $u(t)$ is the solution (IE) with initial data a. By Theorem 1, we know that for any given $T > 0$ we can find a neighbourhood U of $0 \in \mathcal{D}(A^{1/2})$ where S_T is defined everywhere. Hence we can talk about the Fréchet differentiability of S_T at 0. We fix U and T above. For $h \in U$, S_t satisfies the following integral equation by definition

$$S_t h = e^{-tA}h - \int_0^t e^{-(t-s)A} P[(S_s h\cdot\nabla)S_s h + (w\cdot\nabla)S_s h + (S_s\cdot\nabla)w]ds.$$

An inspection indicates that the Fréchet derivative of S_t at $0 \in \mathcal{D}(A^{1/2})$ which we denote by DS_t must satisfy the following integral equation if it exists

$$(4.1) \qquad DS_t h = e^{-tA}h - \int_0^t e^{-(t-s)A} P[(w\cdot\nabla)DS_s h + (DS_s h\cdot\nabla)w]ds.$$

In order to integrate the equation (4.1) we introduce the function space Ψ_t^γ again and define an operator by

$$\Gamma(f)(t) = e^{-tA}h - \int_0^t e^{-(t-s)A} P[(w\cdot\nabla)f(s) + (f(s)\cdot\nabla)w]ds.$$

The operator Γ is well-defined since we have the following estimate

$$\text{(4.2)} \qquad \max\{ \sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}} \| A^{\gamma}\Gamma(f)(s)\| ,$$

$$\sup_{0<s\leqq t} \| A^{1/2}\Gamma(f)(s)\| \} \leq \| A^{1/2}h\| + c(t^{1-\gamma}K_f(t) + t^{1/2}M_f(t)),$$

with positive constant c depending only on γ and w (the Couétte flow). And we have for f_1, $f_2 \in \Psi^{\gamma}_T$,

$$\text{(4.3)} \qquad ||| \Gamma(f_1) - \Gamma(f_2)||| \leqq c_1\tau^{1-\gamma} \| A^{\gamma}w\| (1 + c_2\tau^{\gamma-\frac{1}{2}}) ||| f_1 - f_2 |||$$

with positive constants c_1, c_2 depending only on γ. We choose $\tau > 0$ such that

$$\text{(4.4)} \qquad c_1\tau^{1-\gamma} \|A^{\gamma}w\| (1 + c_2\tau^{\gamma-\frac{1}{2}}) < 1.$$

Then there exists uniquely in Ψ^{γ}_{τ} a solution which we denote by $f(t ; h)$.

Next we shall prove, making use of $f(t ; h)$, that there exists $0 < \tau' < \tau$ such that $S_{\tau'}$ is Fréchet differentiable at 0. We set $g(t ; h) = S_t h$, then we have

$$\text{(4.5)} \qquad f(t ; h) - g(t ; h) = \int_0^t e^{-(t-s)A}\cdot P[(g(s ; h)\cdot\nabla)g(s ; h)]ds$$

$$+ \int_0^t e^{-(t-s)A}\cdot P\{(w\cdot\nabla)(f(s ; h) - g(s ; h)) +$$

$$([f(s ; h) - g(s ; h)]\cdot\nabla)w\}ds.$$

Then what we have to prove is that there exists $\tau' > 0$ such that

$$\| A^{1/2}(f(\tau' ; h) - g(\tau' ; h))\| / \| A^{1/2}h\| \to 0 \text{ as } \| A^{1/2}h\| \to 0.$$

Estimating (4.2), we have

$$|||f(t\ ;\ h) - g(t\ ;\ h)||| \leqq$$

$$c_3\tau'^{1-\gamma}\,\|A^\gamma w\|\,(1 + c_4\tau'^{\gamma-\frac{1}{2}})\ |||f(t\ ;\ h) - g(t\ ;\ h)||| + c_5\tau'^{1-\gamma}\,\|A^\gamma w\|\,L(\tau')^2$$

with positive constants c_3, c_4, c_5 depending only on γ where the norm is for functions $f, g \in \Psi^\gamma_{\tau'}$ and

$$L(t) = \max\{\sup_{0<s\leqq t} s^{\gamma-\frac{1}{2}}\|A^\gamma g(s\ ;\ h)\|,\ \sup_{0<s\leqq t}\|A^{1/2}g(s\ ;\ h)\|\}.$$

If we choose $\tau' > 0$ such that

$$c_3\tau'^{1-\gamma}\,\|A^\gamma w\|\,(1 + c_4\tau'^{\gamma-\frac{1}{2}}) < \rho < 1, \tag{4.6}$$

we have

$$|||f(t\ ;\ h) - g(t\ ;\ h)||| < (1-\rho)^{-1}c_5\tau'^{1-\gamma}\,\|A^\gamma w\|\,L(\tau')^2.$$

This implies that

$$f(\tau'\ ;\ h) - g(\tau'\ ;\ h) = O(\|A^{1/2}h\|^2)$$

which was to be proved. Thus we have proved that $S_{\tau'}$ is Fréchet differentiable at 0. By the estimates (4.2) and (4.4), we see that τ' is determined only γ and w and so, by the chain rule for Fréchet derivatives, we have the following

<u>Theorem 2.</u>

For every $T > 0$ the evolution operator S_t is Fréchet differentiable at $0 \in \mathcal{D}(A^{1/2})$.

5. The eigenvalue probelm.

In this section we study the eigenvalue problem of the linear operator $E = DS_t$ which is the Fréchet derivative at zero of the evolution operator S_t. What we wish to know is whether E has a real positive eigenvalue [redacted] greater than 1 or not. If this is the case, the corresponding Couétte flow must be unstable, or we define that it is unstable.

This problem can be reduced to investigate whether the eigenvalue problem

$$(5.1) \qquad -Au - P\nabla_u w - P\nabla_w y = \lambda u$$

admits real positive eigenvalues or not. It is easy to verify that (5.1) is equivalent to

$$(5.2) \qquad \Delta u - \nabla_u w - \nabla_w u - \operatorname{grad} p = \lambda u, \qquad \operatorname{div} u = 0$$

with a suitable scalar function p. We adopt the cylindrical corrdinate system (r, ϕ, z), where $u = (u_r, u_\phi, u_z) = u_r \frac{\partial}{\partial r} + \frac{u_\phi \partial}{r\partial_\phi} + u_z \frac{\partial}{\partial z}$ $\frac{u_\phi}{r}\frac{\partial}{\partial\phi}$ and the Couétte flow w is expressed as $w = (w_r, w_\phi, w_z) = (A + \frac{B}{r^2})\frac{\partial}{\partial\phi}$.

We recall that

$$A = (R_2^2\Omega_2 - R_1^2\Omega_1)/(R_2^2 - R_1^2), \quad B = R_1^2R_2^2(\Omega_1 - \Omega_2)/(R_2^2 - R_1^2).$$

In the cylindrical coordinate system, (5.2) is expressed as

$$
(5.3)\quad \begin{cases}
(\Delta - \frac{1}{r^2})u_r - \frac{2}{r^2}\frac{\partial \quad _\phi}{\partial \phi} - \mathcal{N}(w)u_r + \frac{w_\phi u_\phi}{r} - \mathcal{N}(u)w_r + \frac{u_\phi w_\phi}{r} - \frac{\partial}{\partial r} = \lambda u_r \\
(\Delta - \frac{1}{r_2})u - \frac{2}{r_2}\frac{\partial u_r}{\partial \phi} - \mathcal{N}(w)u_\phi - \frac{w_\phi u_r}{r} - \mathcal{N}(u)w_\phi - \frac{u_\phi w_r}{r} - \frac{1}{r}\frac{\partial}{\partial \phi} = \lambda u_\phi \\
u_z - \mathcal{N}(w)u_z - \mathcal{N}(u)w_z - \frac{\partial}{\partial z} = \lambda u_z \\
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0
\end{cases}
$$

where

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \mathcal{N}(v) = v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}.
$$

We seek u and p which are independent of the variable ϕ. Then (5.3) reduces to

$$
(5.4)\quad \begin{cases}
(\Delta - \frac{1}{r^2})u_r + 2(A + \frac{B}{r^2})u_\phi - \frac{\partial p}{\partial r} = \lambda u_r \\
(\Delta - \frac{1}{r^2})u_\phi - 2Au_r = \lambda u_\phi \\
\Delta u_z - \frac{\partial p}{\partial z} = \lambda u_z \\
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0
\end{cases}
$$

Introducing a stream function f by the relations $ru_r = \frac{\partial}{\partial z}(rf)$ and $ru_z = -\frac{\partial}{\partial r}(rf)$, we obtain from (5.4)

$$
(5.5)\quad \begin{cases}
L(L - \lambda)f + 2(A + \frac{B}{r^2})u_\phi = 0 \\
(L - \lambda)u_\phi - 2Af = 0
\end{cases}
$$

where we write $L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{1}{r^2}$. On f we impose an additional boundary condition that $f, \frac{\partial}{\partial r}f = 0$ for $r = R_1, R_2$.

We set $f(r,z) = f(r)\cos z$, $u_\phi(r,z) = u(r)\sin z$ with $\sigma = \frac{2\pi}{h}$. The boundary condition is reduced to the condition $f(R_i) = f'(R_i) = u(R_i)$, for $i = 1,2$. Hence we have the following system of linear ordinary differential equations

$$(5.6)\qquad \begin{cases} (\mathcal{D} - \sigma^2)(\mathcal{D}^2 - \sigma^2 - \lambda)f(r) + 2(A + \frac{B}{r^2})\sigma u(r) = 0 \\ (\mathcal{D} - \sigma^2 - \lambda)u(r) + 2Af(r) = 0 \\ f(R_i) = f'(R_i) = u(R_i) = 0, \quad i = 1,2 \end{cases}$$

where we write $\mathcal{D} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}$. In order to investigate the system (5.6), we consider the following two boundary value problems for ordinary differential operators $\mathcal{D} - \mu$ for $\mu \geqq 0$.

$$(\text{BVP-1})\qquad \begin{cases} (\mathcal{D} - \mu)g(r) = \quad (r), \quad r \in (R_1, R_2) \\ g(R_i) = 0, \quad i = 1,2. \end{cases}$$

$$(\text{VBP-2})\qquad \begin{cases} (\mathcal{D} - \mu_1)(\mathcal{D} - \mu_2)g(r) = \quad (r), \quad r \in (R_1, R_2) \\ g(R_i) = g'(R_i) = 0, \quad i = 1,2 \end{cases}$$

with $\mu, \mu_i \geqq 0$.

The next lemma is uselful.

<u>Lemma 4</u>. Let $G(r, r' ; \mu)$ and $H(r, r' ; \mu_1, \mu_2)$ be the Green

functions for (BVP-1) and (BVP-2) respectively. Then G and H are negative valued and positive valued respectively almost everywhere.

This lemma may be proved by an explicit construction of the kernels making use of the Bessel functions but it can be proved by repetition of elementary discussions. The details are left to the reader.

We now return to the system (5.6). From (5.6) we have

$$(5.7)\qquad f(r) = -\int_{R_1}^{R_2} H(r, r' ; \sigma^2 . \sigma^2 + \lambda) 2\sigma (A + \frac{B}{r'^2}) u(r') dr'$$

and

$$(5.8)\qquad u(r) = -\int_{R_1}^{R_2} G(r, r' ; \sigma^2 + \lambda) 2Af(r') dr'.$$

Hence we have

$$(5.9)\qquad f(r) = \int_{R_1}^{R_2} K(r, r' ; \lambda) f(r') dr',$$

where $K(r,s ; \lambda) = 4\sigma \int_{R_1}^{R_2} H(r,r' ; \sigma^2, \sigma^2 + \lambda) A(A + \frac{B}{r'^2}) G(r'.s ; \sigma^2, \lambda) ds.$

From (5.7), (5.8) we have

$$(5.10)\qquad f(r) = \int_{R_1}^{R_2} K(r,s ; \sigma, \lambda) f(s) ds.$$

Now the problem is reduced to that whether the integral operator K defined by the kernel $K(r,s ; \sigma,\lambda)$ has 1 as its eigenvalue or not. For that purpose the following lemma of Jentzsch (See Schmeidler [4]) is useful.

Lemma 5. Let $K(r,s)$ be a continuous kernel on the interval $[R_1, R_2]$ which is positive almost everywhere. Then the integral operator

$Kf(r) = \int_{R_1}^{R_2} K(r,s)f(s)ds$ has a positive eigenvalue.

What we have to do next is to investigate the signature of the

function $k(r) = A(A + \frac{B}{r^2}) = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2}\left[(R_2^2\Omega_2 - R_1^2\Omega_1) + \frac{R_1^2R_2^2(\Omega_1 - \Omega_2)}{r^2}\right]$

For a fixed $\omega = (\omega_1, \omega_2) \in S^1$ (the 1-sphere) we set $(\Omega_1, \Omega_2) =$ $\rho(\omega_1, \omega_2)$, $\rho \geqq 0$, and

$$\ell(r\ ;\ \rho,\omega) = \rho^2 \frac{R_2^2\omega_2 - R_1^2\omega_1}{R_2^2 - R_1^2}\left[(R_2^2\omega_2 - R_1^2\omega_1) + \frac{R_1^2R_2^2(\omega_1 - \omega_2)}{r^2}\right] \equiv \rho^2\tilde{\ell}(r;\omega)$$

Then the kernel K is expressed by

$$(5.12)\quad K(r,s\ ;\ \lambda) = \rho^2 4\sigma \int_{R_1}^{R_2} H(r,r'\ ;\ \sigma^2,\ \sigma^2+\lambda)\tilde{\ell}(r'\ ;\ \omega)K(r',s\ ;\ \sigma^2,\lambda)dr'$$

$$\equiv \rho^2 L(r,s\ ;\lambda,\omega).$$

Hence, making use of Lemma 4 and Lemma 5, we have

<u>Theorem 3.</u>

For every fixed $\omega = (\omega_1, \omega_2) \in S^1$, such that $(R_2^2\omega_2 - R_1^2\omega_1)(\omega_2 - \omega_1) > 0$ there exists $\rho(\omega) > 0$ such that for every $\rho > \rho(\omega)$ the corresponding Couétte flow is unstable under infinitesimal perturbations.

It suffices only to notice that the integral equation (5.10) is reduced to

$$f(r) = \rho^2 \int_{R_1}^{R_2} L(r,s\ ;\ \lambda,\omega)f(s)ds.$$

6. Isolatedness of the Couétte flow.

What we are going to do in this section is to investigate whether or not 0 is an isolated solution of the equation

$$\text{(E)} \qquad Au - P(u\cdot\nabla)u + P(w\cdot\nabla)u + P(u\cdot\nabla)w = 0$$

where w is the Couétte flow. We take $(1 >)\ \gamma\ (> \frac{3}{4})$ and work in $\mathcal{D}(A^{\gamma})$. Assume that $0 \in \mathcal{D}(A^{\gamma})$ is not an isolated solution of (E). Then there exists solutions $u_n (\neq 0)$ of (E) for $n = 1,2,\cdots$ such that $\lim_{n\to\infty} \|A^{\gamma}u_n\| = 0$. Set $\phi_n = u_n / \|A^{\gamma}u_n\|$. Then we have

$$\text{(6.1)} \qquad A\phi_n + \|A^{\gamma}u_n\| P(\phi_n\cdot\nabla)\phi_n + P(w\cdot\nabla)\phi_n + P(\phi_n\cdot\nabla)w = 0.$$

As for the second term, we see that

$$\|P(\phi_n\cdot\nabla)\phi_n\| \leqq C_{\gamma}\|A^{\gamma}\phi_n\|\ \|A^{1/2}\phi_n\| \leqq C_{\gamma}\|A^{\frac{1}{2}-\gamma}\|\ \|A^{\gamma}\phi_n\|^{2} = C_{\gamma}\|A^{\frac{1}{2}-\gamma}\|$$

and so $\lim_{n\to\infty} \|A^{\gamma}u_n\|\ P(\phi_n\cdot\nabla)\phi_n = 0$. Hence

$$\lim_{n\to\infty} \{A\phi_n + P(w\cdot\nabla)\phi_n + P(\phi_n\cdot\nabla)w\} = 0 \quad \text{strongly in } L^2_{\sigma}.$$

And so by the boundedness of $A^{\gamma-1}$ we have

$$\text{(6.2)} \qquad \lim_{n\to\infty} \{A^{\gamma}\phi_n + A^{\gamma-1}[P(w\cdot\nabla)\phi_n + P(\phi_n\cdot\nabla)w]\} = 0.$$

By the compactness of the inclusions $\mathcal{D}(A^{\gamma}) \to \mathcal{D}(A^{1/2}) \to L^2_{\sigma}$ and the equality $\|\nabla\varphi\| = \|A^{1/2}\varphi\|$ for $\varphi \in \mathcal{D}(A^{1/2})$ we see that $\lim_{n} [P(w\cdot\nabla)\phi_n + P(\phi_n\cdot\nabla)w]$ exists strongly in L^2_{σ}, and this implies that $\phi_{\infty} = \lim_{n\to\infty} \phi_n$ exists strongly in $\mathcal{D}(A^{\gamma})$. Hence we have

$$(6.3)\quad \begin{cases} A^{\gamma}\phi_{\infty} + A^{\gamma-1}[P(w\cdot\nabla)\phi_{\infty} + P(\phi_{\infty}\cdot\nabla)w] = 0 \\ \\ \|A^{\gamma}\phi_{\infty}\| = 1. \end{cases}$$

This implies that the operator B defined by $B\varphi = -A^{-1}[P(w\cdot\nabla)\varphi + P(\varphi\cdot\nabla)w]$ in L^2_{σ} with domain of definition $\mathcal{D}(A^{1/2})$ has 1 as its eigenvalue.

We now ~~remember~~ recall the explicit form of w.

$$w = \frac{w_{\phi}}{r}\frac{\partial}{\partial\phi} = \rho\,\frac{1}{R_2^2-R_1^2}\,[R_2^2\omega_2 - R_1^2\omega_1 + \frac{R_1^2R_2^2(\omega_1-\omega_2)}{r^2}]\frac{\partial}{\partial\phi}$$

where $\rho \geqq 0$, $\omega = (\omega_1, \omega_2) \in S^1$. We set $w = \rho W_{\omega}$. Then $B\varphi = -\rho A^{-1}[P(W_{\omega}\cdot\nabla)\varphi + P(\varphi\cdot\nabla)W_{\omega}]$. By the compactness of the operator A^{-1} and the equality $\|\nabla\varphi\| = \|A^{1/2}\varphi\|$ for $\varphi \in \mathcal{D}(A^{1/2})$ we can prove that B can be extended to the whole space L^2_{σ} and the resulting operator is a compact operator. Hence we have

<u>Theorem 4</u>. The Couétte flow is an isolated solution in $W_2^{\gamma}(G_h)$ $(\gamma > \frac{3}{4})$ for almost all $\Omega = (\Omega_1, \Omega_2) \in R^2$.

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