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"Pricing Fixed-Income Securities in an Information-Based Framework"

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Pricing Fixed-Income Securities in an Information-Based Framework

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Abstract

In this paper we introduce a class of information-based models for the pricing of fixed-income securities. We consider a set of continuous-time information processes that describe the flow of information about market factors in a monetary economy. The nominal pricing kernel is at any given time assumed to be given by a function of the values of information processes at that time. By use of a change-of-measure technique we derive explicit expressions for the price processes of nominal discount bonds, and deduce the associated dynamics of the short rate of interest and the market price of risk. The interest rate positivity condition is expressed as a differential inequality. We proceed to the modelling of the price-level, which at any given time is also taken to be a function of the values of the information processes at that time. A simple model for a stochastic monetary economy is introduced in which the prices of nominal discount bonds and inflation-linked notes can be expressed in terms of aggregate consumption and the liquidity benefit generated by the money supply.

Key words: Fixed-income securities, interest rate theory, inflation, inflation-linked securities, non-linear filtering, incomplete information.
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1 Introduction

The key idea of so-called information-based asset pricing (Macrina 2006, Brody *et al.* 2007, 2008a,b, Hughston & Macrina 2008) is that the market filtration should be modeled explicitly in such a way that it is generated by a set of processes that carry information about the future cash flows generated by tradable securities. In particular, one can regard each such cash flow as being a random variable that is in turn given by a function of one or more independent random variables called "market factors" or more succinctly "X-factors". The information processes that generate the market filtration are associated with the various X-factors in such a way that the value of each X-factor is revealed at some designated time by the associated information process. The simplest examples of information processes are those based on Brownian bridges (Brody *et al.* 2007, 2008a, Rutkowski & Yu 2007), and gamma bridges (Brody *et al.* 2008b), which lead to highly tractable asset pricing models; more general information processes can be constructed based on Lévy random bridges (Hoyle *et al.* 2009).

The purpose of the present paper is to present a simple class of information-based models for interest rates, foreign exchange, and inflation. The point of view that we take is the following. We retain the premise that the X-factors represent the fundamental factors, the values of which are revealed from time to time, that determine the cash flows generated by primary securities. We also accept the view that the market filtration is generated collectively by the information processes associated with these factors. In a macroeconomic setting with a dynamic equilibrium, it is appropriate to assume the existence of a universal pricing kernel associated with the choice of a suitable base currency. We shall call this the nominal pricing kernel associated with the given base currency. The pricing kernel is necessarily adapted to the market filtration, and therefore can be given in the present context as a functional of the trajectories of the information processes up to the time at which the value of the pricing kernel is to be determined. A similar property holds for the pricing kernel associated with any other currency or unit of exchange. The models for interest rates and foreign exchange that we develop in the material that follows, are characterized by the following additional assumptions, namely: (a) that the information processes collectively have the Markov property with respect to the market filtration, and (b) that the pricing kernels associated with each currency under consideration can at any given time be expressed as a function of the values taken by the information processes at that time.

In the case of models for inflation, we take a similar point of view, adapting the so-called "foreign exchange analogy" (Hughston 1998, Jarrow & Yildirim 2003, Mercurio 2005, Brody *et al.* 2008, Hinnerich 2008). In this scheme the price level is given by the ratio of the real and the nominal pricing kernels. These in turn are given, in the models developed in the present paper, by functions of the current levels of the relevant information processes.

2 Nominal discount bonds in a one-factor model for interest rates

For simplicity we consider first the somewhat restrictive but nevertheless instructive case of of an economy with a single X-factor and a single information process. The resulting theory can be worked out rather explicitly, and from this example one can then see how the general case can be approached when there are several currencies and many X-factors. In the single-factor case we proceed as follows.

The market will be modelled by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We assume that $\mathbb{P}$ is the "real" probability measure, and that $\{\mathcal{F}_t\}$ is the market filtration. The filtration will be modelled in the following manner. Let time 0 denote the present, and fix a time $U > 0$. We introduce a continuous random variable X_U taking values in $\mathbb{R}$, with probability density $p(x)$. The restriction to a continuous random variable is purely for convenience. With this "X-factor" we associate an information process $\{\xi_{tU}\}$ defined by

$$\xi_{tU} = \sigma t X_U + \beta_{tU}. \tag{2.1}$$

Here σ is an information flow-rate parameter, and the Brownian bridge process $\{\beta_{tU}\}_{0\leq t\leq U}$ is taken to be independent of the market factor X_U. As remarked in Brody *et al.* 2007 (see also Rutkowski & Yu 2007) it is a straightforward exercise making use of well-known properties of the Brownian bridge to show that the process $\{\xi_{tU}\}_{t\geq 0}$ has the Markov property with respect to its own filtration. We shall assume that $\{\xi_{tU}\}$ generates the market filtration, and hence that $\{\mathcal{F}_t\}$ embodies all information available to market participants. Hence for each $t \in [0, U]$ the sigma-algebra $\mathcal{F}_t$ is defined by

$$\mathcal{F}_t = \sigma\left(\{\xi_{sU}\}_{0\leq s\leq t}\right). \tag{2.2}$$

It should be evident in this model that U acts as a kind of "sunset" for the economy, that there is only one piece of information to be revealed, and once it has been revealed then that is, so to speak, the end of the story. This is of course an artifact of the simplicity of our assumptions, and in a more realistic model we can expect the revelation of X-factors proceeding indefinitely into the future, the more distant ones being, generally speaking, less important than the nearer ones.

The pricing kernel $\{\pi_t\}$ will be assumed to be given by a positive function that depends on the time t and the value of the information process at time t. Thus, we have

$$\pi_t = F(t, \xi_{tU}). \tag{2.3}$$

Given the pricing kernel, we can proceed to work out the price processes of various assets. In the simple economy under consideration, the "primary" assets are those that deliver a single cash flow at time U given by an itegrable function $H(X_U)$ that depends on the outcome X_U. The value of such a security at time $t \leq U$ is given by

$$H_t = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}} \left[\pi_U H(X_U) \,|\, \mathcal{F}_t \right], \tag{2.4}$$

where $\mathcal{F}_t$ is the sigma-algebra generated by $\{\xi_{sU}\}_{0 \leq s \leq t}$. For each choice of $H(X_U)$ we obtain a tradable security. We can also consider the discount-bond system associated with the given pricing kernel. Let us write P_{tT} for the price at time t of a bond that pays one unit of currency at time T for $t \leq T \leq U$. Then for each $T \in [0, U]$ we have:

$$P_{tT} = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}} \left[\pi_T \,|\, \mathcal{F}_t \right]. \tag{2.5}$$

Finally, we can consider various "derivative" assets. These deliver prescribed cash flows at one or more times in the interval $(0, U)$ in such a way that these cash flows are determined by the values of the basic assets and the discount bonds at various times. More generally we can also consider "information derivatives" for which the cash flows can depend in an essentially arbitrary way on the information available up to the time of the cash flow. For example, let the payoff of a security at time T be given by $G(T, \xi_{TU})$ where $G(t, \xi)$ is a function of two variables. Then the value of this security at t is given by

$$G_t = \frac{1}{\pi_t} \mathbb{E}^{\mathbb{P}} \left[\pi_T \, G(T, \xi_{TU}) \,|\, \mathcal{F}_t \right]. \tag{2.6}$$

For example the value at time t of a T-maturity option on a primary security takes this form.

Let us consider now in more detail the properties of discount bonds. Recalling that the information process has the Markov property, we see that (2.5) reduces to the following expression:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[F(T,\xi_{TU})\,|\,\xi_{tU}\right]}{F(t,\xi_{tU})}. \tag{2.7}$$

We proceed to work out the conditional expectation. To this end we recall one further property of the information process. This is the existence of the so-called "bridge measure" $\mathbb{B}$. Under the bridge measure (Brody *et al.* 2007) the information process $\{\xi_{tU}\}$ is a Brownian bridge over the interval $[0,U)$. The change-of-measure density martingale for the transformation from $\mathbb{P}$ to $\mathbb{B}$ is given by the process $\{M_t\}_{0\le t<U}$ defined by

$$M_t = \left(\int_{-\infty}^{\infty} p(x)\exp\left[\frac{U}{U-t}\left(\sigma x\,\xi_{tU} - \tfrac{1}{2}\,\sigma^2 x^2 t\right)\right]\mathrm{d}x\right)^{-1}. \tag{2.8}$$

Applying Ito's formula, one can show that

$$\frac{\mathrm{d}M_t}{M_t} = -\frac{\sigma U}{U-t}\,\mathbb{E}^{\mathbb{P}}\left[X_U\,|\,\xi_{tU}\right]\mathrm{d}W_t, \tag{2.9}$$

where the process $\{W_t\}$ defined by

$$W_t = \xi_{tU} + \int_0^t \frac{1}{U-s}\,\xi_{sU}\,\mathrm{d}s - \sigma U\int_0^t \frac{1}{U-s}\,\mathbb{E}^{\mathbb{P}}\left[X_U\,|\,\xi_{sU}\right]\mathrm{d}s. \tag{2.10}$$

is an $(\{\mathcal{F}_t\},\mathbb{P})$ Brownian motion on $[0,U)$. Thus, in the information-based approach the Brownian motions that drive asset prices always arise as "secondary" objects—i.e. innovation processes—rather than as primary drivers. For further details of the change-of-measure martingale $\{M_t\}$ and the related processes appearing in its definition, see Macrina 2006, Chapter 3. Bearing in mind that the random variable M_t can be expressed as a function of t and ξ_{tU}, as given by (2.8), we can without loss of generality introduce a function $f(t,\xi_{tU})$ such that

$$\pi_t = M_t\,f(t,\xi_{tU}), \tag{2.11}$$

and as consequence we obtain

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M_T\,f(T,\xi_{TU})\,|\,\xi_{tU}\right]}{M_t\,f(t,\xi_{tU})}. \tag{2.12}$$

The appearance of the change of measure density in this formula enables us to use the conditional version of Bayes formula to re-express $\{P_{tT}\}$ in terms of an expectation with respect to the bridge measure $\mathbb{B}$:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f(T, \xi_{TU}) \,|\, \xi_{tU}\right]}{f(t, \xi_{tU})}. \tag{2.13}$$

Since the information process is a $\mathbb{B}$-Brownian bridge we know that for each fixed time t the random variable ξ_{tU} is $\mathbb{B}$-Gaussian. Armed with this fact, we proceed as follows. We introduce the random variable Y_{tT} defined by

$$Y_{tT} = \xi_{TU} - \frac{U-T}{U-t}\,\xi_{tU}. \tag{2.14}$$

It is evident that Y_{tT} is $\mathbb{B}$-Gaussian, and a short calculation making use of properties of the Brownian bridge shows that Y_{tT} has mean zero and variance

$$\mathrm{Var}^{\mathbb{B}}\left[Y_{tT}\right] = \frac{(T-t)(U-T)}{U-t}. \tag{2.15}$$

We observe that Y_{tT} is independent of ξ_{tU} under $\mathbb{B}$. This can be checked by calculating the relevant covariance under $\mathbb{B}$. Next we express Y_{tT} in terms of a "standard" normally-distributed variable Y, with mean zero and unit variance. Thus we write

$$Y_{tT} = \nu_{tT} Y, \tag{2.16}$$

where

$$\nu_{tT} = \sqrt{\frac{(T-t)(U-T)}{U-t}}. \tag{2.17}$$

Then we rewrite the expression for the bond price given by (2.13) in terms of Y to obtain

$$P_{tT} = \frac{1}{f(t, \xi_{tU})}\,\mathbb{E}^{\mathbb{B}}\left[f\left(T, \nu_{tT} Y + \frac{U-T}{U-t}\,\xi_{tU}\right) \,\middle|\, \xi_{tU}\right]. \tag{2.18}$$

Since ξ_{tU} is $\mathcal{F}_t$-measurable, and Y is independent of ξ_{tU}, the conditional expectation in (2.18) reduces to a Gaussian integral over the range of Y. Therefore, we obtain:

$$P_{tT} = \frac{1}{\sqrt{2\pi}\, f(t, \xi_{tU})} \int_{-\infty}^{\infty} f\left(T, \nu_{tT}\, y + \frac{U-T}{U-t}\,\xi_{tU}\right) \exp\left(-\tfrac{1}{2}\, y^2\right) \mathrm{d}y. \tag{2.19}$$

We can write the equation above in a more compact way by introducing a function of three variables $\tilde{f}(t,T,\xi)$ that depends on time t, T and ξ in the following way:

$$\tilde{f}(t,T,\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f\left(T,\nu_{tT}\,y+\frac{U-T}{U-t}\,\xi\right)\exp\left(-\tfrac{1}{2}\,y^2\right)\mathrm{d}y \tag{2.20}$$

Then, the bond price process is given by

$$P_{tT}=\frac{\tilde{f}(t,T,\xi_{tU})}{f(t,\xi_{tU})}. \tag{2.21}$$

For any particular choice of f it is straightforward to simulate the dynamics of the bond price since, conditional on the outcome of the underlying factor X_U, the information process is a Gaussian process under $\mathbb{P}$.

3 Pricing kernel dynamics, nominal interest rate, and market price of risk

Let us proceed to derive the dynamics of the pricing kernel $\pi_t=M_t\,f(t,\xi_{tU})$. We apply the Ito product rule to obtain

$$\mathrm{d}\pi_t=f_t\,\mathrm{d}M_t+M_t\,\mathrm{d}f_t+\mathrm{d}M_t\,\mathrm{d}f_t, \tag{3.1}$$

where $f_t=f(t,\xi_{tU})$. The dynamical equation of the change-of-measure density martingale is given by (2.9). We shall assume that $f(t,\xi)$ has a continuous first derivative in t, denoted $\dot{f}(t,\xi)$, and a continuous second derivative in ξ, denoted $f''(t,\xi)$. Hence

$$\mathrm{d}f_t=\dot{f}_t\,\mathrm{d}t+f'_t\,\mathrm{d}\xi_{tU}+\tfrac{1}{2}\,f''_t\,(\mathrm{d}\xi_{tU})^2. \tag{3.2}$$

In terms of the innovations process $\{W_t\}$ defined by (2.10), the dynamical equation for $\{\xi_{tU}\}$ is given by

$$\mathrm{d}\xi_{tU}=\frac{1}{U-t}\left(\sigma\,U\,\mathbb{E}\left[X_U\,|\,\xi_{tU}\right]-\xi_{tU}\right)\mathrm{d}t+\mathrm{d}W_t. \tag{3.3}$$

Thus $(\mathrm{d}\xi_{tU})^2=\mathrm{d}t$, and with expressions (2.8) and (3.3) at hand, a calculation shows that

$$\mathrm{d}\pi_t=$$
$$M_t\left(\dot{f}_t-\frac{\xi_{tU}}{U-t}f'_t+\tfrac{1}{2}f''_t\right)\mathrm{d}t\;+\;M_t\left(f'_t-\frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U\,|\,\xi_{tU}\right]\,f_t\right)\mathrm{d}W_t. \tag{3.4}$$

or equivalently,

$$\frac{\mathrm{d}\pi_t}{\pi_t} = \frac{1}{f_t}\left(\dot{f}_t - \frac{\xi_{tU}}{U-t}\, f'_t + \tfrac{1}{2} f''_t\right)\mathrm{d}t \\ \qquad + \frac{1}{f_t}\left(f'_t - \frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right]\, f_t\right)\mathrm{d}W_t. \tag{3.5}$$

At time t, the drift of the pricing kernel is given by minus the short rate of interest, and the volatility of the pricing kernel is given by minus the market price of risk:

$$\frac{\mathrm{d}\pi_t}{\pi_t} = -r_t\,\mathrm{d}t - \lambda_t\,\mathrm{d}W_t. \tag{3.6}$$

Comparing coefficients, we thus deduce that the short rate and the nominal market price of risk are given respectively by

$$r_t = \frac{1}{f_t}\left(\frac{\xi_{tU}}{U-t}\, f'_t - \tfrac{1}{2}\, f''_t - \dot{f}_t\right), \tag{3.7}$$

and

$$\lambda_t = \frac{\sigma U}{U-t}\,\mathbb{E}\left[X_U \,|\, \xi_{tU}\right] - \frac{f'_t}{f_t}. \tag{3.8}$$

It is natural in the context of some applications to impose the condition that the short rate should be positive. This condition is evidently given by

$$\frac{\xi_{tU}}{U-t}\, f'_t - \tfrac{1}{2}\, f''_t - \dot{f}_t > 0. \tag{3.9}$$

It follows that the interest-rate positivity condition is equivalent to the inequality

$$\frac{x}{U-t}\, f'(t,x) - \tfrac{1}{2} f''(t,x) - \dot{f}(t,x) > 0. \tag{3.10}$$

4 Pricing in a multi-factor setting

We introduce a set of X-factors $\{X_{T_1}, \ldots, X_{T_n}\}$, labeled by a series of dates T_k $(k = 1, \ldots, n)$ such that $0 < T_1 < \ldots < T_n$. With each X-factor we associate an information process $\{\xi_{tT_k}\}$ defined by

$$\xi_{tT_k} = \sigma_k\, t\, X_{T_k} + \beta_{tT_k}. \tag{4.1}$$

We assume that the information processes associated with different X-factors are independent. The market filtration $\{\mathcal{F}_t\}$ is assumed to be generated by the entire collection of information processes:

$$\mathcal{F}_t = \sigma\left(\{\xi_{sT_1}\}_{0\le s\le t}, \ldots, \{\xi_{sT_n}\}_{0\le s\le t}\right). \tag{4.2}$$

Thus as a generalisation of the Markov model introduced in the previous section, we consider the following multi-factor model for π_t:

$$\pi_t = M_t^{(1)} \cdots M_t^{(n)} f(t, \xi_{tT_1}, \ldots, \xi_{sT_n}). \tag{4.3}$$

Here $f(t, \xi_1, \xi_2, \ldots, \xi_n)$ is a function of $n+1$ variables. The processes $\{M_t^{(k)}\}_{k=1,\ldots,n}$ are the $(\mathbb{P}, \{\mathcal{F}_t\})$-martingales defined by

$$\frac{\mathrm{d}M_t^{(k)}}{M_t^{(k)}} = -\frac{\sigma_k T_k}{T_k - t}\, \mathbb{E}\left[X_{T_k} \,|\, \xi_{tT_k}\right] \mathrm{d}W_t^{(k)}, \tag{4.4}$$

where for each k the $\mathbb{P}$-Brownian motion $\{W_t^{(k)}\}$ is defined by

$$W_t^{(k)} = \xi_{tT_k} + \int_0^t \frac{1}{T_k - s}\, \xi_{sT_k} \mathrm{d}s - \sigma_k T_k \int_0^t \frac{1}{T_k - s}\, \mathbb{E}\left[X_{T_k} \,|\, \xi_{sT_k}\right] \mathrm{d}s. \tag{4.5}$$

Since the information processes are independent, it follows that

$$\mathrm{d}W^{(j)} \mathrm{d}W^{(k)} = \delta^{jk}\, \mathrm{d}t. \tag{4.6}$$

Let us focus on the pricing of a nominal discount bond with maturity $T < T_1$. The price of the bond is given by:

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M_T^{(1)} \cdots M_T^{(n)} f\left(T, \xi_{TT_1}, \ldots, \xi_{TT_n}\right) \,\Big|\, \xi_{tT_1}, \ldots, \xi_{tT_n}\right]}{M_t^{(1)} \cdots M_t^{(n)} f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}. \tag{4.7}$$

Here we have used the fact that the information processes are Markovian. Next we note that since the information processes are independent the product of $(\mathbb{P}, \{\mathcal{F}_t\})$-martingales given by $M_t^{(1)} \cdots M_t^{(n)}$ for t in the time interval $[0, T_1)$ is itself an $(\mathbb{P}, \{\mathcal{F}_t\})$-martingale, which induces a bridge measure that has the effect of making all of the information processes Brownian bridges distributionally. More precisely, under the bridge measure each information process has, over the interval $[0, T_1)$, the distribution of a standard Brownian bridge on the interval from 0 to the termination time of the information process. Thus we have

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f\left(T, \xi_{TT_1}, \ldots, \xi_{TT_n}\right) \,\middle|\, \xi_{tT_1}, \ldots, \xi_{tT_n}\right]}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}, \tag{4.8}$$

where all of the relevant random variables are Gaussian. Next we introduce a set of random variables $Y_{tT}^{(1)}, Y_{tT}^{(2)}, \ldots, Y_{tT}^{(n)}$ defined by

$$Y_{tT}^{(k)} = \xi_{TT_k} - \frac{T_k - T}{T_k - t}\, \xi_{tT_k}. \tag{4.9}$$

Since the process $\{\xi_{tT_k}\}$ is a $\mathbb{B}$-Brownian bridge, it follows that $Y_{tT}^{(k)}$ is a Gaussian random variable with mean zero and variance

$$\mathrm{Var}^{\mathbb{B}_k}\left[Y_{tT}^{(k)}\right] = \frac{(T-t)(T_k - T)}{T_k - t}. \tag{4.10}$$

We introduce an n-dimensional set of standard Gaussian variables $(Y_1, \ldots, Y_n)$. The random variable Y_k stands in relationship to $Y_{tT}^{(k)}$ via

$$Y_{tT}^{(k)} = \nu_{tT}^{(k)}\, Y_k, \tag{4.11}$$

where

$$\nu_{tT}^{(k)} = \sqrt{(T-t)(T_k - T)/(T_k - t)}. \tag{4.12}$$

In terms of the standard Gaussian random variables, the bond price at time t can thus be written in the form

$$P_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[f\left(T,\, \nu_{tT}^{(1)} Y_1 + \tfrac{T_1 - T}{T_1 - t}\xi_{tT_1}\,,\, \ldots\,, \nu_{tT}^{(n)} Y_n + \tfrac{T_n - T}{T_n - t}\xi_{tT_n}\right) \mid \xi_{tT_1} \cdots \xi_{tT_n}\right]}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}. \tag{4.13}$$

Finally we observe that under $\mathbb{B}$ the random variables $Y_{tT}^{(k)}$ and ξ_{tT_k} are independent. The expression for the bond price thus reduces to an n-dimensional Gaussian integral:

$$\begin{aligned} P_{tT} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} & \frac{f(T,\, \nu_{tT}^{(1)}\, y_1 + \frac{T_1 - T}{T_1 - t}\, \xi_{tT_1}\,,\, \ldots\,,\, \nu_{tT}^{(n)}\, y_n + \frac{T_n - T}{T_n - t}\, \xi_{tT_n})}{f(t, \xi_{tT_1}, \ldots, \xi_{tT_n})} \\ & \times \frac{1}{\left(\sqrt{2\pi}\right)^n} \exp\left[-\tfrac{1}{2}\left(y_n^2 + \ldots + y_1^2\right)\right] \mathrm{d}y_1 \cdots \mathrm{d}y_n. \end{aligned} \tag{4.14}$$

That is to say, we obtain an expression of the form

$$P_{tT} = \frac{\tilde{f}\left(t, T, \xi_{tT_1}, \dots, \xi_{tT_n}\right)}{f\left(t, \xi_{tT_1}, \dots, \xi_{tT_n}\right)}, \tag{4.15}$$

where we introduce the transformed function

$$\begin{aligned} &\tilde{f}(t, T, \xi_1, \xi_2, \dots, \xi_n) \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f\left(T,\, \nu_{tT}^{(1)}\, y_1 + \frac{T_1 - T}{T - t}\, \xi_1\,,\, \dots\,,\, \nu_{tT}^{(n)}\, y_n + \frac{T_n - T}{T - t}\, \xi_n\right) \\ &\qquad\qquad \times \frac{1}{\left(\sqrt{2\pi}\right)^n} \exp\left[-\tfrac{1}{2}\left(y_n^2 + \dots + y_1^n\right)\right] \mathrm{d}y_n \cdots y_1. \end{aligned} \tag{4.16}$$

Let us consider now the class of functions $f(t, \xi_1, \dots, \xi_n)$ for which the pricing kernel is a supermartingale. We thus need to derive the dynamics of the pricing kernel and in particular to work out the drift. Therefore we shall assume that the function $f(t, \xi_1, \dots, \xi_n)$ belongs to class $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n)$, and we let $\dot{f}$ denote the first derivative with respect to t, $\partial_k f$ the first derivative with respect to the k-th coordinate, and $\partial_{kk} f$ the second derivative with respect to the k-th coordinate. Then in the multi-factor setting the dynamical equation of the pricing kernel is given by:

$$\begin{aligned} \frac{\mathrm{d}\pi_t}{\pi_t} &= \frac{1}{f_t}\left[\dot{f} + \sum_{k=1}^{n}\left(\tfrac{1}{2}\,\partial_{kk} f_t - \frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t\right)\right]\mathrm{d}t \\ &\quad + \frac{1}{f_t}\sum_{k=1}^{n}\left(\partial_k f_t - \frac{\sigma_k T_k}{T_k - t}\, X_{tT_k}\, f_t\right)\mathrm{d}W_t^k, \end{aligned} \tag{4.17}$$

where $X_{tT_k} = \mathbb{E}[X_{T_k} \,|\, \xi_{tT_k}]$. The multi-factor short rate process is therefore given by

$$r_t = \frac{1}{f_t}\left[\sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t - \frac{1}{2}\,\partial_{kk} f_t\right) - \dot{f}_t\right], \tag{4.18}$$

and for the k-th component of the market price of risk vector we obtain

$$\lambda_t^k = \frac{1}{f_t}\left(\frac{\sigma_k T_k}{T_k - t}\, X_{tT_k} f_t - \partial_k f_t\right). \tag{4.19}$$

Were one to impose the condition of a positive short rate process, then the following condition would need to be satisfied:

$$\sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t - \tfrac{1}{2}\,\partial_{kk} f_t\right) - \dot{f}_t > 0. \tag{4.20}$$

A sufficient condition for (4.20) to hold is that the function $f(t, \xi_1, \ldots, \xi_n)$ satisfies

$$\sum_{k=1}^{n}\left[\frac{x_k}{T_k - t}\,\partial_k f(t,\xi_1,\ldots,\xi_n) - \tfrac{1}{2}\,\partial_{kk} f(t,\xi_1,\ldots\,\xi_n)\right] - \dot{f}(t,\xi_1,\ldots,\xi_n) > 0. \tag{4.21}$$

The case of a multiple-currency environment, with a system of interest rates for each currency, can be handled similarly. We consider a set of $N+1$ currencies, writing $\{\pi_t\}$ for the pricing kernel of the "domestic" or "base" currency, and $\{\pi_t^i\}$, $i = 1, \ldots, N$, for the pricing kernels of the N foreign currencies. We introduce a collection of n information processes, and we assume that each of the pricing kernels is given by a function of the current levels of the information processes. The prices associated with the N foreign currencies, expressed in units of the domestic currency, are then given by the ratios of the various foreign pricing kernels to the domestic pricing kernel. Normally we would expect to have $n \geq 2N+1$ for a realistic model.

5 Inflation-linked products

The technique used for the pricing of nominal discount bonds can be adopted to the pricing of inflation-linked assets. In particular we focus on the pricing of inflation-linked discount bonds. We denote the price level (e.g. the consumer price index) by $\{C_t\}_{t\geq 0}$, and note the relationship between the nominal pricing kernel $\{\pi_t\}$, the real pricing kernel $\{\pi_t^R\}$, and the price level process. This is given by

$$C_t = \frac{\pi_t^R}{\pi_t}. \tag{5.1}$$

We take the view that the dynamics of the price level should be derived from the dynamics of the pricing kernels. We return to this point shortly, when we introduce the elements of a stochastic monetary economy. We keep in mind that once models for the nominal and the real pricing kernels have been constructed, then the dynamics of the price level follows as a result of (5.1).

It will be convenient for our purpose to define an inflation-linked discount bond as a bond that at its maturity T generates a single cash flow equal to the price level C_T prevailing at that time. Thus, the price process $\{Q_{tT}\}_{0\leq t\leq T}$

of an inflation-linked discount bond is given by the relation

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi_T\, C_T \,|\, \mathcal{F}_t\right]}{\pi_t}. \tag{5.2}$$

Using relationship (5.1) we can write this alternatively as

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[\pi^R_T \,|\, \mathcal{F}_t\right]}{\pi_t}. \tag{5.3}$$

We shall construct models for the nominal and real pricing kernels following the approach presented in the earlier sections. For simplicity let us assume the existence of a pair of independent X-factors X_{T_1} and X_{T_2}, where $0 \leq t \leq T < T_1 < T_2$. Then we introduce a pair of Brownian-bridge information processes $\{\xi_{tT_1}\}$ and $\{\xi_{tT_2}\}$, and we assume that the filtration is generated by these independent processes. We consider the following models for the respective pricing kernels:

$$\pi_t = M^{(1)}_t M^{(2)}_t\, f\left(t, \xi_{tT_1}, \xi_{tT_2}\right), \tag{5.4}$$

$$\pi^R_t = M^{(1)}_t M^{(2)}_t\, g\left(t, \xi_{tT_1}, \xi_{tT_2}\right), \tag{5.5}$$

where $\{M^{(1)}_t\}$ and $\{M^{(2)}_t\}$ are the change-of-measure density martingales that are used to transform from the real measure to the bridge measure. In terms of the functions f and g, the price of the inflation-linked bond is given by

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{P}}\left[M^{(1)}_T M^{(2)}_T\, g(T, \xi_{TT_1}, \xi_{TT_2}) \,\middle|\, \xi_{tT_2}, \xi_{tT_2}\right]}{M^{(1)}_t M^{(2)}_t\, f\left(t, \xi_{tT_1}, \xi_{tT_2}\right)}. \tag{5.6}$$

Here we have made use of the Markov property of the information processes. Then we change the measure from $\mathbb{P}$ to $\mathbb{B}$ to obtain

$$Q_{tT} = \frac{\mathbb{E}^{\mathbb{B}}\left[g(T, \xi_{TT_1}, \xi_{TT_2}) \,\middle|\, \xi_{tT_2}, \xi_{tT_1}\right]}{f(t, \xi_{tT_1}, \xi_{tT_2})}. \tag{5.7}$$

Thus the conditional expectation reduces to a Gaussian integral and the price process $\{Q_{tT}\}_{0\leq t\leq T}$ can be expressed as follows:

$$Q_{tT} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{g(T, \nu^{(1)}_{tT} y_1 + \frac{T_1-T}{T_1-t}\,\xi_{tT_1}, \nu^{(2)}_{tT} y_2 + \frac{T_2-T}{T_2-t}\,\xi_{tT_2})}{f(t, \xi_{tT_1}, \xi_{tT_2})} \times \exp\left[-\tfrac{1}{2}\left(y_1^2 + y_2^2\right)\right] \mathrm{d}y_1 \mathrm{d}y_2. \tag{5.8}$$

In such a setting the nominal discount bond has the following price:

$$P_{tT} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(T, \nu^{(1)}_{tT} y_1 + \frac{T_1-T}{T_1-t}\,\xi_{tT_1}, \nu^{(2)}_{tT} y_2 + \frac{T_2-T}{T_2-t}\,\xi_{tT_2})}{f(t, \xi_{tT_1}, \xi_{tT_2})} \times \exp\left[-\tfrac{1}{2}\left(y_1^2 + y_2^2\right)\right] \mathrm{d}y_1 \mathrm{d}y_2. \tag{5.9}$$

The dynamics of the real pricing kernel can be computed analogously to that of the nominal pricing kernel. There is however an important difference. Since the real interest rate may be positive or negative, there is no "supermartingale condition" on g. By inserting the expressions for the nominal and the real pricing kernels in (5.1) and applying Ito's quotient rule, one obtains the dynamics of the price level $\{C_t\}$. Next we give the dynamics for the price level in a more general situation in which the pricing kernels are of the following form:

$$\pi_t = M^{(1)}_t \cdots M^{(n)}_t\, f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right), \tag{5.10}$$

$$\pi^R_t = M^{(1)}_t \cdots M^{(n)}_t\, g\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right). \tag{5.11}$$

In this case the price level $\{C_t\}$ is given by

$$C_t = \frac{g\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}{f\left(t, \xi_{tT_1}, \ldots, \xi_{tT_n}\right)}. \tag{5.12}$$

For the dynamics of $\{C_t\}$ we obtain:

$$\begin{aligned}\frac{\mathrm{d}C_t}{C_t} = \Bigg\{ & \frac{1}{f_t}\left[\sum_{k=1}^{n} \frac{\xi_{tT_k}}{T_k - t}\,\partial_k f_t - \tfrac{1}{2}\sum_{k=1}^{n} \partial_k\partial_k f_t - \dot{f}\right] \\ & - \frac{1}{g_t}\left[\sum_{k=1}^{n} \frac{\xi_{tT_k}}{T_k - t}\,\partial_k g_t - \tfrac{1}{2}\sum_{k=1}^{n} \partial_k\partial_k g_t - \dot{g}\right] \\ & + \sum_{k=1}^{n} \frac{\sigma_k T_k}{T_k - t}\, X_{tT_k}\left(\frac{1}{g_t}\partial_k g_t - \frac{1}{f_t}\partial_k f_t\right) \\ & - \frac{1}{g_t f_t}\sum_{k=1}^{n} \partial_k g_t\, \partial_k f_t + \frac{1}{f_t^2}\sum_{k=1}^{n} \left(\partial_k f_t\right)^2 \Bigg\}\,\mathrm{d}t \\ + \sum_{k=1}^{n} & \left(\frac{1}{g_t}\,\partial_k g_t - \frac{1}{f_t}\,\partial_k f_t\right)\mathrm{d}W^{(k)}_t, \end{aligned} \tag{5.13}$$

where $X_{tT_k} = \mathbb{E}\,[X_{T_k}\,|\,\xi_{tT_k}]$. For this calculation we have used the relationship

$$\mathrm{d}\xi_{tT_k} = \frac{1}{T_k - t}\,(\sigma_k\,T_k\,X_{tT_k} - \xi_{tT_k})\,\mathrm{d}t + \mathrm{d}W^{(k)}_t. \tag{5.14}$$

The relative drift of the price level is given by the instantaneous inflation rate determined by the Fisher equation: $I_t = r_t - r^R_t + \lambda_t\left(\lambda_t - \lambda^R_t\right)$. The relative volatility of the price level on the other hand is given by $\lambda_t - \lambda^R_t$. Verification of these results is achieved by calculating the dynamics of the nominal and the real pricing kernels. In particular, we have (3.6) and

$$\frac{\mathrm{d}\pi^R_t}{\pi^R_t} = -r^R_t\,\mathrm{d}t - \lambda^R_t\,\mathrm{d}W_t. \tag{5.15}$$

A calculation then shows that

$$\begin{aligned}\frac{\mathrm{d}\pi_t}{\pi_t} = \frac{1}{f_t}\left[\dot{f}_t - \sum_{k=1}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k\,f_t - \tfrac{1}{2}\partial_k\partial_k\,f_t\right)\right]\mathrm{d}t \\ + \frac{1}{f_t}\sum_{k=1}^{n}\left(\partial_k f_t - \frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\,f_t\right)\mathrm{d}W^{(k)}_t\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}\frac{\mathrm{d}\pi^R_t}{\pi^R_t} = \frac{1}{g_t}\left[\dot{g}_t - \sum_{k=m}^{n}\left(\frac{\xi_{tT_k}}{T_k - t}\,\partial_k\,g_t - \tfrac{1}{2}\partial_k\partial_k\,g_t\right)\right]\mathrm{d}t \\ + \frac{1}{g_t}\sum_{k=m}^{n}\left(\partial_k g_t - \frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\,g_t\right)\mathrm{d}W^{(k)}_t.\end{aligned} \tag{5.17}$$

Thus we form the difference between the k-th components of the nominal and the real market prices of risk to obtain:

$$\begin{aligned}\lambda^{(k)}_t - \lambda^{R\ (k)}_t &= \frac{1}{f_t}\left(\frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\,f_t - \partial_k f_t\right) - \frac{1}{g_t}\left(\frac{\sigma_k T_k}{T_k - t}\,X_{tT_k}\,g_t - \partial_k g_t\right) \\ &= \frac{1}{g_t}\,\partial_k g_t - \frac{1}{f_t}\,\partial_k f_t.\end{aligned} \tag{5.18}$$

This verifies that the volatility vector of the price level process is $\lambda_t - \lambda^R_t$. In the case of the drift of the price level, one sees that the first two summands are the nominal and the real interest rate processes thus forming the difference

$r_t - r_t^R$. It can be shown that the remaining terms in the drift reduce to $\lambda_t(\lambda_t - \lambda_t^R)$ by multiplying the expression for the nominal risk premium process and the difference (5.18). This shows that, indeed, the dynamics of the price level can be written in the form

$$\frac{\mathrm{d}C_t}{C_t} = \left[r_t - r_t^R + \lambda_t(\lambda_t - \lambda_t^R)\right]\mathrm{d}t + (\lambda_t - \lambda_t^R)\,\mathrm{d}W_t, \tag{5.19}$$

where $\{W_t\}$ is an n-dimensional Brownian motion.

6 Stochastic monetary economy

So far we have indicated how the pricing of fixed income assets, in particular the nominal and inflation-linked discount bond system, can be modelled in an information-based framework. We have shown how the nominal and real pricing kernels, and thus the price level, can be modelled in terms of the so-called information processes. It is our goal now to consider the relationship between the two pricing kernels, and to develop a simple macroeconomic model for the pricing kernels based on (a) the liquidity benefit of the money supply, and (b) the rate of consumption of goods and services. This will be carried out in the context of the information-based pricing theory developed in the previous sections. A macroeconomic asset pricing model that suits the present investigation is one presented in Hughston & Macrina (2008) in which expected utility, derived from the consumption of goods and services and from the liquidity benefit of money supply, is maximised over a finite period of time. Let us briefly summarise this model for a stochastic monetary economy and, by doing so, translate the discrete-time results in Hughston & Macrina (2008) to a continuous-time formulation.

There are three exogenously-specified stochastic processes that form the ingredients of such an economy: (1) the real per capita rate of consumption of goods and services $\{k_t\}$, (2) the per capita money supply $\{m_t\}$, and (3) the rate of liquidity benefit $\{\eta_t\}$ provided per unit of money supply. The product $\eta_t m_t$ is the instantaneous benefit rate in units of cash derived from the presence of the money supply level at time t. The goal of a representative agent in such an economy is to find the optimal strategy over a certain period of time in, on the one hand, consuming goods and services at a certain rate and, on the other hand, in maintaining a certain level of benefit derived from the availability of money. Since the liquidity benefit $\{\eta_t\}$ is measured in nominal units, we use the price level $\{C_t\}$ to convert its units to units of good and services. The "real" liquidity benefit (in units of good and services) of money supply $\{l_t\}$ is thus

$$l_t = \frac{\eta_t m_t}{C_t}. \tag{6.1}$$

The agent's rate of utility derived from real consumption and real liquidity benefit is modelled by a bivariate utility function $U(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ of the Sidrauski (1967) type satisfying $U_x > 0$, $U_{xx} < 0$, $U_y > 0$, $U_{yy} < 0$ and $U_{xx}U_{yy} > (U_{xy})^2$. The consumption strategy that delivers the agent the highest level of total expected utility over the period $[0, T]$ is found by maximising

$$J = \mathbb{E}\left[\int_0^T \mathrm{e}^{-\gamma t} U(k_t, l_t)\mathrm{d}t\right], \tag{6.2}$$

where the agent possesses a limited budget defined by

$$H = \mathbb{E}\left[\int_0^T \pi_t\left(k_t C_t + \eta_t m_t\right)\mathrm{d}t\right]. \tag{6.3}$$

Here the parameter γ is a psychological discount factor which for simplicity we take to be constant. Summarizing the representative agent's maximization problem, one could say that in equilibrium, the price level $\{C_t\}$ and the real rate of consumption $\{k_t\}$ will be adjusted to each other in order to maximise the total expected utility derivable by the economy as a whole.

A closed-form solution for this maximization problem can be found in the case for which the utility function is of a separable bivariate logarithmic type:

$$U(x, y) = a \ln(x) + b \ln(y), \tag{6.4}$$

where a and b are constant. In this case we obtain the following expressions for the real and the nominal pricing kernels:

$$\pi_t^R = \frac{a_t}{k_t} \qquad \text{and} \qquad \pi_t = \frac{b_t}{\eta_t\, m_t}, \tag{6.5}$$

where $a_t = \exp(-\gamma t)\, a/\lambda$ and $b_t = \exp(-\gamma t)\, b/\lambda$. The price level is then given by

$$C_t = \frac{b}{a}\,\frac{\eta_t m_t}{k_t}. \tag{6.6}$$

Next we establish a link between this specific model for a stochastic monetary economy with the information-based approach to the modelling of the pricing kernel presented in the previous sections. At this stage we revert for simplicity to a "low-dimensional" example in which the economy is driven by two factors. In the language of information-based pricing this means that we introduce a pair of information processes $\{\xi_{tT_k}\}_{k=1,2}$ associated with macroeconomic X-factors X_{T_1} and X_{T_2}. The results that follow generalise to a higher-dimensional setting with n information processes. We focus on the case where the nominal and the real pricing kernels are of the form

$$\pi_t = M_t^{(1)} M_t^{(2)}\, f(t, \xi_{tT_1}, \xi_{tT_2}) \quad \text{and} \quad \pi_t^R = M_t^{(1)} M_t^{(2)}\, g(t, \xi_{tT_1}, \xi_{tT_2}), \tag{6.7}$$

where $\{M_t^{(1)}\}$ and $\{M_t^{(2)}\}$ are the $\mathbb{P}$-martingales defined by (4.4). We assume that the real rate of consumption $\{k_t\}$, the money supply $\{m_t\}$ and the nominal rate of specific liquidity benefit $\{\eta_t\}$ are given by functions of the form

$$k_t = k(t, \xi_{tT_1}, \xi_{tT_2}), \quad m_t = m(t, \xi_{tT_1}, \xi_{tT_2}), \quad \eta_t = \eta(t, \xi_{tT_1}, \xi_{tT_2}). \tag{6.8}$$

By comparison with the models for the nominal and the real pricing kernels (6.5) we thus obtain the following relationships:

$$f(t, \xi_{tT_1}, \xi_{tT_2}) = \frac{b_t}{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}. \tag{6.9}$$

and

$$g(t, \xi_{tT_1}, \xi_{tT_2}) = \frac{a_t}{M_t^{(1)} M_t^{(2)}\, k(t, \xi_{tT_1}, \xi_{tT_2})} \tag{6.10}$$

By applying the results (5.8) and (5.9) we are then able to work out the bond prices that result in a stochastic monetary economy in which inflation is regarded a purely monetary phenomenon and asset prices fluctuate according to emerging information about macroeconomic factors influencing the economy. The two-factor price process of an inflation-linked bond is given by:

$$\begin{aligned} Q_{tT} = {} & \frac{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}{b_t} \\ & \times \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a_T \, \exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right] \, \mathrm{d}y_1 \, \mathrm{d}y_2}{M_T^{(1)}\left(z(y_1)\right) M_T^{(2)}\left(z(y_2)\right) k\left(T, z(y_1), z(y_2)\right)}, \end{aligned} \tag{6.11}$$

where

$$z(y_k) = \nu_{tT}^{(k)}\, y_k + \frac{T_k - T}{T_k - t}\, \xi_{tT_k}, \qquad k = 1, 2. \tag{6.12}$$

The corresponding nominal discount bond system is given by

$$P_{tT} = \frac{M_t^{(1)} M_t^{(2)}\, \eta(t, \xi_{tT_1}, \xi_{tT_2})\, m(t, \xi_{tT_1}, \xi_{tT_2})}{b_t}$$
$$\times \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{a_T \exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right] \, \mathrm{d}y_1 \, \mathrm{d}y_2}{M_T^{(1)}\left(z(y_1)\right) M_T^{(2)}\left(z(y_2)\right) \eta\left(T, z(y_1), z(y_2)\right) m(T, z(y_1), z(y_2))}. \tag{6.13}$$

Similar formulae can be derived for a separable power-utility function. In such a situation, the nominal pricing kernel $\{\pi_t\}$ will also depend explicitly on the rate of consumption $\{k_t\}$.

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References.

1. D. C. Brody, J. Crosby & H. Li (2008) Convexity adjustments in inflation-linked derivatives. Risk Magazine.

2. D. C. Brody, L. P. Hughston & A. Macrina (2007) Beyond hazard rates: a new framework for credit-risk modelling. In *Advances in Mathematical Finance, Festschrift Volume in Honour of Dilip Madan*, edited by R. Elliott, M. Fu, R. Jarrow & J.-Y. Yen. Birkhäuser, Basel and Springer, Berlin.

3. D. C. Brody, L. P. Hughston & A. Macrina (2008a) Information-based asset pricing. International Journal of Theoretical and Applied Finance Vol. 11, 107-142.

4. D. C. Brody, L. P. Hughston & A. Macrina (2008b) Dam rain and cumulative gain. Proceedings of the Royal Society London A, Vol. 464, No. 2095, 1801-1822.

5. M. Hinnerich (2008) Inflation-indexed swaps and swaptions. Journal of Banking and Finance, Vol. 32, No. 11, 2293-2306.

6. L. P. Hughston (1998) Inflation derivatives. Merrill Lynch and King's College London report, with added note (2004).

7. L. P. Hughston & A. Macrina (2008) Information, inflation, and interest. Advances in Mathematics of Finance. Banach Center Publications, Vol. 83. Institute of Mathematics, Polish Academy of Science, Warsaw.

8. R. Jarrow & Y. Yildirim (2003) Pricing treasury inflation protected securities and related derivative securities using an HJM model. Journal of Financial and Quantitative Analysis, No. 38, 409.

9. A. Macrina (2006) An information-based framework for asset pricing: X-factor theory and its applications. PhD thesis, King's College London.

10. F. Mercurio (2005) Pricing inflation-indexed derivatives. Journal of Quantitative Finance, No. 5, 289-302.

11. M. Rutkowski & N. Yu (2007) An extension of the Brody-Hughston-Macrina approach to modeling of defaultable bonds. International Journal of Theoretical and Applied Finance 10, 557-589.

12. M. Sidrauski (1967) Rational choice and patterns of growth in a monetary economy. American Economic Review 2, 534-544.