# Nonexponential decay law of unstable multilevel quantum-systems at long times

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不安定多準位系における生存確率 $S(t)$ の長時間での崩壊様式を $N$ 準位 Friedrichs モデルに基づき解析する．不安定多準位上にまたがった任意の初期状態に対し，$S(t)$ の長時間での漸近形を求め，それが初期状態の不安定準位の占有の仕方にどのように依存するかを明らかにする．このとき特に $S(t)$ の漸近形を最大化する初期状態が存在することを指摘する．一方その漸近展開の第一項目を消してしまう特別な初期状態も存在する．この場合，従来知られている崩壊則よりも速い崩壊則が得られることが期待される．そこで実際に水素原子からの光子の自然放出過程を例に，この速い崩壊則を含む崩壊様式の初期状態依存性を数値的に確認する．

## 1 Introduction

In the beginning, the study on the unstable systems was aimed at the explanation of the exponential-decay law for the radioactive decay, the spontaneous emission from the atoms, and so forth. On the other hand, in the middle of the last century, the possible deviation from the exponential-decay law was pointed out both for short times and for long times [1]. In a recent experiment, a short-time deviation was successfully observed, while the long-time deviation has still not been detected, even though expected for a variety of unstable systems which have a continuum of the lower-bounded energy spectrum. The traditional study on the long-time deviation often resorts to the single lowest-level approximation (SLA) of the atoms, and it could be verified as long as that level is quite separate from the higher ones. However, the multilevel treatment of the study has a possibility of another advantage: the choice of coherently superposed initial-states extending over various levels. Such multilevel effects on the temporal behavior are still not well studied, and much less examined with respect to nonexponential decay at long times. Recently, we however examined such a long-time behavior of the survival probability $S(t)$, incorporating the initial-state dependence, based on the N-level Friedrichs model, and clarified how the asymptotic form of $S(t)$, that follows a power-decay law, depends on the initial states [2]. In this study, we also numerically confirm these results for $S(t)$ by considering the spontaneous emission process for the hydrogen atom interacting with the electromagnetic (EM) field. We demonstrate the $t^{-4}$-decay of $S(t)$ theoretically obtained in [3] and a faster decay pointed out in [2]. The latter is estimated like $t^{-8}$ as a power-decay law. The analytic results herein are owning to Ref. [2].

## 2 Friedrichs model and the long-time behavior of $S(t)$

The $N$-level Friedrichs model describes the couplings between the discrete spectrum and the continuous spectrum. The model Hamiltonian is defined by $H = H_0 + \lambda V$, where $H_0 = \sum_{n=1}^{N} \omega_n |n\rangle\langle n| + \int_0^\infty d\omega\ \omega|\omega\rangle\langle\omega|$ and $\lambda V = \lambda \sum_{n=1}^{N} \int_0^\infty d\omega\ [v_n^*(\omega)|\omega\rangle\langle n| + v_n(\omega)|n\rangle\langle\omega|]$. They denote the free and the interaction Hamiltonian with coupling constant $\lambda$, respectively. The eigenvalues $\omega_n$ of $H_0$ were supposed not to be degenerate, i.e., $\omega_n < \omega_{n'}$ for $n < n'$. Both $|n\rangle$ and $|\omega\rangle$ are the bound and scattering eigenstates of $H_0$, respectively, and compose the completely orthonormal system. $v_n(\omega)$ denotes the form factor characterizing the transition between $|n\rangle$ and $|\omega\rangle$. In the latter discussion, we analyze the model with the assumption that the form factor $v_n(\omega)$ is analytic in a complex domain including the cut $(0, \infty)$, square integrable, and behaves

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like $v_n(\omega) \simeq q_n \omega^{p_n}$ as $\omega \to 0$, where $p_n$ is a positive constant while $q_n$ is an appropriate one. These conditions are satisfied by several systems involving the spontaneous emission process of photons and the photodetachment process of electrons The initial unstable-state $|\psi\rangle$ of our interest is an arbitrary superposition of $|n\rangle$,

$$|\psi\rangle = \sum_{n=1}^{N} c_n |n\rangle, \tag{1}$$

where $c_n$'s are complex numbers satisfying the normalization condition $\sum_{n=1}^{N} |c_n|^2 = 1$. Then, the survival probability $S(t)$ of the initial state $|\psi\rangle$, that is, the probability of finding the initial state in the state at a later time $t$, is defined by $S(t) = |A(t)|^2$, where $A(t)$ denotes the survival amplitude of $|\psi\rangle$, i.e., $A(t) = \langle\psi|e^{-itH}|\psi\rangle$. The Hamiltonian $H$ in general has the possibility of possessing not only the scattering eigenstates $|\psi_\omega^{(\pm)}\rangle$, but also the bound eigenstates. However, the emitted particles detected in the decaying process are only brought from the initial component associated with the scattering eigenstates. We shall here confine ourselves to studying the decaying part of $A(t)$, denoted by the same symbols as

$$A(t) = \int_0^\infty d\omega |\langle\psi_\omega^{(\pm)}|\psi\rangle|^2 e^{-it\omega}. \tag{2}$$

In order to estimate the long-time behavior of $A(t)$, we need to obtain the scattering eigenstates $|\psi_\omega^{(\pm)}\rangle$. For this model, it is actually accomplished by solving the Lippmann-Schwinger equation, $|\psi_\omega^{(\pm)}\rangle = |\omega\rangle + \lambda(\omega \pm i0 - H_0)^{-1} V |\psi_\omega^{(\pm)}\rangle$. Then, we can estimate the low-energy behavior of $|\langle\psi_\omega^{(\pm)}|\psi\rangle|^2$ and evaluate the long-time behavior of $A(t)$ by using the asymptotic method for the Fourier integral,

$$A(t) = \lambda^2 \frac{\Gamma(2p+1)}{(it)^{2p+1}} |\langle\chi|\psi\rangle|^2 + o(t^{-2p-1}), \quad \text{as } t \to \infty, \tag{3}$$

where $i^{2p+1} = e^{i(2p+1)\pi/2}$ and $\Gamma(z+1) = \int_0^\infty dx x^z e^{-x}$. We have here introduced an auxiliary vector defined by

$$|\chi\rangle \equiv \sum_{n=1}^{N} f_n |n\rangle, \tag{4}$$

with $f_n = \tilde{q}_n/\omega_n + O(\lambda^2)$, where $\tilde{q}_n = q_n$ for $p_n = p$ or 0 for $p_n \neq p$, and $p = \min\{p_n\}$. With use of the $\tilde{q}_n$ instead of $q_n$, we extracted only the dominant part of $F_n^{(\pm)}(\omega)$ at small $\omega$. We clearly perceive $A(t) \sim t^{-2p-1}$, the power-decay law. It is also worth noticing that the dependence on the initial states surely appears in Eq. (3) through the factor $|\langle\chi|\psi\rangle|^2$.

## 3 The initial-states dependence of the long-time behavior of $S(t)$

In this section, we shall examine the long-time behavior of $S(t)$ with the various initial-states. Let us first consider the higher-level effects on the long-time behavior of $A(t)$ that starts from the localized initial state at the lowest level. For such an initial state, i.e., $c_n = \delta_{n1}$, $|\langle\chi|\psi\rangle|^2 = (|q_1|^2/\omega_1^2)[1 + O(\lambda^2)] + o(t^{-2p-1})$, where we supposed that $\tilde{q}_1 \neq 0$. It is worth noting that there are no factors related to the higher levels explicitly, which implies that the long-time asymptotic behavior of $A(t)$ could agree with that in the SLA (i.e., $N = 1$) for a sufficiently small $\lambda$. We can also find a special superposition of discrete states $|n\rangle$ that maximizes the asymptotic form of $A(t)$ at long times. With resort to the Schwarz inequality, we see that the maximum of the factor $\langle\chi|\psi\rangle$ is just attained by if and only if $|\psi\rangle = c|\chi\rangle/\|\chi\|$, where $c$ is an arbitrary complex number with $|c| = 1$. Therefore, preparing the initial state $|\psi\rangle$ parallel to $|\chi\rangle$, we can maximize the asymptotic form of $A(t)$ at long times. On the other hand, there are another kind of initial states that are coherently superposed to eliminate the factor $\langle\chi|\psi\rangle$, which is realized by the initial states orthogonal to $|\chi\rangle$, i.e., $\langle\chi|\psi\rangle = 0$. In this case, the first term in the right-hand side of Eq. (3) becomes zero. This fact may imply that $A(t)$ for such an orthogonal state asymptotically decays faster than $t^{-2p-1}$. Before concluding this section, we point out that the initial state extended over discrete states $|n\rangle$ has the possibility of increasing the intensity of $A(t)$ more than a localized one would. This possibility may be justly recognized by remembering the hermiticity of the Hamiltonian which allows an repopulation process of the decayed state at a later time $t$.

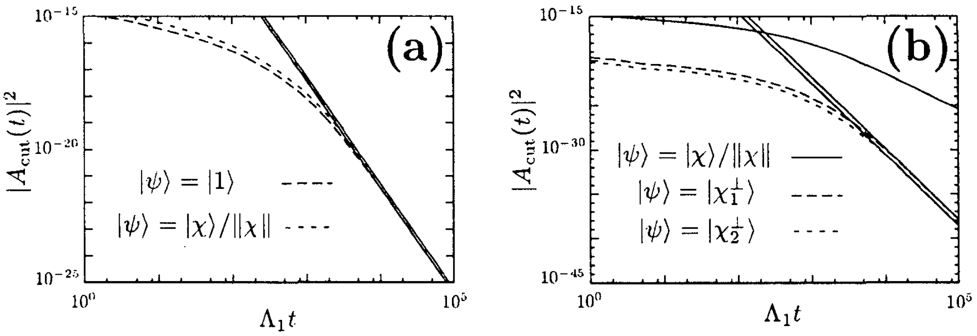


Figure 1: (a) $|A_{\rm cut}(t)|^2$ for the initial states $|\psi\rangle = |1\rangle$ and $|\chi\rangle/\|\chi\|$, and their corresponding asymptotes predicted by Eq. (3) (solid lines). (b) $|A_{\rm cut}(t)|^2$ for the initial states $|\psi\rangle = |\chi\rangle/\|\chi\|$, $|\chi_1^\perp\rangle$, and $|\chi_2^\perp\rangle$. For comparison, two straight lines parallel to $t^{-8}$ (solid lines) are also depicted.

## 4 An application to the excited states of the hydrogen atom

In order to illustrate our analysis, we consider the spontaneous emission process for the hydrogen atom interacting with the EM field [3]. We suppose that $|n\rangle = |(n+1)p\rangle \otimes |0\rangle$, where $|(n+1)p\rangle$ and $|0\rangle$ denote the $(n+1)p$-state of the atom and the vacuum state of the field respectively, and also $|\omega\rangle = |1s\rangle \otimes |1_\omega\rangle$, where $|1s\rangle$ and $|1_\omega\rangle$ denote the $1s$-state of the atom and the one-photon state respectively. In this case, an initially excited atom makes a transition to the ground state with the emission of a photon. We here choose only three excited levels: the $2p$ state, $3p$ state, and $4p$ state. Then, the form factors for the $2p-1s$, $3p-1s$, and $4p-1s$ transitions become [4],

$$\begin{aligned} v_1^*(\omega) &= i\Lambda_1^{-1/2}(\omega/\Lambda_1)^{1/2}[1+(\omega/\Lambda_1)^2]^{-2}, & (5)\\ v_2^*(\omega) &= i81\Lambda_1^{-1/2}(\omega/\Lambda_2)^{1/2}[1+2(\omega/\Lambda_2)^2][1+(\omega/\Lambda_2)^2]^{-3}/128\sqrt{2}, & (6)\\ v_3^*(\omega) &= i54\sqrt{3}\Lambda_1^{-1/2}(\omega/\Lambda_3)^{1/2}[45+146(\omega/\Lambda_3)^2+125(\omega/\Lambda_3)^4][1+(\omega/\Lambda_3)^2]^{-4}/15625, & (7)\end{aligned}$$

where $\Lambda_1 = 8.498 \times 10^{18}\ s^{-1}$, $\Lambda_2 = (8/9)\Lambda_1\ s^{-1}$, and $\Lambda_3 = (10/12)\Lambda_1\ s^{-1}$ are the cut-off constants. Note that these form factors have different forms, however all of them behave like $\omega^{1/2}$ at small energy. The other parameters are given by $\lambda^2 = 6.435 \times 10^{-9}$, $\omega_n = \frac{4}{3}\Omega[1-(n+1)^{-2}]$, and $\Omega = 1.55 \times 10^{16} s^{-1}$. As was emphasized in Ref. [3], these form-factors are surely analytic results without any approximation. The Hamiltonian $H$ is then derived within the four-level approximation and the rotating-wave approximation.

In the following, we shall compare the long-time asymptotic-form of $A(t)$ predicted by Eq. (3) and that of $A_{\rm cut}(t)$, the latter of which we evaluate numerically. $A_{\rm cut}(t)$ is defined by

$$A_{\rm cut}(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} \langle\psi|(H-z)^{-1}|\psi\rangle e^{-izt} dz. \tag{8}$$

The contour $\mathcal{C}$ runs clockwise around the half line $\{re^{7\pi i/4}|0 \le r < \infty\}$ in the complex energy plane. This contour lies on the first Riemann sheet when it goes below the half line, and gets into the second Riemann sheet when it above the half line. $A_{\rm cut}(t)$ is related to $A(t)$ through the equation, $A(t) = A_{\rm cut}(t) - \sum_{z_p} \mathrm{Res}\,(\langle\psi|(H-z)^{-1}|\psi\rangle e^{-izt}, z_p)$, where $z_p$ is in general the complex pole of $\langle\psi|(H-z)^{-1}|\psi\rangle$ located in the region between the half lines $[0,\infty)$ and $\{re^{-\pi i/4}|0 \le r < \infty\}$ in the second Riemann sheet. In the weak-coupling case considered here, each of $z_p$ is in the neighborhood of $\omega_n$, and thus the asymptotic form of $A_{\rm cut}(t)$ and that of $A(t)$ are expected to exhibit the same behavior at long times, when the power decay dominates over the exponential decay [5]. Let us first restrict ourselves to the two initial states: the localized state at the $2p$ level $|1\rangle$ and the maximizing state $|\chi\rangle/\|\chi\|$. Figure 1 (a) shows the time evolution of $|A_{\rm cut}(t)|^2$ and the asymptote of $|A(t)|^2$ for these initial-states. It is clearly seen that $|A_{\rm cut}(t)|^2$'s for these initial states approach to the corresponding asymptotes of $|A(t)|^2$ parallel to $t^{-4}$, however the difference between them is very small [2]. At $t = 10^5\Lambda_1^{-1}$, we obtain $|A_{\rm cut}(t)/A_{\rm asp}(t)|^2 \simeq 0.999$ for these initial states. Is is worth stressing that this time is very earlier than $1/\gamma_1 \simeq 1.36 \times 10^{10}\Lambda_1^{-1}$ the lifetime of the $2p$ state [3], where $\gamma_1 = 2\pi\lambda^2|v_1(\omega_1)|^2 + O(\lambda^4) \simeq 6.268 \times 10^8 s^{-1}$ [3]. We next choose the two special states, $|\chi_1^\perp\rangle$ and $|\chi_2^\perp\rangle$, as an initial state $|\psi\rangle$. They are defined by

$$|\chi_1^\perp\rangle = C_1[f_2^*|1\rangle - f_1^*|2\rangle], \quad |\chi_2^\perp\rangle = C_2[f_1 f_3^*|1\rangle + f_2 f_3^*|2\rangle - (|f_1|^2 + |f_2|^2)|2\rangle], \tag{9}$$

where $C_1$ and $C_2$ are the normalization constants. Then, the relations that $\langle\chi|\chi_1^\perp\rangle = 0$, $\langle\chi|\chi_2^\perp\rangle = 0$, and $\langle\chi_1^\perp|\chi_2^\perp\rangle = 0$ are satisfied. Figure 1 (b) shows that the time evolution of $|A_{\rm cut}(t)|^2$ for these initial states and for the maximizing initial state $|\chi\rangle$. We clearly find that, as was seen in Fig. 1 (a), $|A_{\rm cut}(t)|^2$ for $|\chi\rangle$ asymptotically decays like $t^{-4}$ (solid curve), whereas $|A_{\rm cut}(t)|^2$ for other initial state follow another decay-law faster than $t^{-4}$ (long-dashed and short-dashed curves). They seem to be fitted with the power law $t^{-8}$. For a comparison, we also depict in Fig. 1 (b) the two straight lines $150.0 \times (\Lambda_1 t)^{-8}$ and $30.0 \times (\Lambda_1 t)^{-8}$ (solid lines), to which $|A_{\rm cut}(t)|^2$ for the initial state $|\chi_1^\perp\rangle$ and $|\chi_2^\perp\rangle$ approach respectively in this time region.

# 5 Concluding remarks

We have considered the long-time behavior of the unstable multilevel systems and examined the asymptotic behavior of $S(t)$ for an arbitrary initial state in the long-time region, where $S(t)$ obeys a power-decay law. In particular, we have also discovered two kinds of special initial-state. One of them maximizes the asymptotic form of $S(t)$ at long times. The other initial state eliminates the first term of the asymptotic expansion of $S(t)$. We numerically confirm the previous results for $S(t)$ [2] in consideration of the spontaneous emission process for the hydrogen atom. Then, we find not only the $t^{-4}$-decay of $S(t)$ but also a faster decay, which is fitted by a power-decay law $t^{-8}$. These results mean that the long-time behavior is determined by not only the small-energy behavior of the form factors but also the initial unstable-states. Such relations between the initial states and the power decay law were already studied with respect to the asymptotic behavior of wave packets for finite-range potential systems [6].

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