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ON THE NUMERICAL SOLUTIONS OF HARMONIC, BIHARMONIC AND SIMILAR EQUATIONS BY THE DIFFERENCE METHOD NOT THROUGH SUCCESSIVE APPROXIMATIONS

BY

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Errata

Page 7,	line 6,	for	$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ (printed "lin")	read	$\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$
〃 9,		〃	Fig. 5	〃	Fig. 6
〃 11,	line 8,	〃	t_2,	〃	t_1,
〃 15,	〃 11,	〃	Coefficients	〃	coefficients
〃 18,	〃 1,	〃	numbers	〃	coefficients
〃 20,	〃 3,	〃	$\{(2-\lambda h^2)-2\}$	〃	$\{(2-\lambda h^2)^2-2\}$

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Contents

1. Introduction

To calculate the harmonic, biharmonic and similar functions by the finite difference method, it is difficult, in many cases, to solve the algebraic linear equations deduced by it and in these cases successive approximations are used. This is known as the method of iteration or relaxation. But it is also troublesome and arduous to get exact values by this method which satisfy the equations. The author has tried to solve the algebraic linear equations deduced by the difference method not according to successive approximations. Since the finite difference method of itself is an approximate method, it seemed to be unimportant to solve the algebraic equations exactly. However, if we can solve them by only one course of computation and get the exact values, it is convenient to estimate the errors by the finite difference and we are released from the trouble of computing the same equations many times.

In the following the author explains a method to solve the above equations exactly on Laplace's and Poisson's equations, each of which is one of the simplest partial differential equation and the most important in engineering. After that, the method is developed to solve the biharmonic equation and eigenvalue problems.

Although this method is not always applicable to any problem, and difficult when the boundary conditions are complicated, in many cases it is easier than the iteration or relaxation method. In this paper the rectangular domains are mainly considered.

2. Laplace's and Poisson's equations

(1) Principles of the method

Since the Laplace equation is a special case of the Poission equation, the method is explained on the latter, that is

$$\nabla^2 w = f(x, y), \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad \cdots\cdots\cdots(21)$$

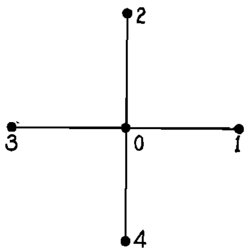

Fig. 1.

As well known, to determine a value which satisfies the above equation by finite difference, 5 points in the domain must be related as (Fig. 1)

$$4w_0 - w_1 - w_2 - w_3 - w_4 = C_0. \quad \cdots\cdots\cdots\cdots(22)$$

If n points are arranged in one row as Fig. 2, the difference equations in this case are as follows.

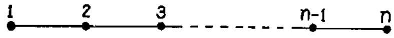

Fig. 2.

$$\begin{cases} 4w_1 - w_2 = C_1 \\ 4w_2 - w_1 - w_3 = C_2 \\ 4w_3 - w_2 - w_4 = C_3 \\ \quad \cdots\cdots \\ 4w_{n-1} - w_{n-2} - w_n = C_{n-1} \\ 4w_n - w_{n-1} = C_n. \end{cases} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(23)$$

The values C_j ($j=1, 2, 3\cdots\cdots n$) are known factors composed of the right-hand term of the original equation and the boundary values.

w_1 which satisfies the equation (23) can be expressed by the next formula.

$$a_{n+1}w_1 = a_nC_1 + a_{n-1}C_2 + \cdots\cdots + a_2C_{n-1} + a_1C_n, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(24)$$

$$\begin{cases} a_{j+1} = \kappa a_j - a_{j-1}, \\ a_1 = 1, \quad a_2 = \kappa = 4. \end{cases} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(25)$$

In general, the values w_j($j=1, 2, 3\cdots\cdots n$) are shown by the following matrix.

$$a_{n+1}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & a_{n-3}\cdots\cdots a_2 & a_1 \\ a_{n-1} & a_2a_{n-1} & a_2a_{n-2} & a_2a_{n-3}\cdots\cdots & a_2 \\ a_{n-2} & a_2a_{n-2} & a_3a_{n-3} & a_3a_{n-3}\cdots\cdots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots\cdots\cdots\cdots\cdots\cdots & a_n \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{pmatrix}. \qquad \cdots\cdots\cdots\cdots(26)$$

These results are easily deduced from the equation (23).

When there are $2n$ points in 2 rows as in Fig. 3, we can calculate the values w_{ij} ($i=1, 2$; $j=1, 2\cdots\cdots n$) like the above case, combining the values w_{1j} and w_{2j}. If we put

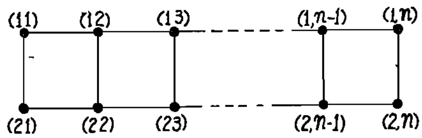

Fig. 3.

$$\begin{cases} u_j = w_{1j} + w_{2j} \\ \bar{u}_j = w_{1j} - w_{2j}, \end{cases} \qquad \cdots\cdots\cdots\cdots\cdots\cdots(27)$$

the same equations as (23) result for u_j and $\bar{u}_j$, instead of w_j, but the coefficients are different. For instance for the unknowns w_{11}, w_{21},

$$\begin{cases} 4w_{11} - w_{21} - w_{12} = C_{11} \\ 4w_{21} - w_{11} - w_{22} = C_{21}. \end{cases}$$

From these equations

$$3u_1-u_2=C_{11}+C_{21},\quad 5\bar{u}_1-\bar{u}_2=C_{11}-C_{21}. \qquad\cdots\cdots\cdots\cdots(28)$$

Now we can get the values u_j, $\bar{u}_j$ after we put the value κ as 3 and 5 into the formula (26) and replace the value C_j by $C_{1j}+C_{2j}$ and $C_{1j}-C_{2j}$. It is very easy to determine the values w_{ij} from the values u_j, $\bar{u}_j$ by the equation (27).

When the points are arranged in more than 2 rows, we can calculate the values w_{ij} in the same way as above, combining suitably the unknowns in one column. If the points are given in 3 rows, the values of κ are

$$\kappa=4,\quad 4-\sqrt{2},\quad 4+\sqrt{2},$$

and we can determine the values $\bar{u}_j$, t_j, $\bar{t}_j$ by the next equations.

$$\begin{cases} \bar{u}_j=w_{1j}-w_{3j} \\ t_j=w_{1j}+\sqrt{2}\,w_{2j}+w_{3j} \\ \bar{t}_j=w_{1j}-\sqrt{2}\,w_{2j}+w_{3j} \end{cases}$$

After $\bar{u}_j$, t_j, $\bar{t}_j$ were determined, it is very easy to get the velues $w_{ij}(i=1, 2, 3\,;\ j=1, 2,\cdots\cdots n)$ from them as in the case when there are 2 rows.

As seen from the equation (24), a_j can be expressed by κ and these relations are shown in Table 1. Expressing the number of rows by m, how to combine the unknowns to transform the equation into the form of equation (23) and what values of κ should be taken for in each case is shown in Table 2, when the rows are from 1 to 5. The values of $\kappa^{\nu}(\nu=1, 2, 3, 4\cdots\cdots)$ which are necessary to compute a_j by Table 1 are shown in Table 3.

Table 1.

α_1	1
α_2	κ
α_3	κ^2-1
α_4	$\kappa^3-2\kappa$
α_5	$\kappa^4-3\kappa^2+1$
α_6	$\kappa^5-4\kappa^3+3\kappa$
α_7	$\kappa^6-5\kappa^4+6\kappa^2-1$
α_8	$\kappa^7-6\kappa^5+10\kappa^3-4\kappa$
α_9	$\kappa^8-7\kappa^6+15\kappa^4-10\kappa^2+1$
α_{10}	$\kappa^9-8\kappa^7+21\kappa^5-20\kappa^3+5\kappa$

(2) Extension to infinite points arranged

When there are many points or n is large, the value of a_n becomes very great and the following considerations are available. From the equation (24),

$$w_1=\frac{a_n}{a_{n+1}}C_1+\frac{a_{n-1}}{a_{n+1}}C_2+\cdots\cdots+\frac{a_2}{a_{n+1}}C_{n-1}+\frac{a_1}{a_{n+1}}C_n. \qquad\cdots\cdots\cdots\cdots(29)$$

In this case, n is so large that a_{n+1} and a_n are far greater than a_2, a_1. That

Table 2.

m	κ	Combinations of the unknowns	
1	4	$w_{1j}=u_t$	$u_j=w_{1i}$
2	3	$u_j=w_{1j}+w_{2j}$	$w_{1j}=\frac{1}{2}(u_j+\bar{u}_j)$
	5	$\bar{u}_j=w_{1j}-w_{2j}$	$w_{2j}=\frac{1}{2}(u_j-\bar{u}_j)$
3	4	$\bar{u}_j=w_{1j}-w_{3j}$	$w_{1j}=\frac{1}{4}(t_j+\bar{t}_j)+\frac{1}{2}\bar{u}_j$
	$4-\sqrt{2}$	$t_j=w_{1j}+\sqrt{2}\,w_{2j}+w_{3j}$	$w_{2j}=\frac{1}{2\sqrt{2}}(t_j-\bar{t}_j)$
	$4+\sqrt{2}$	$\bar{t}_j=w_{1j}-\sqrt{2}\,w_{2j}+w_{3j}$	$w_{3j}=\frac{1}{4}(t_j+\bar{t}_j)-\frac{1}{2}\bar{u}_j$
4	$\frac{1}{2}(7-\sqrt{5})$	$t_j=w_{1j}+w_{4j}+\frac{1+\sqrt{5}}{2}(w_{2j}+w_{3j})$	$w_{1j}=\frac{1}{4}\left(\frac{-t_j+\bar{t}_j+s_j-\bar{s}_j}{\sqrt{5}}+t_j+\bar{t}_j+s_j+\bar{s}_j\right)$
	$\frac{1}{2}(7+\sqrt{5})$	$\bar{t}_j=w_{1j}+w_{4j}+\frac{1-\sqrt{5}}{2}(w_{2j}+w_{3j})$	$w_{2j}=\frac{1}{2\sqrt{5}}(t_j-\bar{t}_j+s_j-\bar{s}_j)$
	$\frac{1}{2}(9-\sqrt{5})$	$s_j=w_{1j}-w_{4j}-\frac{1-\sqrt{5}}{2}(w_{2j}-w_{3j})$	$w_{3j}=\frac{1}{2\sqrt{5}}(t_j-\bar{t}_j-s_j+\bar{s}_j)$
	$\frac{1}{2}(9+\sqrt{5})$	$\bar{s}_j=w_{1j}-w_{4j}-\frac{1+\sqrt{5}}{2}(w_{2j}-w_{3j})$	$w^4{}_j=\frac{1}{4}\left(\frac{-t_j+\bar{t}_j-s_j+\bar{s}_j}{\sqrt{5}}+t_j+\bar{t}_j-s_j-\bar{s}_j\right)$
5	3	$t_j=w_{1j}+w_{2j}-w_{4j}-w_{5j}$	$w_{1j}=\frac{1}{4}(t_j+\bar{t}_j)+\frac{1}{3}t'_j+\frac{1}{12}(s_j+\bar{s}_j)$
	4	$t'_j=w_{1j}-w_{3j}+w_{5j}$	$w_{2j}=\frac{1}{4}(t_j-\bar{t}_j)+\frac{1}{4\sqrt{3}}(s_j-\bar{s}_j)$
	5	$\bar{t}_j=w_{1j}-w_{2j}+w_{4j}-w_{5j}$	$w_{3j}=\frac{1}{6}(s_j+\bar{s}_j)-\frac{1}{3}t'_j$
	$4-\sqrt{3}$	$s_j=w_{1j}+\sqrt{3}\,w_{2j}+2w_{3j}+\sqrt{3}\,w_{4j}+w_{5j}$	$w_{4j}=\frac{1}{4\sqrt{3}}(s_j-\bar{s}_j)-\frac{1}{4}(t_j-\bar{t}_j)$
	$4+\sqrt{3}$	$\bar{s}_j=w_{1j}-\sqrt{3}\,w_{2j}+2w_{3j}-\sqrt{3}\,w_{4j}+w_{5j}$	$w_{5j}=\frac{1}{3}t'_j+\frac{1}{12}(s_j+\bar{s}_j)-\frac{1}{4}(t_j+\bar{t}_j)$

Table 3. The values of κ^{ν}.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
κ	3	4	5	$4\pm\sqrt{2}$	$4\pm\sqrt{3}$	$\frac{1}{2}(7\pm\sqrt{5})$	$\frac{1}{2}(9\pm\sqrt{5})$
κ^2	9	16	25	$18\pm8\sqrt{2}$	$19\pm8\sqrt{3}$	$\frac{1}{2}(27\pm7\sqrt{5})$	$\frac{1}{2}(43\pm9\sqrt{5})$
κ^3	27	64	125	$88\pm50\sqrt{24}$	$100\pm51\sqrt{3}$	$\frac{1}{2}(112\pm38\sqrt{5})$	$\frac{1}{2}(216\pm62\sqrt{5})$
κ^4	81	256	625	$452\pm288\sqrt{2}$	$553\pm304\sqrt{3}$	$\frac{1}{2}(487\pm189\sqrt{5})$	$\frac{1}{2}(1,127\pm387\sqrt{5})$
κ^5	243	1,024	3,125	$2,384\pm1,604\sqrt{2}$	$3,124\pm1,769\sqrt{3}$	$\frac{1}{2}(2,177\pm905\sqrt{5})$	$\frac{1}{2}(6,039\pm2,305\sqrt{5})$
κ^6	729	4,096	15,625	$12,744\pm8,800\sqrt{2}$	$17,803\pm10,200\sqrt{3}$	$\frac{1}{2}(9,882\pm4,256\sqrt{5})$	$\frac{1}{2}(32,938\pm13,392\sqrt{5})$
κ^7	2,187	16,384	78,125	$68,576\pm47,944\sqrt{2}$	$101,812\pm58,603\sqrt{3}$	$\frac{1}{2}(45,227\pm19,837\sqrt{5})$	$\frac{1}{2}(181,701\pm76,733\sqrt{5})$
κ^8	6,561	65,536	390,625	$370,192\pm260,352\sqrt{2}$	$583,057\pm336,224\sqrt{3}$	$\frac{1}{2}(207,887\pm92,043\sqrt{5})$	$\frac{1}{2}(1,009,487\pm436,149\sqrt{5})$
κ^9	19,683	262,144	1,953,125	$2,001,472\pm1,411,600\sqrt{2}$	$3,340,900\pm1,927,953\sqrt{3}$	$\frac{1}{2}(957,712\pm426,094\sqrt{5})$	$\frac{1}{2}(5,633,064\pm2,467,414\sqrt{5})$
κ^{10}	59,049	1,048,576	9,765,625	$10,829,088\pm7,647,872\sqrt{2}$	$19,147,459\pm11,052,712\sqrt{3}$	$\frac{1}{2}(4,417,227\pm1,970,185\sqrt{5})$	$\frac{1}{2}(31,517,323\pm13,919,895\sqrt{5})$

is to say, we may compute the several terms at the beginning of the right-hand side of the formula (29) and neglect other terms. This means that the values at the points far from the points w_1 do not influence the value of w_1.

If n is infinity, the coefficient of the first term at the right-hand side of the equation (29) takes the value,

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = k = 0.267949, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(210)$$

when $\kappa = 4$. This can be obtained easily from the relation (25). Since n is large,

$$\frac{a_{n+1-\gamma}}{a_{n+1}} = \frac{a_n}{a_{n+1}} \frac{a_{n-1}}{a_n} \cdots\cdots \frac{a_{n+1-\gamma}}{a_{n+2-\gamma}} \doteqdot k^{\gamma}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(211)$$

and we can simplify the formua (29) into the following form.

$$w_1 = kC_1 + k^2C_2 + k^3C_3 + \cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots(212)$$

In Fig. 2, the points are arranged in one row on a straight line but the equation (29) or (212) is applicable to every case in which the points are on any type of lines, provided the line does not cross over itself. When the many points are arranged on a ring, as Fig. 4, the values of a point on that ring is obtained by the next equation which is got from the formula (29).

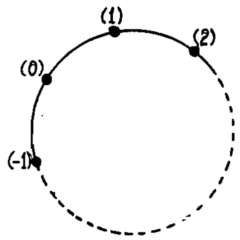

Fig. 4.

$$(a^2_{n+1} - a^2_n) w_0 = (a_{n+1}a_n + a_n a_1) C_0 + (a_{n+1}a_{n-1} + a_n a_2)(C_1 + C_{-1}) + \cdots\cdots \cdots(213)$$

Here C_0, C_1, C_{-1},$\cdots\cdots$ are the given values at each point. From the above equation, the coefficient of $C_{\gamma-1}$ is

$$k'_{\gamma} = \frac{a_n a_{n-\gamma+1} + a_n a_{\gamma}}{a_{n+1}{}^2 + a_n{}^2}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(214)$$

or by the relation (211),

$$k'_{\gamma} = \frac{k^{\gamma}}{1-k^2}, \qquad \cdots\cdots\cdots\cdots\cdots\cdots(215)$$

and the equation (213) is written as

$$w_0 = k'_1 C_0 + k'_2 (C_1 + C_{-1}) + k'_3 (C_2 + C_{-2}) + \cdots\cdots. \qquad \cdots\cdots\cdots\cdots(216)$$

We can determine by this formula the unknown values in an infinite line.

We have considered above the case when the points are arranged on only one row but this principle can be extended to cases of many rows as in the case of section (**1**).

(3) Applications

(i) An example shown as Fig. 5, is as follows. In this case $m=n=3$, so we must take κ as the value 4, $4\pm\sqrt{2}$ from Table 2 and calculate α_1, α_2, α_3, α_4 at first. As α_j are expressed by κ in Table 1, we can get these values at once. The values of the first column which satisfy Poisson's equation are

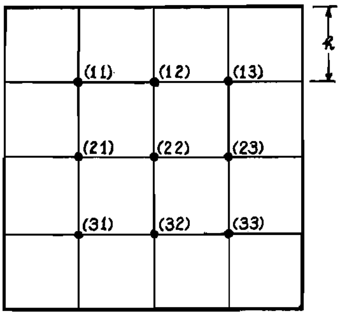

Fig. 5.

$$\alpha_4\xi_1=\alpha_3\theta_1+\alpha_2\theta_2+\alpha_1\theta_3, \cdots\cdots\cdots\cdots(217)$$

where ξ_1 represents one of $\bar{u}_1$, t_1, $\bar{t}_1$ and θ_1, θ_2, θ_3 are suitable combinations of given values C_{ij}. After we have got the values $\bar{u}_1$, t_1, $\bar{t}_1$ then w_{11}, w_{21}, w_{31} can be determined from the relations shown in the last column of Table 2. The values in the second column of the figure are also computed by the next formula as above.

$$\alpha_4\xi_2=\alpha_2\theta_1+\alpha_2^2\theta_2+\alpha_2\theta_3,\cdots\cdots\cdots(218)$$

or employing the next relation

$$\alpha_2^2=\alpha_3+\alpha_1,$$

the equation (218) is

$$\alpha_4\xi_2=\alpha_2\theta_1+(\alpha_3+\alpha_1)\theta_2+\alpha_2\theta_3.$$

$\bar{u}_2$, t_2, $\bar{t}_2$ can be determined from this formula.

The results are shown by Table 4* and in these tables each number is coefficient of known factor C_{ij}. If the values C_{ij} were given, we could get the values w_{ij} at once by them. The shape of this example is symmetrical, so we can get all

Table 4. The coefficients β_{ij} for Poisson's equation when $m=n=3$,
$\beta_0 w_{ij}=\sum\beta_{ij}C_{ij}$, $\beta_0=224$.

(11)

i \ j	1	2	3
1	67	22	7
2	22	14	6
3	7	6	3

(12)

i \ j	1	2	3
1	22	74	22
2	14	28	14
3	6	20	6

(22)

i \ j	1	2	3
1	14	28	14
2	28	84	28
3	14	28	14

* On Laplace's equation, similar nesults were already given by H. Liebmann.

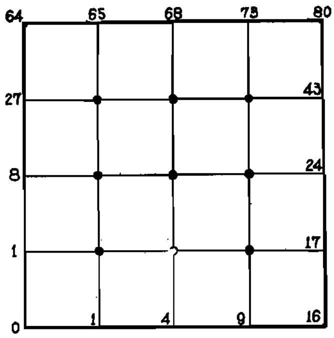

Fig. 5.

values from 3 tables for points (11), (12), (22) and tables for other points are not necessary.

This method should be compared with the relaxation method by this example. In Mr. G. Allen's book*, an example of the Laplace equation is shown. The boundary values of this example are as in Fig. 6 and the results by the relaxation method are shown in Table 5,(2). By the author's method, we can get the values of w_{11} as follows, using Table 4.

$$224w_{11}=67(65+27)+22(68+8)+7(73+43+1+1)+6(24+4)+3(17+9),$$

or

$$w_{11}=\frac{8908}{224}.$$

The values of other points are calculatable in this manner by the table, and the results are shown in Table 5.(1).

Table 5.

Point	(1)	(2)
(11)	$\frac{8,908}{224}=39.7679$	40
(12)	$\frac{10,264}{224}=45.8214$	46
(13)	$\frac{10,700}{224}=47.7679$	48
(21)	$\frac{4,760}{224}=21.2500$	21
(22)	$\frac{6,216}{224}=27.7500$	28
(23)	$\frac{6,552}{224}=29.2500$	29
(31)	$\frac{2,124}{224}=9.4821$	9
(32)	$\frac{3,288}{224}=14.6786$	15
(33)	$\frac{3,916}{224}=17.4821$	17

These calculations from the beginning are a little more troublesome than the relaxation method, but when the values shown in Table 4 were obtained beforehand, it is far easier by this method and we can get exact values. By the relaxation method, it is very

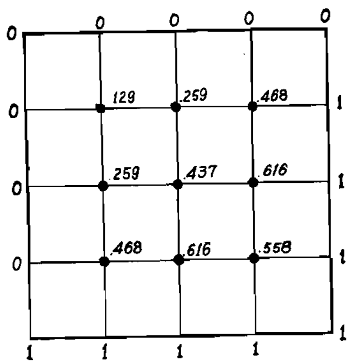

Fig. 7.

* D. N. de G. Allen, Relaxation Method, p. 57, 1954.

elaborate work to get exact values. Moreover Table 4 is available to the problems of different boundary values, provided the shape of the domain remains unchanged.

When the boundary values are given as shown in Fig. 7, the values of the inner points are determined as shown in the figure by the coefficients in Table 4.

(ii) The next example is similar to (i) but $m=4$ and $n=4$. In this case the values of κ are obtained from Table 2 as $\frac{1}{2}(7\pm\sqrt{5})$, $\frac{1}{2}(9\pm\sqrt{5})$ and the results are shown as Table 6.

Table 6. The cofficients β_{ij} for Poisson's equation,

$\beta_0 w_{ij}=\sum\beta_{ij}C_{ij}$, $\beta_0=6{,}600$.

(11)

$i \backslash j$	1	2	3	4
1	1,987	674	251	88
2	674	458	242	101
3	251	242	158	74
4	88	101	74	37

(12)

$i \backslash j$	1	2	3	4
1	674	2,238	762	251
2	458	916	559	242
3	242	409	316	158
4	101	162	138	74

(22)

$i \backslash j$	1	2	3	4
1	458	916	559	242
2	916	2,647	1,078	409
3	559	1,078	697	316
4	242	409	316	158

3. Biharmonic equations

(1) Principles of the method

We will treat in this paragraph the next differential equation,

$$\nabla^4 w = f(x,\ y). \qquad \cdots\cdots\cdots\cdots(31)$$

When the unknowns are arranged as shown in Fig. 2, we can see easily the solution of this equation by comparing it with the Poisson equation. From the solution of the latter by finite differences shown as the equation (24),

$$a^2_{n+1}\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_n & a_{n-1} & a_{n-2}\cdots\cdots a_2 & a_1 \\ a_{n-1} & \cdots\cdots\cdots\cdots\cdots\cdots & & a_2 \\ a_{n-2} & \cdots\cdots\cdots\cdots\cdots\cdots & & a_3 \\ \vdots & & & \vdots \\ a_1 & a_2 & a_3 \cdots\cdots a_{n-1} & a_n \end{pmatrix}^2 \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} \qquad \cdots\cdots\cdots\cdots(32)$$

Or we write briefly

$$\gamma_{n+1}\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \gamma_n & \gamma_{n-1} & \gamma_{n-2}\cdots\cdots\gamma_2 & \gamma_1 \\ \gamma'_n & \gamma'_{n-1} & \gamma'_{n-2}\cdots\cdots\gamma'_2 & \gamma'_1 \\ \gamma''_r & \gamma''_{n-1}\cdots\cdots\cdots\cdots & \gamma''_2 & \gamma_1'' \\ \vdots & & \vdots & \vdots \\ \gamma^{(n)}{}_n & \gamma^{(n)}{}_{n-1} \cdots\cdots\cdots & \gamma^{(n)}{}_2 & \gamma^{(n)}{}_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}. \qquad \cdots\cdots\cdots\cdots(33)$$

The coefficients γ_j are expressed by κ in Table 7 when $n=1$ to 5. When there are more than one row, the values w_{ij} can be obtained as in the case of the Poisson equation, combining the unknowns in one column.

Here the boundary conditions must be considered to make available the formula (32) or (33). In the case of bending problems of flat plate freely suppprted at the boundary, the original equation (31) should be rewritten as follows.

$$\nabla^2 w = v, \quad \nabla^2 v = f(x,\ y),$$

and on the boundaries $v=0$, $f(x,\ y)=0$, so (33) becomes directly the solution of the equation (31) if we consider C_{ij} as the values of $h^2 f(x,\ y)$ at each point.

However generally we must correct the formula (33) to be consistent with the boundary conditions. This is in some cases very easy but in others very difficult or troublesome. For the developement of this method, it is important to investigate how to simplify the conditions.

As mentioned above, the calculation of the supported plate is easiest, the author explains this problem as an example, at first, and then, the square plate which is supported at two edges and fixed at the other, and finally the square plate fixed at the every boundary.

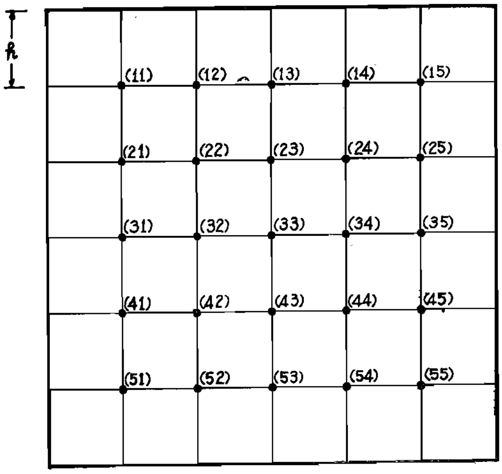

Fig. 8.

(2) Application

(i) The freely supported square plate

As shown by Fig. 8, 25 points are taken in the plate and $m=n=5$. Since $n=5$ in this case, from Table 7

$$
\left\{
\begin{aligned}
\gamma_6 &= \kappa^{10}+22\kappa^6-8\kappa^8-24\kappa^4+9\kappa^2\\
\gamma_5 &= \kappa^8-5\kappa^6+8\kappa^4-3\kappa^2+3\\
\gamma_4 &= 2\kappa^7-8\kappa^5+10\kappa^3\\
\gamma_3 &= 3\kappa^6-9\kappa^4+9\kappa^2-3\\
\gamma_2 &= 4\kappa^5-8\kappa^3\\
\gamma_1 &= 5\kappa^4-12\kappa^2+3.
\end{aligned}
\right.
$$

We must take here κ as 3, 4, 5, $4\pm\sqrt{3}$ because of $m=5$ and we can obtain

Table 7.

n	γ_j	$f(\alpha)$	$f(\kappa)$
1	γ_2	α_2^2	κ^2
	γ_1	α_1^2	1
2	γ_3	α_3^2	$\kappa^4-2\kappa^2+1$
	γ_2	$\alpha_2^2+\alpha_1^2$	κ^2+1
	γ_1	$2\alpha_1\alpha_2$	2κ
3	γ_4	α_4^2	$\kappa^6-4\kappa^4+4\kappa^2$
	γ_3	$\alpha_3^2+\alpha_2^2+\alpha_1^2$	$\kappa^4-\kappa^2+2$
	γ_2	$\alpha_2\alpha_3+\alpha_2^2+\alpha_2$	$2\kappa^3$
	γ_1	$2\alpha_1\alpha_3+\alpha_2^2$	$3\kappa^2-2$
4	γ_5	α_5^2	$\kappa^8-6\kappa^6+11\kappa^4-6\kappa^2+1$
	γ_4	$\alpha_4^2+\alpha_3^2+\alpha_2^2+\alpha_1^2$	$\kappa^6-3\kappa^4+3\kappa^2+2$
	γ_3	$\alpha_3\alpha_4+\alpha_2\alpha_3^2+\alpha_2^3+\alpha_2$	$2\kappa^5-4\kappa^3+4\kappa$
	λ_2	$\alpha_2\alpha_4++2\alpha_2^2\alpha_3+\alpha_3$	$3\kappa^4-3\kappa^2-1$
	γ_1	$2\alpha_4+2\alpha_2\alpha_3$	$4\kappa^3-6\kappa$
5	γ_6	α_6^2	$\kappa^{10}+22\kappa^6-8\kappa^8-24\kappa^4+9\kappa^2$
	γ_5	$\alpha_5^2+\alpha_4^2+\alpha_3^2+\alpha_2^2+\alpha_1^2$	$\kappa^8-5\kappa^6+8\kappa^4-3\kappa^2+3$
	γ_4	$\alpha_4\alpha_5+\alpha_2\alpha_4^2+\alpha_2\alpha_3^2+\alpha_2^3+\alpha_2^3+\alpha_2$	$2\kappa^7-8\kappa^5+10\kappa^3$
	γ_3	$\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_3^3+\alpha_3\alpha_2^2+\alpha_3$	$3\kappa^6-9\kappa^4+9\kappa^2-3$
	γ_2	$\alpha_2\alpha_5+2\alpha_2^2\alpha_4+\alpha_2\alpha_3^2+\alpha_4$	$4\kappa^5-8\kappa^3$
	γ_1	$2\alpha_5+2\alpha_2\alpha_4+\alpha_3^2$	$5\kappa^4-12\kappa^2+3$

n	γ'	$f(\gamma)$
3	γ'_3	γ_2
	γ'_2	$\gamma_3+\gamma_1$
	γ'_1	γ_2
4	γ_4'	γ_3
	γ_3'	$\gamma_4+\gamma_2$
	γ_2'	$\gamma_3+\gamma_1$
	γ_1'	γ_2
5	γ_5'	γ_4
	γ_4'	$\gamma_5+\gamma_3$
	γ_3'	$\gamma_4+\gamma_2$
	γ_2'	$\gamma_2+\gamma_1$
	γ_1'	γ_2

Table 8.

the coefficients for t_1, t'_1, $\bar{t}_1$, s_1, $\bar{s}_1$.

To get the values of the second column w_{i2}, we should use the second row of the matrix (33), but we must not calculate the values by κ again and the relation shown in Table 8 are available. So it is very easy also to get the values w_{i2} after we have determined the values t_2, t_1', $\bar{t}_1$, s_1, $\bar{s}_1$. The values of the third column w_{i3} are also revealed by the relations shown in Table 9. As the shape of the plate

n	γ''	$f(\gamma)$
5	γ_5''	γ_3
	γ_4''	$\gamma_4+\gamma_2$
	γ_3''	$\gamma_5+\gamma_3+\gamma_1$
	γ_2''	$\gamma_4+\gamma_2$
	γ_1''	γ_3

Table 9.

Table 10. The coefficient β_{ij} for the freely supperted plate when $m=n=5$, $w_{ij}=\sum\beta_{ij}h^4f_{ij}(x, y)$.

(11)

i \ j	1	2	3	4	5
1	0.127567	0.104265	0.069655	0.041679	0.019436
2	0.104265	0.116362	0.092475	0.060764	0.029656
3	0.069655	0.092475	0.082655	0.058413	0.029643
4	0.041679	0.060764	0.058413	0.043360	0.022625
5	0.019436	0.029656	0.029643	0.022625	0.011999

(12)

i \ j	1	2	3	4	5
1	0.104265	0.197222	0.145944	0.089091	0.041679
2	0.116362	0.196740	0.177126	0.122130	0.060764
3	0.092475	0.152310	0.150888	0.112298	0.058413
4	0.060764	0.100092	0.104124	0.081038	0.043360
5	0.029656	0.049079	0.052280	0.041642	0.022625

(13)

i \ j	1	2	3	4	5
1	0.069655	0.145944	0.216658	0.145944	0.069655
2	0.092475	0.177126	0.226396	0.177126	0.092475
3	0.082655	0.150888	0.181953	0.150888	0.082655
4	0.058413	0.104124	0.122717	0.104124	0.058413
5	0.029643	0.052280	0.061078	0.052280	0.029643

(22)

i \ j	1	2	3	4	5
1	0.116362	0.196740	0.177126	0.122130	0.060764
2	0.196740	0.349532	0.296832	0.201389	0.100092
3	0.177126	0.296832	0.281250	0.203168	0.104124
4	0.122130	0.201389	0.203168	0.153940	0.081038
5	0.060764	0.100092	0.104124	0.081038	0.043360

(23)

i \ j	1	2	3	4	5
1	0.092475	0.177126	0.226396	0.177126	0.092475
2	0.152310	0.296832	0.398611	0.296832	0.152310
3	0.150888	0.281250	0.349112	0.281250	0.150888
4	0.112298	0.203168	0.243031	0.203168	0.112298
5	0.058413	0.104124	0.122717	0.104124	0.058413

(33)

i \ j	1	2	3	4	5
1	0.082655	0.150888	0.181953	0.150888	0.082655
2	0.150888	0.281250	0.349112	0.281250	0.0150888
3	0.181953	0.349112	0.459689	0.349112	0.181953
4	0.150888	0.281250	0.349112	0.281250	0.0150888
5	0.082655	0.150888	0.181953	0.150888	0.082655

is symmetrical, it is not necessary to calculate the values of the fourth and the fifth column and the results are shown by Table 10.

(ii) The square plate freely supported at two edges and fixed at other edges

As shown as Fig. 9, on the freely supported edge we must put

$$w_0=0, \quad w_2=-w_1,$$

and on the fixed edge

$$w_0=0. \quad w_3=w_1.$$

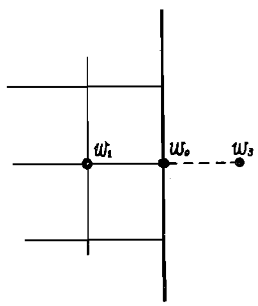

Fig. 9.

Then the difference of the boundary conditions between the supported edge and the fixed edge is $2w_1$. If we replace C_{1j} by $C_{1j}-2w_{1j}$ and C_{5j} by $C_{5j}-2w_{5j}$ in the case (i), we can get the coefficients of C_{ij} in this case.

$$
\left\{
\begin{array}{lll}
\gamma_6=\kappa^{10}-4\kappa^8+6\kappa^6+9\kappa^2+36, & & \\
\gamma_5=\kappa^8-3\kappa^6+4\kappa^4+3\kappa^2+15, & \gamma'_5=\gamma_4, & \gamma''_5=\gamma_3 \\
\gamma_4=2\kappa^7-4\kappa^5+6\kappa^3+12\kappa, & \gamma'_4=\kappa^8+2\kappa^6-\kappa^4+18\kappa^2+24, & \gamma''_4=\gamma'_3 \\
\gamma_3=3\kappa^6-3\kappa^4+9\kappa^2-9, & \gamma'_3=2\kappa^7+4\kappa^5+6\kappa^3+12\kappa, & \gamma_3''=\kappa^8+2\kappa^6+12\kappa^4 \\
 & & \qquad +6\kappa^2+27 \\
\gamma_2=4\kappa^5-12\kappa, & \gamma'_2=3\kappa^6+8\kappa^4-3\kappa^2-12, & \gamma''_2=\gamma'_3 \\
\gamma_1=5\kappa^4-12\kappa^2+3, & \gamma'_1=\gamma_2, & \gamma''_1=\gamma_3.
\end{array}
\right.
$$

By these relations we can get the following results (Table 11).

(iii) The fixed square plate

In the results of (ii) we should replace C_{i1} by $C_{i1}-2w_{i1}$ and C_{i5} by C_{i5}

Table 11. The Coefficients β_{ij} for the plate freely supported at the two edges and fixed at the other when $m=n=5$,

$$w_{ij}=\sum \beta_{ij} h^4 f_{ij}(x, y).$$

(11)

$i \backslash j$	1	2	3	4	5
1	0.088142	0.053154	0.024886	0.010479	0.003725
2	0.064658	0.058443	0.036463	0.018892	0.007684
3	0.037877	0.042813	0.031635	0.018511	0.008126
4	0.018809	0.023869	0.019330	0.012071	0.005511
5	0.006164	0.008153	0.006791	0.004301	0.001972

(12)

$i \backslash j$	1	2	3	4	5
1	0.053154	0.113028	0.063633	0.028611	0.010479
2	0.058443	0.101120	0.077335	0.044146	0.018892
3	0.042813	0.069512	0.061324	0.039761	0.018511
4	0.023869	0.038138	0.035940	0.024840	0.012071
5	0.008153	0.012956	0.012454	0.008763	0.004301

(13)

$i \backslash j$	1	2	3	4	5
1	0.024886	0.063633	0.116753	0.063633	0.024886
2	0.036463	0.077335	0.108405	0.077335	0.036463
3	0.031635	0.061324	0.077638	0.061324	0.031635
4	0.019330	0.035940	0.043650	0.035940	0.019330
5	0.006791	0.012454	0.014928	0.012454	0.006791

(21)

i \ j	1	2	3	4	5
1	0.064658	0.058443	0.036463	0.018892	0.007684
2	0.152620	0.126708	0.079955	0.043209	0.018293
3	0.105471	0.111283	0.080422	0.047177	0.020928
4	0.055507	0.067017	0.052811	0.032654	0.014902
5	0.018809	0.023869	0.019330	0.012071	0.005511

(22)

i \ j	1	2	3	4	5
1	0.058443	0.101120	0.077335	0.044146	0.018892
2	0.126708	0.232575	0.169917	0.098248	0.043209
3	0.111283	0.185893	0.158460	0.101350	0.047177
4	0.067017	0.108318	0.099671	0.067713	0.032654
5	0.023869	0.038138	0.035940	0.024340	0.012071

(23)

i \ j	1	2	3	4	5
1	0.036463	0.077335	0.108804	0.077335	0.036463
2	0.079955	0.169917	0.250868	0.169917	0.079955
3	0.080422	0.158460	0.206821	0.158460	0.080422
4	0.052811	0.099671	0.123220	0.099671	0.052811
5	0.019330	0.035940	0.043649	0.035940	0.019330

(31)

i \ j	1	2	3	4	5
1	0.037877	0.042813	0.031635	0.018511	0.008126
2	0.105471	0.111283	0.080422	0.047177	0.020928
3	0.174521	0.156995	0.106707	0.061259	0.027014
4	0.105471	0.111283	0.080422	0.047177	0.020928
5	0.037877	0.042813	0.031635	0.018511	0.008126

(32)

i \ j	1	2	3	4	5
1	0.042813	0.069512	0.061324	0.039761	0.018511
2	0.111283	0.185893	0.158460	0.101350	0.047177
3	0.156995	0.281227	0.218253	0.133720	0.061259
4	0.111283	0.185893	0.158460	0.101350	0.047177
5	0.042813	0.069512	0.061324	0.039761	0.018511

(33)

$i \backslash j$	1	2	3	4	5
1	0.031635	0.061324	0.077638	0.061324	0.031635
2	0.080422	0.158460	0.206821	0.158460	0.080422
3	0.106707	0.218253	0.308241	0.218253	0.106707
4	0.080422	0.158460	0.206821	0.158460	0.080422
5	0.031635	0.061324	0.077638	0.061324	0.031635

$-2w_{i5}$ and the coefficients of the fixed plate can be obtained. When we replace the factor C_{i1} and C_{i5} as above, each equation has 10 unknowns but according to the symmetric law we can transform the equations containing only 3 unknowns and eliminate them. The results are shown by Table 12.

Table 12. The coefficients β_{ij} for the fixed plate when $m=n=5$, $w_{ij}=\sum\beta_{ij}h^4f_{ij}(x, y)$.

(11)

$i \backslash j$	1	2	3	4	5
1	0.069243	0.039208	0.016795	0.006033	0.001433
2	0.039208	0.035198	0.020687	0.009314	0.002634
3	0.016795	0.020687	0.015044	0.007786	0.002374
4	0.006033	0.009314	0.007786	0.004349	0.001333
5	0.001433	0.002634	0.002374	0.001333	0.000358

(12)

$i \backslash j$	1	2	3	4	5
1	0.039208	0.101412	0.055575	0.022658	0.006033
2	0.035804	0.079089	0.060313	0.031158	0.009708
3	0.021404	0.046438	0.042246	0.024868	0.008291
4	0.009708	0.021923	0.022024	0.013832	0.004638
5	0.002634	0.006563	0.006958	0.004392	0.001333

(13)

$i \backslash j$	1	2	3	4	5
1	0.016795	0.055575	0.109240	0.055575	0.016795
2	0.021404	0.060588	0.092446	0.060588	0.021404
3	0.015940	0.042604	0.058856	0.042604	0.015940
4	0.008291	0.022229	0.029704	0.022229	0.008291
5	0.002374	0.006957	0.009357	0.006957	0.002374

(22)

i \ j	1	2	3	4	5
1	0.035197	0.079089	0.060587	0.031158	0.009315
2	0.079089	0.184960	0.131917	0.068453	0.021923
3	0.060587	0.131917	0.113621	0.065769	0.022229
4	0.031158	0.068453	0.065769	0.040591	0.013832
5	0.009315	0.021923	0.022229	0.013832	0.004348

(23)

i \ j	1	2	3	4	5
1	0.020687	0.060313	0.092446	0.060313	0.020687
2	0.046437	0.131917	0.213438	0.131917	0.046437
3	0.042603	0.113773	0.162095	0.113773	0.042603
4	0.024868	0.065769	0.089017	0.065769	0.024868
5	0.007786	0.022024	0.029704	0.022024	0.007786

(33)

i \ j	1	2	3	4	5
1	0.015044	0.042245	0.058856	0.042245	0.015044
2	0.042245	0.113621	0.162095	0.113621	0.042245
3	0.058856	0.162095	0.252132	0.162095	0.058856
4	0.042245	0.113621	0.162095	0.113621	0.042245
5	0.015044	0.042245	0.058856	0.042245	0.015044

The values shown in above tables are considered to be influence numbers of the plate for deflections or they express the deflections of every points when at a point, a unit load is applied. We can get the deflections of the plate under any given load by these tables.

(iv) Above examples are all square plates, so an example of a rectangular plate when $m=3$, $n=4$ should be shown. Since the procedures are the same as in the above cases, the results alone will be shown by Table 13.

Table 13. The coefficients β_{ij} for the freely supported plate when $m=3$, $n=4$, $w_{ij}=\sum \beta_{ij} h^4 f_{ij}(x, y)$.

(11)

i \ j	1	2	3	4
1	0.119198	0.089855	0.052242	0.023295
2	0.086979	0.087504	0.059292	0.028475
3	0.041864	0.048464	0.035782	0.017984

(12)

$i \backslash j$	1	2	3	4
1	0.089855	0.171440	0.113150	0.052242
2	0.087504	0.146271	0.115979	0.059292
3	0.048464	0.077646	0.066448	0.035782

(21)

$i \backslash j$	1	2	3	4
1	0.086979	0.087504	0.059292	0.028475
2	0.161062	0.138319	0.088025	0.041279
3	0.086979	0.087504	0.059292	0.028475

(22)

$i \backslash j$	1	2	3	4
1	0.087504	0.146271	0.115978	0.059292
2	0.138319	0.249086	0.179598	0.088025
3	0.087504	0.146271	0.115978	0.059292

4. Eigenvalue problems

The principles of the calculations are mentioned by examples.

(1) A differential equation related to the problems of the transverse vibration of a stretched string is

$$\frac{d^2w}{dx^2}+\lambda w=0.$$

By the difference method the above equation is

$$(2-\lambda h^2)w_0-w_1-w_3=0.$$

When we take 3 points in the string as Fig. 10 and assume $w=0$ at the ends, from the formula (26),

$$a_4\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}=0,$$

or $$a_4=0.$$

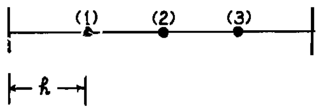

Fig. 10.

a_4 is expressed by κ in Table 1 and then

$$\kappa^3 - 2\kappa = 0.$$

We must take κ as $(2-\lambda h^2)$ in this case and the above equation is

$$(2-\lambda h^2)\{(2-\lambda h^2)-2\}=0,$$

or $$(2-\lambda h^2)(2-4\lambda h^2+\lambda^2 h^4)=0.$$

Then $$\lambda h^2 = 2, \quad 2\pm\sqrt{2}.$$

These are the eigenvalues approximated by the difference method.

When we take 5 points in the string, we may put

$$a_6 = 0$$

From Table 1, $$\kappa^5 - 4\kappa^3 + 3\kappa = 0$$

Substituting $\kappa = 2-\lambda h^2$ into this equation

$$(2-\lambda h^2)\{(2-\lambda h^2)^2-1\}\{(2-\lambda h^2)^2-3\}=0.$$

The roots of this equation are

$$\lambda h^2 = 1,\ 2,\ 3,\ 2\pm\sqrt{3}.$$

The lowest value of λh^2 is $2-\sqrt{3}$ and if we take the length of the string as 1, then $h=1/6$. So

$$\lambda = \frac{2-\sqrt{3}}{(1/6)^2} = 9.65.$$

This is nearly equal to the correct value π^2. When the eigenvalues have been determined, it is very easy to get the corresponding modes.

(2) The next example is on the vibration of a stretched mambrane. The differential equation in this case is

$$\nabla^2 w + \lambda w = 0.$$

If we take 2×7 points in a rectangular membrane shown as Fig. 11, and $w=$

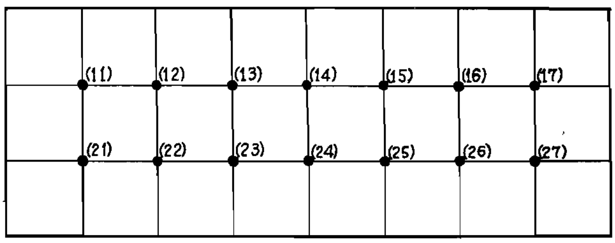

Fig. 11.

0 at the boundary,

$$m=2, \quad n=7.$$

Then from Table 1,

$$a_8=\kappa^7-6\kappa^5+10\kappa^3-4\kappa=0,$$

or

$$\kappa(\kappa^2-2)(\kappa^4-4\kappa^2+2)=0.$$

Because of $m=2$, the values of κ are two kinds, one is $3-\lambda h^2$ and another is $5-\lambda h^2$. We put these values into the above equation and get

$$\lambda h^2=3, \quad 3\pm\sqrt{2}, \quad 3\pm\sqrt{2\pm\sqrt{2}} \qquad \text{when} \quad u_j\neq 0, \ \bar{u}_j=0$$
$$\lambda h^2=5, \quad 5\pm\sqrt{2}, \quad 5\pm\sqrt{2\pm\sqrt{2}} \qquad \text{when} \quad u_j=0, \ \bar{u}_j\neq 0.$$

5. Conclusion

The author has developed a method to solve the partial differential equation by the difference method and shown some examples. Although the method is not yet sufficiently considered or studied in detail and there are some points to be corrected, it is believed that the method can be applied to many problems in engineering. The tables in this paper will be available to calculate the functions in practice. To make this method more useful, it is necessary to prepare more tables.

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