Brézis-Gallouët-Wainger type inequality with a double logarithmic term in the Hölder space: its sharp constants and extremal functions

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Abstract

We investigate the sharp constants in a Brézis-Gallouët-Wainger type inequality with a double logarithmic term in the Hölder space in a bounded domain in $\mathbb{R}^n$. Ibrahim, Majdoub and Masmoudi gave the sharp constant in the 2-dimensional case. We make precise estimates to give the sharp constants, and pass to the case of higher dimensions $n \geq 2$. We can also show that the inequality with fixed constants including the sharp ones admits an extremal function under a suitable condition when the domain is a ball.

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1 Introduction and main results

The purpose of this paper is to consider a Brézis-Gallouët-Wainger type inequality with a double logarithmic term in the Hölder space. Ibrahim, Majdoub and Masmoudi [8] obtained its sharp constant in the 2-dimensional case. In this paper, we examine a similar type inequality with a slightly general form for any bounded domain $\Omega \subset \mathbb{R}^n$ and higher dimensions $n \geq 2$. We treat only real-valued functions.

First we recall the Sobolev embedding theorem in the critical case. For $1 < p < \infty$, it is well known that the embedding $W^{n/p,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ holds for any $p \leq q < \infty$, and does not hold for $q = \infty$, i.e., one cannot estimate the L^∞-norm by the $W^{n/p,p}$-norm. However, the Brézis-Gallouët-Wainger inequality states that the L^∞-norm can be estimated by the $W^{n/p,p}$-norm with the partial aid of the $W^{s,r}$-norm with $s > n/r$ and $1 \leq r \leq \infty$ as follows:

$$\|u\|_{L^\infty(\mathbb{R}^n)}^{p/(p-1)} \leq \lambda(1 + \log(1 + \|u\|_{W^{s,r}(\mathbb{R}^n)})) \tag{1.1}$$

holds for all $u \in W^{n/p,p}(\mathbb{R}^n) \cap W^{s,r}(\mathbb{R}^n)$ with $\|u\|_{W^{n/p,p}(\mathbb{R}^n)} = 1$, where λ is a positive constant independent of u. Note that the embedding $W^{s,r}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ holds for s and r specified as above. Originally, Brézis-Gallouët [4] proved (1.1) for the case $n = p = r = s = 2$. Later on, Brézis-Wainger [5] obtained (1.1) for the general case, and remarked that the power $p/(p-1)$ in (1.1) is optimal in the sense that one cannot replace it by any larger power. Ozawa [15] improved (1.1) so that the Sobolev norm $\|u\|_{W^{s,r}(\mathbb{R}^n)}$ in (1.1) can be replaced with the homogeneous Sobolev norm $\|u\|_{\dot{W}^{s,r}(\mathbb{R}^n)}$. However, it seems that little is known about the sharp constants in Brézis-Gallouët-Wainger type inequalities.

In this paper, we restrict our attention to the case $p = n$, and consider the problem for a bounded domain $\Omega \subset \mathbb{R}^n$. We regard any function on Ω as the function on $\mathbb{R}^n$ by the zero-extension on $\mathbb{R}^n \setminus \Omega$. Then the inequality (1.1) holds for all $u \in W_0^{1,n}(\Omega) \cap W^{s,r}(\mathbb{R}^n)$ with $\|u\|_{W^{1,n}(\Omega)} = 1$, where $s > n/r$ and $1 \leq r \leq \infty$. Note that a $W_0^{1,n}(\Omega)$-norm is equivalent to $\|\nabla u\|_{L^n(\Omega)}$, and (1.1) also holds with $\|\nabla u\|_{L^n(\mathbb{R}^n)} = 1$. Here,

$$\|\nabla u\|_{L^n(\Omega)} = \| \, |\nabla u| \, \|_{L^n(\Omega)} = \left\| \left(\sum_{k=1}^n \left(\frac{\partial u}{\partial x_k} \right)^2 \right)^{1/2} \right\|_{L^n(\Omega)} \quad \text{for } u \in W_0^{1,n}(\Omega). \tag{1.2}$$

Furthermore, if $s > 0$ and $n/s < r < n/(s-1)_+$, then the embedding $W^{s,r}(\mathbb{R}^n) \hookrightarrow \dot{C}^{0,\alpha}(\mathbb{R}^n)$ holds with $\alpha = s - n/r$. We also note that $\|u\|_{\dot{C}^{0,\alpha}(\Omega)} = \|u\|_{\dot{C}^{0,\alpha}(\mathbb{R}^n)}$ for $u \in \dot{C}^{0,\alpha}(\Omega) \cap C_0(\Omega)$, where $C_0(\Omega) = \{u \in C(\mathbb{R}^n);\, u = 0 \text{ on } \mathbb{R}^n \setminus \Omega\}$, and $\dot{C}^{0,\alpha}(\Omega)$ denotes the subspace of the homogeneous Hölder space of order α endowed with the seminorm

$$\|u\|_{\dot{C}^{0,\alpha}(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

with $0<\alpha\leq 1$. Then a slightly stronger inequality

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leq C(1+\log(1+\|u\|_{\dot{C}^{0,\alpha}(\Omega)})) \tag{1.3}$$

for $u\in W_0^{1,n}(\Omega)\cap\dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)}=1$, can be an object of our study. In the case $n=2$, Ibrahim-Majdoub-Masmoudi [8] proved such inequalities of the type (1.3) and gave their sharp constant. They formulated and proved their principal results for the case $\Omega=B_1$ as in Theorem A below though they remarked how to modify the results for an arbitrary bounded domain Ω. Here, B_1 denotes the unit open ball centered at the origin.

Theorem A. (Ibrahim-Majdoub-Masmoudi [8, Theorems 1.3 and 1.4]). Let $n=2$ and $0<\alpha<1$.

(i) If $\lambda_1>1/(2\pi\alpha)$, then there exists a constant $C>0$ such that

$$\|u\|_{L^\infty(B_1)}^2 \leq \lambda_1\log(\|u\|_{\dot{C}^{0,\alpha}(B_1)}+C) \tag{1.4}$$

holds for all $u\in W_0^{1,2}(B_1)\cap\dot{C}^{0,\alpha}(B_1)$ with $\|\nabla u\|_{L^2(B_1)}=1$. Furthermore, if $\lambda_1\leq 1/(2\pi\alpha)$, then the inequality (1.4) with any constant $C>0$ does not hold for some $u\in W_0^{1,2}(B_1)\cap\dot{C}^{0,\alpha}(B_1)$ with $\|\nabla u\|_{L^2(B_1)}=1$.

(ii) If $\lambda_1=1/(2\pi\alpha)$, then there exists a constant $C>0$ such that

$$\|u\|_{L^\infty(B_1)}^2 \leq \lambda_1\log(e^3+C\|u\|_{\dot{C}^{0,\alpha}(B_1)}(\log(2e+\|u\|_{\dot{C}^{0,\alpha}(B_1)}))^{1/2}) \tag{1.5}$$

holds for all $u\in W_0^{1,2}(B_1)\cap\dot{C}^{0,\alpha}(B_1)$ with $\|\nabla u\|_{L^2(B_1)}=1$. Furthermore, if $\lambda_1<1/(2\pi\alpha)$, then the inequality (1.5) with any constant $C>0$ does not hold for some $u\in W_0^{1,2}(B_1)\cap\dot{C}^{0,\alpha}(B_1)$ with $\|\nabla u\|_{L^2(B_1)}=1$.

Theorem A (i) claims that $\lambda_1=1/(2\pi\alpha)$ is the sharp constant for the inequality (1.4), and (1.4) does not hold when λ_1 is just the sharp one. However, since the right hand side of (1.5) behaves like

$$\lambda_1\log\|u\|_{\dot{C}^{0,\alpha}(B_1)}+\frac{\lambda_1}{2}\log(\log\|u\|_{\dot{C}^{0,\alpha}(B_1)})+O(1) \text{ as } \|u\|_{\dot{C}^{0,\alpha}(B_1)}\to\infty,$$

Theorem A (ii) essentially claims that the inequality holds also for the sharp constant $\lambda_1=1/(2\pi\alpha)$ if we add a certain weak term in the right hand side.

In this paper, we generalize the inequalities above twofold. One is to deal with higher dimensions $n\geq 2$, and the other is to give two sharp constants of the coefficients of single and double logarithms in the inequality (1.5). Instead of the inequalities (1.4) and (1.5), we introduce a new formulation of the inequality:

$$\|u\|_{L^\infty(\Omega)}^{n/(n-1)} \leq \lambda_1 \log(1+\|u\|_{\dot{C}^{0,\alpha}(\Omega)}) + \lambda_2 \log(1+\log(1+\|u\|_{\dot{C}^{0,\alpha}(\Omega)})) + C \quad (1.6)$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$, where Ω is an arbitrary bounded domain in $\mathbb{R}^n$. We are here concerned with the sharpness of both constants λ_1 and λ_2, where C is a constant which may depend on Ω, α, λ_1 and λ_2. We remark that the power $n/(n-1)$ on the left hand side of (1.6) is also optimal in the sense that one cannot replace it by any larger power (see also Remark 3.5 below).

Our main purpose is to determine the sharp constants for λ_1 and λ_2 in (1.6). Note that these sharp constants may depend on the definition of $\|\nabla u\|_{L^n(\Omega)}$; there are several manners to define $\|\nabla u\|_{L^n(\Omega)}$. In what follows, we choose (1.2) as the definition of $\|\nabla u\|_{L^n(\Omega)}$, and then we shall show that $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$ are the sharp constants in (1.6) as described in the theorems below. Here and below, we denote

$$\Lambda_1 = \frac{1}{\omega_{n-1}^{1/(n-1)}}, \ \Lambda_2 = \frac{\Lambda_1}{n} = \frac{1}{n\omega_{n-1}^{1/(n-1)}}$$

and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$. More precisely, we have the following theorems, which essentially include Theorem A because $\Lambda_1 = 1/(2\pi)$ and $\Lambda_2 = \Lambda_1/2$ in the case $n = 2$.

Theorem 1.1. *Let $n \geq 2$, $0 < \alpha \leq 1$ and Ω be a bounded domain in $\mathbb{R}^n$. Assume that either*

$$\text{(I) } \lambda_1 > \frac{\Lambda_1}{\alpha} \ (\textit{and } \lambda_2 \in \mathbb{R}) \quad \textit{or} \quad \text{(II) } \lambda_1 = \frac{\Lambda_1}{\alpha} \textit{ and } \lambda_2 \geq \frac{\Lambda_2}{\alpha}$$

holds. Then there exists a constant C such that the inequality (1.6) *holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

Theorem 1.2. *Let $n \geq 2$, $0 < \alpha \leq 1$ and Ω be a bounded domain in $\mathbb{R}^n$. Assume that either*

$$\text{(III) } \lambda_1 < \frac{\Lambda_1}{\alpha} \ (\textit{and } \lambda_2 \in \mathbb{R}) \quad \textit{or} \quad \text{(IV) } \lambda_1 = \frac{\Lambda_1}{\alpha} \textit{ and } \lambda_2 < \frac{\Lambda_2}{\alpha}$$

holds. Then for any constant C, the inequality (1.6) *does not hold for some $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$.*

In Theorem A, it is not mentioned whether the power 1/2 of the inner logarithmic factor on the right hand side of (1.5) is optimal or not. On the other hand, we can assert that the power $1/2$ in (1.5) must be optimal by virtue of Theorem 1.2 (IV).

We are also interested in the existence of an extremal function of the inequality (1.6). Here, for fixed λ_1 and λ_2 such that (1.6) holds, we introduce the notion of the best constant and an extremal function as follows. We call

$$C_0 = \sup\{F[u; \lambda_1, \lambda_2];\ u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega),\ \|\nabla u\|_{L^n(\Omega)} = 1\}$$

the *best constant* for (1.6), where $F[u; \lambda_1, \lambda_2]$ is defined by

$$F[u; \lambda_1, \lambda_2] = \|u\|_{L^\infty(\Omega)}^{n/(n-1)} - \lambda_1 \log(1 + \|u\|_{\dot{C}^{0,\alpha}(\Omega)}) - \lambda_2 \log(1 + \log(1 + \|u\|_{\dot{C}^{0,\alpha}(\Omega)})).$$

We also call u_0 an *extremal function* of (1.6) if $C_0 = F[u_0; \lambda_1, \lambda_2]$. Since the inequality (1.6) corresponds to the critical embedding, we cannot expect any compactness property for treating that maximizing problem, and it is difficult to ensure the existence of an extremal function, in general. However, in the case that Ω is an open ball $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$, we can find an extremal function in some cases.

Theorem 1.3. *Let $n \geq 2$, $0 < \alpha \leq 1$, $R > 0$ and $\Omega = B_R$. Fix $\lambda_1, \lambda_2 \geq 0$ satisfying the assumption* (I) *or* (II) *in Theorem 1.1. If the best constant C_0 for the inequality* (1.6) *(with $\Omega = B_R$) is positive, then there exists an extremal function $u_0 \in W_0^{1,n}(B_R) \cap \dot{C}^{0,\alpha}(B_R)$ with $\|\nabla u_0\|_{L^n(B_R)} = 1$ of* (1.6).

Now we give some remarks on our results. The following remark is concerned with Theorems 1.1 and 1.2.

Remark 1.4. When we consider the inequality (1.6) without the double logarithmic term, i.e., $\lambda_2 = 0$, Theorem 1.1 (I) and Theorem 1.2 (III) claim that Λ_1/α is the sharp constant for λ_1, and (1.6) with $\lambda_1 = \Lambda_1/\alpha$ (and $\lambda_2 = 0$) fails to hold by virtue of Theorem 1.2 (IV). Hence, only in this case, it is essentially meaningful to consider the inequality with the double logarithmic term. Then Theorem 1.1 (II) and Theorem 1.2 (IV) claim that Λ_2/α is the sharp constant for λ_2 in the case $\lambda_1 = \Lambda_1/\alpha$, and (1.6) holds with these sharp constants. Therefore, even in the crucial case $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$, it is no more meaningful to consider an inequality with any weaker term such as the triple logarithmic term; see also Remark 3.6 below.

The following remark is concerned with Theorem 1.3.

Remark 1.5. (i) The assumption of the positivity of the best constant C_0 for the inequality (1.6) (with $\Omega = B_R$) in Theorem 1.3 seems to be technical.

(ii) In some cases, the best constant C_0 for the inequality (1.6) is actually positive, and hence there exists an extremal function of (1.6). In fact, if $1 - \alpha$, $\lambda_1 - \Lambda_1/\alpha$ and $\lambda_2 - \Lambda_2/\alpha_2$ are nonnegative and sufficiently small, then the best constant C_0 for (1.6) (with $\Omega = B_R$) is positive, provided that n and $1/R$ are not so large. In Section 4, we shall observe this fact especially in the case $R = 1$.

We here mention that Ozawa [15] gave another proof of the Brézis-Gallouët-Wainger inequality (1.1). First he refined a Gagliardo-Nirenberg inequality, which states that

$$\|u\|_{L^q(\mathbb{R}^n)} \le Cq^{1-1/p}\|u\|_{L^p(\mathbb{R}^n)}^{p/q}\|(-\Delta)^{n/(2p)}u\|_{L^p(\mathbb{R}^n)}^{1-p/q} \tag{1.7}$$

holds for all $u \in W^{n/p,p}(\mathbb{R}^n)$ with $p \le q < \infty$, where $1 < p < \infty$ and the constant C is independent of q. Then, by applying (1.7), he proved the Brézis-Gallouët-Wainger inequality (1.1). We note that the growth order $q^{1-1/p}$ of the coefficient on the right hand side as $q \to \infty$ is optimal, and (1.7) was originally obtained by Ogawa [13] in the case $n = p = 2$.

Furthermore, Kozono-Ogawa-Taniuchi [10] and Ogawa [14] recently studied similar estimates to (1.1) in Besov and Triebel-Lizorkin spaces, or BMO, and applied them to the Navier-Stokes equations and the Euler equations.

On the other hand, the Trudinger-Moser estimate is known as a dual inequality of the Brézis-Gallouët-Wainger inequality, which is the exponential type inequality characterizing the Sobolev critical case. As far as we know, the sharp constant of the Brézis-Gallouët-Wainger inequality is little known, while we can find some papers concerning the Trudinger-Moser estimate; see for instance Adachi-Tanaka [1], Kozono-Sato-Wadade [11] and references therein. In general, Brézis-Gallouët-Wainger type estimates can be obtained by the Trudinger-Moser estimate without giving their sharp constants; see [15] and [12].

Here we outline the proof of our results. First we note that the inequality (1.6) holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_{L^n(\Omega)} = 1$ if and only if there exists a constant C such that

$$\begin{aligned}\left(\frac{\|u\|_{L^\infty(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right)^{n/(n-1)} &- \lambda_1 \log\left(1 + \frac{\|u\|_{\dot{C}^{0,\alpha}(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right) \\ &-\lambda_2 \log\left(1 + \log\left(1 + \frac{\|u\|_{\dot{C}^{0,\alpha}(\Omega)}}{\|\nabla u\|_{L^n(\Omega)}}\right)\right) \le C\end{aligned} \tag{1.8}$$

holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega) \setminus \{0\}$. A scaling argument enables us to reduce the matters to the case $\Omega = B_1$. The key point of the proof of Theorems 1.1 and 1.2 is that we can explicitly determine the minimizer of the minimizing problem with a unilateral constraint

$$\inf\{\|\nabla u\|_{L^n(B_1)}^n;\ u \in W_0^{1,n}(B_1), u \ge h_\tau \text{ a.e. on } B_1\} \tag{1.9}$$

for $0 < \tau \leq 1$. Here the obstacle function h_τ is given by

$$h_\tau(x) = \tilde{h}_\tau(|x|) = 1 - \left(\frac{|x|}{T_\tau}\right)^\alpha \quad \text{for } x \in \mathbb{R}^n, \tag{1.10}$$

where

$$T_\tau = \tau \left(\alpha \log \frac{1}{\tau} + 1\right)^{1/\alpha}.$$

This approach is based on the argument by Ibrahim-Majdoub-Masmoudi [8] in the case $n = 2$. Since $W_0^{1,n}(B_1)$ is not a Hilbert space for $n \geq 3$, we are not able to use several tools for treating such a variational problem. Unlike the case in $W_0^{1,2}(B_1)$, little seems to be known on its regularity of a minimizer in the space $W_0^{1,n}(B_1)$ for $n \geq 3$, and we are not able to assume any regularity property of a minimizer. However, because of the uniqueness of a minimizer, it is radially symmetric and continuous on $\bar{B}_1 \setminus \{0\}$. Furthermore, we can show that the minimizer $u_\tau^\sharp$ is n-harmonic on the region $\{u_\tau^\sharp > h_\tau\}$. Then we can explicitly determine the shape of the minimizer with the aid of elementary one-dimensional calculi. Although we cannot assume any regularity of the minimizer, the explicit representation of the minimizer implies the C^1-regularity on $\bar{B}_1 \setminus \{0\}$ as a conclusion. Our method consists of calculating the norms of the minimizer and a simple scale argument. On the other hand, Ibrahim-Majdoub-Masmoudi [8] made use of the C^1-regularity of the minimizer and the theory of the rearrangement of functions to obtain Theorem A.

The organization of this paper is as follows. In Section 2, we investigate the minimizing problem (1.9). Then we can give the proof of Theorems 1.1 and 1.2, which will be described in Section 3. In Section 4, for λ_1 and λ_2 such that (1.6) holds, we consider the existence of an extremal function of (1.6) with the best constant C_0 in the case $\Omega = B_R$. In Section 5, we prove a few lemmas of elementary calculi which we stated in Sections 2 and 3. Section 6 is an appendix, where we give the proof of a certain inequality concerned with the Hölder seminorm of the rearrangement of a function.

2 Minimizing problem

Throughout this paper, let the dimension $n \geq 2$ and $0 < \alpha \leq 1$. In what follows, for simplicity we shall omit putting down n and α as subscripts of constants or functions to indicate the dependency. First of all, we introduce some function spaces. Let Ω be a bounded domain in $\mathbb{R}^n$. In what follows, we regard a function u on Ω as the function on $\mathbb{R}^n$ extended by $u = 0$ on $\mathbb{R}^n \setminus \Omega$, and we denote

$$\|u\|_p = \|u\|_{L^p(\mathbb{R}^n)}, \ \|\nabla u\|_p = \|\, |\nabla u|\, \|_p$$

for $1 \le p \le \infty$,

$$\|u\|_{(\alpha)} = \|u\|_{\dot{C}^{0,\alpha}(\mathbb{R}^n)} = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}},$$

for simplicity. Note that we have

$$\|\nabla u\|_p = \|\nabla u\|_{L^p(\Omega)}, \ \|u\|_{(\alpha)} = \|u\|_{\dot{C}^{0,\alpha}(\Omega)}$$

for all $u \in W_0^{1,p}(\Omega)$, and $u \in \dot{C}^{0,\alpha}(\Omega) \cap C_0(\Omega)$, respectively. We also note that the norm of $W_0^{1,p}(\Omega)$ is equivalent to $\|\nabla u\|_p$ if Ω is bounded and $1 \le p < \infty$, because of the Poincaré inequality. We denote by B_R the open ball in $\mathbb{R}^n$ centered at the origin of radius $R > 0$, i.e., $B_R = \{x \in \mathbb{R}^n;\ |x| < R\}$.

In order to prove our results, we examine a problem of minimizing $\|\nabla u\|_n^n$ with a unilateral constraint. More generally, for $1 < p < \infty$, we formulate the following minimizing problem:

$$m[\Omega, h] = \inf\{\|\nabla u\|_p^p;\ u \in K[\Omega, h]\}, \qquad (\mathrm{M}^p;\ \Omega, h)$$

where the obstacle h is a measurable function on Ω and

$$K[\Omega, h] = \{u \in W_0^{1,p}(\Omega);\ u \ge h \ \text{ a.e. on } \Omega\}.$$

In this section, we prove three auxiliary lemmas. The first one ensures the existence of a unique minimizer whenever the set $K[\Omega, h]$ is nonempty. Since the functional $K[\Omega, h] \ni u \mapsto \|\nabla u\|_p^p \in [0, \infty)$ is continuous, strictly convex, coercive, and $K[\Omega, h]$ is convex, (weakly) closed, we can obtain existence and uniqueness of the minimizer with the aid of [6, Chapter II, Proposition 1.2] as follows.

Lemma 2.1. *Let $1 < p < \infty$, Ω be a bounded domain in $\mathbb{R}^n$, h be a measurable function defined on Ω, and assume that $K[\Omega, h]$ is nonempty. Then there exists a unique minimizer $u^\sharp = u^\sharp[\Omega, h] \in K[\Omega, h]$ of $(\mathrm{M}^p;\ \Omega, h)$, that is, $\|\nabla u^\sharp\|_p^p = m[\Omega, h]$.*

Next we verify that the minimizer is p-harmonic on the (open) set $\{u^\sharp > h\}$ in the weak sense. We can prove the lemma below by a similar argument as in [7]. This property is well known for the case $p = 2$; see e.g. [7] and [9].

Lemma 2.2. *Let $1 < p < \infty$, Ω be a bounded domain in $\mathbb{R}^n$ and $h \in C(\bar{\Omega})$. Assume that $K[\Omega, h]$ is nonempty and the minimizer $u^\sharp = u^\sharp[\Omega, h]$ of $(\mathrm{M}^p;\ \Omega, h)$ is continuous on $\hat{\Omega}$ for some open subset $\hat{\Omega}$ of Ω. Then it holds*

$$\int_{O[\Omega,\hat{\Omega},h]} |\nabla u^\sharp(x)|^{p-2} \nabla u^\sharp(x) \cdot \nabla \phi(x)\, dx = 0 \ \text{ for all } \phi \in C_c^1(O[\Omega, \hat{\Omega}, h]), \quad (2.1)$$

where

$$O[\Omega,\hat{\Omega},h]=\{x\in\hat{\Omega};\ u^\sharp(x)>h(x)\}.$$

Proof. (a) First we show the *variational inequality*

$$\int_\Omega |\nabla u^\sharp(x)|^{p-2}\nabla u^\sharp(x)\cdot(\nabla u(x)-\nabla u^\sharp(x))dx\geq 0 \ \text{ for all } u\in K[\Omega,h]. \tag{2.2}$$

Since $K[\Omega,h]$ is convex, it holds $u^\sharp+\theta(u-u^\sharp)\in K[\Omega,h]$ for all $0<\theta\leq 1$ and $u\in K[\Omega,h]$. Then we have

$$\begin{aligned}0&\leq\frac{1}{\theta}(\|\nabla(u^\sharp+\theta(u-u^\sharp))\|_p^p-\|\nabla u^\sharp\|_p^p)\\&=\frac{1}{\theta}\int_\Omega(|\nabla(u^\sharp+\theta(u-u^\sharp))(x)|^p-|\nabla u^\sharp(x)|^p)dx\\&\to p\int_\Omega|\nabla u^\sharp(x)|^{p-2}\nabla u^\sharp(x)\cdot(\nabla u(x)-\nabla u^\sharp(x))dx \ \text{ as } \theta\searrow 0.\end{aligned}$$

(b) Note that $O[\Omega,\hat{\Omega},h]$ is open. For a fixed $\phi\in C^1_{\mathrm{c}}(O[\Omega,\hat{\Omega},h])\setminus\{0\}$, we set

$$\theta_0=\frac{\min\{u^\sharp(x)-h(x);\ x\in\operatorname{supp}\phi\}}{\|\phi\|_\infty},$$

and then θ_0 is positive. Moreover, we have $u^\sharp\pm\theta_0\phi\in K[\Omega,h]$. Indeed,

$$\mp\theta_0\phi(y)\leq\theta_0\|\phi\|_\infty=\min\{u^\sharp(x)-h(x);\ x\in\operatorname{supp}\phi\}\leq u^\sharp(y)-h(y)$$

for all $y\in\operatorname{supp}\phi$, which implies that $u^\sharp\pm\theta_0\phi\geq h$ a.e. on Ω. Substituting $u=u^\sharp\pm\theta_0\phi$ into (2.2) yields

$$\pm\theta_0\int_{O[\Omega,\hat{\Omega},h]}|\nabla u^\sharp(x)|^{p-2}\nabla u^\sharp(x)\cdot\nabla\phi(x)dx\geq 0,$$

and (2.1) follows. □

The goal of this section is to prove the following fact, which explicitly gives the minimizer $u^\sharp_\tau$ of the specific minimizing problem $(\mathrm{M}^n;\ B_1,h_\tau)$ with a parameter $0<\tau\leq 1$, where h_τ is defined by (1.10). We also denote

$$K_\tau=K[B_1,h_\tau]=\{u\in W^{1,n}_0(B_1);\ u\geq h_\tau \ \text{ a.e. on } B_1\}.$$

Lemma 2.3. *For any $0<\tau\leq 1$, the unique minimizer $u^\sharp_\tau$ of $(\mathrm{M}^n;\ B_1,h_\tau)$ is given by*

$$u^\sharp_\tau(x)=\tilde{u}^\sharp_\tau(|x|)=\begin{cases}h_\tau(x) & \text{for } x\in\bar{B}_\tau,\\ \alpha\left(\dfrac{\tau}{T_\tau}\right)^\alpha\log\dfrac{1}{|x|} & \text{for } x\in B_1\setminus B_\tau.\end{cases} \tag{2.3}$$

The aim of this section is to prove Lemma 2.3. We need a lemma and several propositions.

Proposition 2.4. *Let $h \in C(\bar{B}_1)$ be a radially symmetric function and assume that $K[B_1, h]$ is nonempty.*

(i) *The minimizer $u^\sharp = u^\sharp[B_1, h]$ of $(\mathrm{M}^n;\, B_1, h)$ is radially symmetric and continuous on $\bar{B}_1 \setminus \{0\}$.*

(ii) *The set $O = O[B_1, B_1 \setminus \{0\}, h]$ can be decomposed into a disjoint (at most countable) union $\{O^{(j)}\}_{j=1}^J$ of annuli with $J \in \mathbb{N} \cup \{0, \infty\}$, that is,*

$$O = \bigcup_{j=1}^{J} O^{(j)},\ O^{(j)} = \{r\omega;\, a^{(j)} < r < b^{(j)},\, \omega \in S^{n-1}\} = (a^{(j)}, b^{(j)}) \times S^{n-1},$$

where $0 \le a^{(j)} < b^{(j)} \le 1$, and $\{(a^{(j)}, b^{(j)})\}_{j=1}^J$ is disjoint.

(iii) *For each j, there exist two constants $c^{(j)}, \bar{c}^{(j)} \in \mathbb{R}$ such that*

$$u^\sharp(x) = \tilde{u}^\sharp(|x|) = c^{(j)} \log \frac{1}{|x|} + \bar{c}^{(j)} \quad \textit{for } x \in O^{(j)}.$$

Proof. (i) The minimizer $u^\sharp$ of $(\mathrm{M}^n;\, B_1, h)$ is radially symmetric because of the uniqueness. Then we can write $u^\sharp(x) = \tilde{u}^\sharp(|x|)$ for $x \in \bar{B}_1$ by introducing a one-variable function $\tilde{u}^\sharp$. Since $\tilde{u}^\sharp \in W^{1,n}_{\mathrm{loc}}((0,1])$, the Sobolev embedding theorem in one dimension implies that $\tilde{u}^\sharp$ is continuous on $(0, 1]$, and hence $u^\sharp$ is continuous on $\bar{B}_1 \setminus \{0\}$.

(ii) By virtue of (i), there exists an open set $\tilde{O}$ in $(0,1)$ such that $O = \tilde{O} \times S^{n-1}$. Hence there exist disjoint (at most countable) open intervals $\{(a^{(j)}, b^{(j)})\}_{j=1}^J$ such that $\tilde{O} = \bigcup_{j=1}^J (a^{(j)}, b^{(j)})$. Then the assertion holds by putting $O^{(j)} = (a^{(j)}, b^{(j)}) \times S^{n-1}$.

(iii) Since the function $\mathbb{R}^n \ni x \mapsto \tilde{\phi}(|x|) \in \mathbb{R}$ belongs to $C^1_{\mathrm{c}}(O^{(j)})$ for all $\tilde{\phi} \in C^1_{\mathrm{c}}((a^{(j)}, b^{(j)}))$, we have from (2.1) that

$$\omega_{n-1} \int_{a^{(j)}}^{b^{(j)}} |(\tilde{u}^\sharp)'(r) r|^{n-2} (\tilde{u}^\sharp)'(r) r \tilde{\phi}'(r)\, dr = 0 \ \text{ for all } \tilde{\phi} \in C^1_{\mathrm{c}}((a^{(j)}, b^{(j)})).$$

By applying [3, Lemme VIII.1], there exists a constant $c_j \in \mathbb{R}$ such that

$$|(\tilde{u}^\sharp)'(r) r|^{n-2} (\tilde{u}^\sharp)'(r) r = -|c^{(j)}|^{n-2} c^{(j)} \ \text{ for a.e. } a^{(j)} < r < b^{(j)}.$$

Since the function $\mathbb{R} \ni s \mapsto |s|^{n-2} s \in \mathbb{R}$ is bijective, we have

$$(\tilde{u}^\sharp)'(r) r = -c^{(j)} \ \text{ for a.e. } a^{(j)} < r < b^{(j)}.$$

Therefore, there exists a constant $\bar{c}^{(j)} \in \mathbb{R}$ such that $\tilde{u}^\sharp(r) = c^{(j)} \log(1/r) + \bar{c}^{(j)}$ for $a^{(j)} < r < b^{(j)}$, and then $u^\sharp(x) = c^{(j)} \log(1/|x|) + \bar{c}^{(j)}$ for $x \in O^{(j)}$. □

Proposition 2.5. *Let* $0 < \tau \le 1$*,* $c, \bar{c} \in \mathbb{R}$ *and* $0 < a < b \le 1$*. If* $\tilde{u}(r) = c\log(1/r) + \bar{c}$ *for* $a \le r \le b$ *and* $\tilde{h}_\tau(a) = \tilde{u}(a)$*,* $\tilde{h}_\tau(b) = \tilde{u}(b)$*, then* $\tilde{h}_\tau > \tilde{u}$ *on* (a, b)*.*

Proof. Since $(\tilde{h}_\tau - \tilde{u})(a) = (\tilde{h}_\tau - \tilde{u})(b) = 0$ and

$$(r(\tilde{h}_\tau - \tilde{u})')'(r) = -\frac{\alpha^2}{T_\tau^\alpha}\frac{1}{r^{1-\alpha}} < 0 \text{ for } a < r < b,$$

we conclude that $\tilde{h}_\tau - \tilde{u} > 0$ on (a, b) by using the maximum principle. □

Proposition 2.6. *For any* $0 < \tau \le 1$ *and* $0 < a \le 1$*, define*

$$w_{\tau,a}(x) = \tilde{w}_{\tau,a}(|x|) = \begin{cases} h_\tau(x) & \text{for } x \in \bar{B}_a, \\ \dfrac{1 - (a/T_\tau)^\alpha}{\log(1/a)} \log\dfrac{1}{|x|} & \text{for } x \in B_1 \setminus B_a. \end{cases}$$

(i) *There hold* $w_{\tau,a} \in W_0^{1,n}(B_1)$ *and*

$$\|\nabla w_{\tau,a}\|_n^n = \omega_{n-1}\alpha^{n-1}\left(\frac{(a/T_\tau)^{n\alpha}}{n} + \frac{|1 - (a/T_\tau)^\alpha|^n}{(\alpha\log(1/a))^{n-1}}\right) \text{ for } \tau \le a \le 1.$$

(ii) *It holds* $w_{\tau,a} \in K_\tau$ *if and only if* $\tau \le a \le 1$*.*

Proof. (i) We can show the assertion by the direct calculation.

(ii) Define

$$\psi_\tau(a) = \frac{1 - (a/T_\tau)^\alpha}{\log(1/a)} \text{ for } 0 < a \le T_\tau.$$

Then we can easily show that $\psi_\tau(a) \to 0$ as $a \searrow 0$, $\psi_\tau(T_\tau) = 0$, ψ_τ increases on $(0, \tau)$ and decreases on (τ, T_τ). Hence for any $0 < a < \tau$, there exists $\tau < r_a < T_\tau$ uniquely such that $\psi_\tau(a) = \psi_\tau(r_a)$. This implies that $\tilde{w}_{\tau,a}(a) = \tilde{h}_\tau(a)$, $\tilde{w}_{\tau,a}(r_a) = \tilde{w}_{\tau,r_a}(r_a) = \tilde{h}_\tau(r_a)$ and

$$\tilde{w}_{\tau,a}(r) = \psi_\tau(a)\log\frac{1}{r} < \tilde{h}_\tau(r) \text{ for } a < r < r_a$$

by virtue of Proposition 2.5. This means $w_{\tau,a} \notin K_\tau$.

On the other hand, we can easily show that $\tilde{w}_{\tau,a} \ge \tilde{h}_\tau$ on $(0, 1)$ for $\tau \le a \le 1$, and hence $w_{\tau,a} \in K_\tau$ for $\tau \le a \le 1$. □

Proposition 2.7. *For any* $0 < \tau \le 1$*, there exists* $\tau \le a_\tau \le 1$ *uniquely such that* $u_\tau^\sharp = w_{\tau,a_\tau}$ *on* B_1*. In particular,* $u_1^\sharp = h_1$ *on* B_1*.*

Proof. We denote $O_\tau = O[B_1, B_1 \setminus \{0\}, h_\tau]$ as in Lemma 2.2 (or Proposition 2.4) and $O_\tau = \tilde{O}_\tau \times S^{n-1}$. Furthermore, the argument in the proof of Proposition 2.4 (ii) enables the decomposition $\tilde{O}_\tau = \bigcup_{j=1}^{J_\tau}(a_\tau^{(j)}, b_\tau^{(j)})$ with $J_\tau \in \mathbb{N} \cup \{0, \infty\}$, where $0 \le a_\tau^{(j)} < b_\tau^{(j)} \le 1$, and $\{(a_\tau^{(j)}, b_\tau^{(j)})\}_{j=1}^{J_\tau}$ is disjoint.

(a) First we show that either $\tilde{O}_\tau$ is empty or $\tilde{O}_\tau = (a_\tau, 1)$ with some $0 < a_\tau < 1$. To prove this, we have only to show that $J_\tau = 1$ and that $0 < a_\tau^{(1)} < b_\tau^{(1)} = 1$. If $0 < a_\tau^{(j)} < b_\tau^{(j)} < 1$ for some j, then $\tilde{u}_\tau^\sharp(a_\tau^{(j)}) = \tilde{h}_\tau(a_\tau^{(j)})$ and $\tilde{u}_\tau^\sharp(b_\tau^{(j)}) = \tilde{h}_\tau(b_\tau^{(j)})$, and it follows from Proposition 2.4 (iii) and Proposition 2.5 that

$$\tilde{u}_\tau^\sharp(r) = c_\tau^{(j)} \log \frac{1}{r} + \bar{c}_\tau^{(j)} < \tilde{h}_\tau(r) \ \text{ for } a_\tau^{(j)} < r < b_\tau^{(j)},$$

which contradicts the definition of $\tilde{O}_\tau$. If $0 = a_\tau^{(j)} < b_\tau^{(j)} \le 1$ for some j, then Proposition 2.4 (iii) implies $\|\nabla u_\tau^\sharp\|_{L^n(O_\tau^{(j)})} = \infty$, which is a contradiction. Consequently, the claim is proved.

(b) The case $0 < \tau < 1$. Since $\tilde{u}_\tau^\sharp(1) = 0 > \tilde{h}_\tau(1)$, we see that $\tilde{O}_\tau$ is nonempty and $\tilde{O}_\tau = (a_\tau, 1)$ with some $0 < a_\tau < 1$. It follows from Proposition 2.6 (ii) that $\tau \le a_\tau < 1$. From the continuity of $\tilde{u}_\tau^\sharp$ on $(0, 1]$ and Proposition 2.4 (iii), we have $u_\tau^\sharp = w_{\tau, a_\tau}$ on B_1.

(c) The case $\tau = 1$. Suppose that $\tilde{O}_1$ is nonempty, i.e. $\tilde{O}_1 = (a_1, 1)$ with some $0 < a_1 < 1$. As we argued in (b), we have $\tau \le a_1$ and $u_1^\sharp = w_{1,a_1}$ on B_1. Then it follows $\tau \le a_1 < 1$, which contradicts $\tau = 1$. Therefore, $\tilde{O}_1$ is empty, and hence $u_1^\sharp = h_1 = w_{1,1}$. □

We can determine a_τ in Proposition 2.7 once we accept the following lemma, which will be proved in Section 5.

Lemma 2.8. *For $\rho > 0$, define*

$$H(\sigma; \rho) = \frac{\sigma^n}{n} + \frac{(1-\sigma)^n}{(\rho - \log(\sigma(\rho+1)))^{n-1}} \ \textit{ for } \frac{1}{\rho+1} \le \sigma \le 1.$$

Then for any $\rho > 0$, $H(\sigma; \rho)$ attains its minimum only at $\sigma = 1/(\rho+1)$.

We are now in a position to prove Lemma 2.3.

Proof of Lemma 2.3. (a) In view of Proposition 2.7, we may assume $0 < \tau < 1$. By the definition of $u_\tau^\sharp$, we can characterize a_τ in Proposition 2.7 as

$$\|\nabla w_{\tau, a_\tau}\|_n^n = \min_{\tau \le a \le 1} \|\nabla w_{\tau, a}\|_n^n. \tag{2.4}$$

By virtue of Proposition 2.6 (i), we have that

$$\|\nabla w_{\tau, a}\|_n^n > \|\nabla w_{\tau, T_\tau}\|_n^n \ \text{ for } T_\tau < a \le 1,$$

and hence $\tau \le a_\tau \le T_\tau$.

(b) By virtue of Lemma 2.8, we have that

$$H\left(\frac{(a/\tau)^\alpha}{\alpha\log(1/\tau)+1};\alpha\log\frac{1}{\tau}\right) \ge H\left(\frac{1}{\alpha\log(1/\tau)+1};\alpha\log\frac{1}{\tau}\right) \quad \text{for } \tau \le a \le T_\tau$$

and that the equality holds only if $a = \tau$. Then we obtain

$$\begin{aligned}
\|\nabla w_{\tau,a}\|_n^n &= \omega_{n-1}\alpha^{n-1}\left(\frac{(a/T_\tau)^{n\alpha}}{n} + \frac{(1-(a/T_\tau)^\alpha)^n}{(\alpha\log(1/a))^{n-1}}\right) \\
&= \omega_{n-1}\alpha^{n-1} H\left(\frac{(a/\tau)^\alpha}{\alpha\log(1/\tau)+1};\alpha\log\frac{1}{\tau}\right) \\
&\ge \omega_{n-1}\alpha^{n-1} H\left(\frac{1}{\alpha\log(1/\tau)+1};\alpha\log\frac{1}{\tau}\right) \\
&= \|\nabla w_{\tau,\tau}\|_n^n \quad \text{for } \tau \le a \le T_\tau,
\end{aligned}$$

and $a_\tau = \tau$ follows. Therefore, we conclude that $u_\tau^\sharp = w_{\tau,\tau}$ on B_1. □

Remark 2.9. As is mentioned in the introduction, we cannot assume that the minimizer $u_\tau^\sharp$ is of class C^1 in $B_1 \setminus \{0\}$. However, in our argument, we obtained $a_\tau = \tau$ so that (2.4) holds. As a conclusion, the minimizer has the C^1-regularity except for the origin. In fact, we see that $w_{\tau,a} \in C^1(B_1 \setminus \{0\})$ if and only if $a = \tau$.

Remark 2.10. We can calculate the norms of $u_\tau^\sharp$ as

$$\|u_\tau^\sharp\|_\infty = 1,\ \|\nabla u_\tau^\sharp\|_n^n = \left(\frac{\alpha}{\Lambda_1}\right)^{n-1}\frac{\alpha\log(1/\tau)+1/n}{(\alpha\log(1/\tau)+1)^n},$$
$$\|u_\tau^\sharp\|_{(\alpha)} = \frac{1}{T_\tau^\alpha} = \frac{1}{\tau^\alpha(\alpha\log(1/\tau)+1)}.$$

Although these are straightforward and elementary, we shall include the verification of the third equality for the sake of completeness. First we note that

$$\|u_\tau^\sharp\|_{(\alpha)} = \sup_{0\le\rho<r<1}\frac{\tilde{u}_\tau^\sharp(\rho)-\tilde{u}_\tau^\sharp(r)}{(r-\rho)^\alpha}$$

since $u_\tau^\sharp$ is radially symmetric. Next we see that

$$\frac{\tilde{u}_\tau^\sharp(0)-\tilde{u}_\tau^\sharp(r)}{(r-0)^\alpha} = \frac{\tilde{h}_\tau(0)-\tilde{h}_\tau(r)}{r^\alpha} = \frac{1}{T_\tau^\alpha} \quad \text{for } 0 < r \le \tau.$$

Using the inequality

$$\alpha\log s \le s^\alpha - 1 \le (s-1)^\alpha \quad \text{for } s \ge 1,$$

we easily obtain

$$\frac{\tilde{u}_\tau^\sharp(\rho) - \tilde{u}_\tau^\sharp(r)}{(r-\rho)^\alpha} \le \frac{\tilde{h}_\tau(\rho) - \tilde{h}_\tau(r)}{(r-\rho)^\alpha} = \frac{1}{T_\tau^\alpha} \frac{(r/\rho)^\alpha - 1}{(r/\rho - 1)^\alpha} \le \frac{1}{T_\tau^\alpha}$$
$$\text{for } 0 \le \rho \le \tau,\ \rho < r \le 1,$$
$$\frac{\tilde{u}_\tau^\sharp(\rho) - \tilde{u}_\tau^\sharp(r)}{(r-\rho)^\alpha} = \frac{1}{T_\tau^\alpha} \left(\frac{\tau}{\rho}\right)^\alpha \frac{\alpha \log(r/\rho)}{(r/\rho - 1)^\alpha} \le \frac{1}{T_\tau^\alpha} \quad \text{for } \tau \le \rho < r \le 1.$$

Therefore, we have $\|u_\tau^\sharp\|_{(\alpha)} = 1/T_\tau^\alpha$.

3 Sharp constants for λ_1 and λ_2

In this section, we prove Theorems 1.1 and 1.2. We use the notation

$$\ell(s) = \log(1+s) \ \text{ for } s \ge 0, \tag{3.1}$$

for simplicity and then $\ell \circ \ell(s) = \log(1 + \log(1+s))$ for $s \ge 0$. In order to examine whether (1.8) holds or not, we may assume $\lambda_1 \ge 0$ and define

$$F[u; \lambda_1, \lambda_2] = \left(\frac{\|u\|_\infty}{\|\nabla u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) - \lambda_2 \ell \circ \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right)$$
$$\text{for } u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega) \setminus \{0\}$$

and

$$F^*[\lambda_1, \lambda_2; \Omega] = \sup\{F[u; \lambda_1, \lambda_2];\ u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega) \setminus \{0\}\}$$
$$\text{for } \lambda_1 \ge 0,\ \lambda_2 \in \mathbb{R}.$$

Note that

$$F[cu; \lambda_1, \lambda_2] = F[u; \lambda_1, \lambda_2] \ \text{ for all } c \in \mathbb{R} \setminus \{0\}.$$

Then Theorems 1.1 and 1.2 are equivalent to the following:

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the following hold:*

(i) *For any $\lambda_1 > \Lambda_1/\alpha$ and $\lambda_2 \in \mathbb{R}$, it holds $F^*[\lambda_1, \lambda_2; \Omega] < \infty$;*
(ii) *For any $\lambda_2 \ge \Lambda_2/\alpha$, it holds $F^*[\Lambda_1/\alpha, \lambda_2; \Omega] < \infty$;*
(iii) *For any $0 \le \lambda_1 < \Lambda_1/\alpha$ and $\lambda_2 \in \mathbb{R}$, it holds $F^*[\lambda_1, \lambda_2; \Omega] = \infty$;*
(iv) *For any $\lambda_2 < \Lambda_2/\alpha$, it holds $F^*[\Lambda_1/\alpha, \lambda_2; \Omega] = \infty$.*

The aim of this section is to prove Lemma 3.1. Let us first reduce our problem on a general bounded domain Ω to that on the unit open ball B_1. We set

$$\hat{K} = \{u \in W_0^{1,n}(B_1) \cap \dot{C}^{0,\alpha}(B_1);\ \|u\|_\infty = u(0) = 1\}$$

and

$$\hat{F}^*[\lambda_1, \lambda_2] = \sup\{F[u; \lambda_1, \lambda_2];\ u \in \hat{K}\} \ \text{ for } \lambda_1 \geq 0,\ \lambda_2 \in \mathbb{R}.$$

Let s_+ denote the positive part of $s \in \mathbb{R}$, i.e., $s_+ = \max\{s, 0\}$.

Proposition 3.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}$. Then, $\hat{F}^*[\lambda_1, \lambda_2] < \infty$ holds if and only if $F^*[\lambda_1, \lambda_2; \Omega] < \infty$.*

Proof. (a) First we show that $\hat{F}^*[\lambda_1, \lambda_2] < \infty$ implies $F^*[\lambda_1, \lambda_2; \Omega] < \infty$. For any $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega) \setminus \{0\}$, which is regarded as a function on $\mathbb{R}^n$ by the zero-extension on $\mathbb{R}^n \setminus \Omega$, there exists $z_u \in \Omega$ such that $\|u\|_\infty = |u(z_u)| > 0$. We set

$$v_u(x) = \frac{\operatorname{sgn} u(z_u)}{\|u\|_\infty} u(d_\Omega x + z_u) \ \text{ for } x \in \mathbb{R}^n,$$

where $d_\Omega = \operatorname{diam} \Omega = \sup\{|x - y|;\ x, y \in \Omega\}$. Then we have $v_u \in \hat{K}$ and

$$\|\nabla v_u\|_n = \frac{\|\nabla u\|_n}{\|u\|_\infty},\ \|v_u\|_{(\alpha)} = d_\Omega^\alpha \frac{\|u\|_{(\alpha)}}{\|u\|_\infty}.$$

Since $\max\{\ell(st), \ell(s+t)\} \leq \ell(s) + \ell(t)$ for $s, t \geq 0$, we have

$$\begin{aligned}
&F[u; \lambda_1, \lambda_2] \\
&= \left(\frac{\|v_u\|_\infty}{\|\nabla v_u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{1}{d_\Omega^\alpha} \frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) - \lambda_2 \ell \circ \ell\left(\frac{1}{d_\Omega^\alpha} \frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) \\
&\leq \left(\frac{\|v_u\|_\infty}{\|\nabla v_u\|_n}\right)^{n/(n-1)} - \lambda_1 \ell\left(\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) + \lambda_1 \ell(d_\Omega^\alpha) \\
&\quad - \lambda_2 \ell \circ \ell\left(\frac{\|v_u\|_{(\alpha)}}{\|\nabla v_u\|_n}\right) + |\lambda_2| \ell \circ \ell(d_\Omega^{\alpha \operatorname{sgn} \lambda_2}) \\
&= F[v_u; \lambda_1, \lambda_2] + \lambda_1 \ell(d_\Omega^\alpha) + |\lambda_2| \ell \circ \ell(d_\Omega^{\alpha \operatorname{sgn} \lambda_2}) \\
&\leq \hat{F}^*[\lambda_1, \lambda_2] + \lambda_1 \ell(d_\Omega^\alpha) + |\lambda_2| \ell \circ \ell(d_\Omega^{\alpha \operatorname{sgn} \lambda_2}) \\
&\qquad\qquad \text{for } u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega) \setminus \{0\}.
\end{aligned}$$

Therefore, if $\hat{F}^*[\lambda_1, \lambda_2] < \infty$, then $F^*[\lambda_1, \lambda_2; \Omega] < \infty$.

(b) Next we show that $\hat{F}^*[\lambda_1, \lambda_2] = \infty$ implies $F^*[\lambda_1, \lambda_2; \Omega] = \infty$, conversely. Fix $z_0 \in \Omega$ and $R_0 > 0$ such that $B = \{x \in \mathbb{R}^n;\ |x - z_0| < 1/R_0\} \subset \Omega$. Assume that $\hat{F}^*[\lambda_1, \lambda_2] = \infty$. Then there exists a sequence $\{v_j\}_{j=1}^\infty \subset \hat{K}$ such that $F[v_j; \lambda_1, \lambda_2] \to \infty$ as $j \to \infty$. If we set $u_j(x) = v_j(R_0(x - z_0))$ for $x \in \mathbb{R}^n$, then $u_j \in W_0^{1,n}(B) \cap \dot{C}^{0,\alpha}(B) \subset W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ and we have

$$\|u_j\|_\infty = \|v_j\|_\infty,\ \|\nabla u_j\|_n = \|\nabla v_j\|_n,\ \|u_j\|_{(\alpha)} = R_0^\alpha \|v_j\|_{(\alpha)}.$$

A similar calculation as in (a) yields

$$F[v_j; \lambda_1, \lambda_2] \leq F[u_j; \lambda_1, \lambda_2] + \lambda_1 \ell(R_0^\alpha) + |\lambda_2| \ell \circ \ell(R_0^{\alpha \operatorname{sgn} \lambda_2}),$$

and it follows $F[u_j;\lambda_1,\lambda_2]\to\infty$ as $j\to\infty$. Therefore, we obtain $F^*[\lambda_1,\lambda_2;\Omega]=\infty$. □

For $\kappa>0$ and $\mu_1,\mu_2\geq 0$, define

$$G_\kappa(s;\mu_1,\mu_2)=\left(\frac{(s+1)^n}{s+1/n}\right)^{1/(n-1)}-\mu_1\ell\left(\frac{\kappa e^s}{(s+1/n)^{1/n}}\right)\\ -\frac{\mu_2}{n}\ell\circ\ell\left(\frac{\kappa e^s}{(s+1/n)^{1/n}}\right)\quad\text{for } s\geq 0.$$

Then we can show a relation between $\hat{F}^*[\lambda_1,\lambda_2]$ and $G_\kappa(s;\mu_1,\mu_2)$ as follows. The idea of the proof is essentially due to [8].

Proposition 3.3. *For any $\lambda_1\geq 0$ and $\lambda_2\in\mathbb{R}$, it holds*

$$\hat{F}^*[\lambda_1,(\lambda_2)_+]\leq\frac{\Lambda_1}{\alpha}\sup_{s\geq 0}G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+. \tag{3.2}$$

Proof. (a) We show that

$$\hat{K}=\bigcup_{0<\tau\leq 1}\hat{K}_\tau, \tag{3.3}$$

where

$$\hat{K}_\tau=\left\{u\in K_\tau\cap\dot{C}^{0,\alpha}(B_1);\ \|u\|_{(\alpha)}=\frac{1}{T_\tau^\alpha},\ \|u\|_\infty=u(0)=1\right\}.$$

It is trivial that $\hat{K}_\tau\subset\hat{K}$ for all $0<\tau\leq 1$. Conversely, for any $u\in\hat{K}$, we have

$$\|u\|_{(\alpha)}\geq\sup_{x\in\partial B_1}\frac{|u(x)-u(0)|}{|x|^\alpha}=1,$$

and

$$u(x)=1-|u(x)-u(0)|\geq 1-\|u\|_{(\alpha)}|x|^\alpha\quad\text{for } x\in\bar{B}_1.$$

Then, $u\in\hat{K}_\tau$ with $1/T_\tau^\alpha=\|u\|_{(\alpha)}\geq 1$, and hence we obtain (3.3).

(b) Next we show that

$$F[u;\lambda_1,(\lambda_2)_+]\leq F[u_\tau^\sharp;\lambda_1,\lambda_2]_+\quad\text{for } u\in\hat{K}_\tau. \tag{3.4}$$

Note that $\|\nabla u\|_n\geq\|\nabla u_\tau^\sharp\|_n$ for all $u\in K_\tau$. We also remark that $u_\tau^\sharp\in\hat{K}_\tau$ because $\|u_\tau^\sharp\|_{(\alpha)}=1/T_\tau^\alpha$ and $\|u_\tau^\sharp\|_\infty=u_\tau^\sharp(0)=1$. Since the functions

$$(0,\infty)\ni s\mapsto s^{n/(n-1)}\ell\left(\frac{1}{s}\right)\in(0,\infty),\ (0,\infty)\ni s\mapsto s^{n/(n-1)}\ell\circ\ell\left(\frac{1}{s}\right)\in(0,\infty)$$

are both increasing, we have

$$
\begin{aligned}
&\|\nabla u\|_n^{n/(n-1)} F[u;\lambda_1,(\lambda_2)_+] \\
&= 1-\lambda_1\|\nabla u\|_n^{n/(n-1)}\ell\left(\frac{1}{T_\tau^\alpha}\frac{1}{\|\nabla u\|_n}\right)-(\lambda_2)_+\|\nabla u\|_n^{n/(n-1)}\ell\circ\ell\left(\frac{1}{T_\tau^\alpha}\frac{1}{\|\nabla u\|_n}\right)\\
&\le 1-\lambda_1\|\nabla u_\tau^\sharp\|_n^{n/(n-1)}\ell\left(\frac{1}{T_\tau^\alpha}\frac{1}{\|\nabla u_\tau^\sharp\|_n}\right)-\lambda_2\|\nabla u_\tau^\sharp\|_n^{n/(n-1)}\ell\circ\ell\left(\frac{1}{T_\tau^\alpha}\frac{1}{\|\nabla u_\tau^\sharp\|_n}\right)\\
&= \|\nabla u_\tau^\sharp\|_n^{n/(n-1)} F[u_\tau^\sharp;\lambda_1,\lambda_2]\\
&\le \|\nabla u\|_n^{n/(n-1)} F[u_\tau^\sharp;\lambda_1,\lambda_2]_+ \quad \text{for } u\in\hat{K}_\tau,
\end{aligned}
$$

which implies (3.4).

(c) It follows from Remark 2.10 that

$$
F[u_\tau^\sharp;\lambda_1,\lambda_2]=\frac{\Lambda_1}{\alpha}G_{(\Lambda_1/\alpha)^{1-1/n}}\left(\alpha\log\frac{1}{\tau};\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)\quad\text{for } 0<\tau\le 1. \tag{3.5}
$$

Combining (3.3)–(3.5) yields

$$
\begin{aligned}
\sup_{u\in\hat{K}} F[u;\lambda_1,(\lambda_2)_+] &\le \sup_{0<\tau\le 1}\sup_{u\in\hat{K}_\tau} F[u;\lambda_1,(\lambda_2)_+]\\
&\le \sup_{0<\tau\le 1} F[u_\tau^\sharp;\lambda_1,\lambda_2]_+\\
&= \frac{\Lambda_1}{\alpha}\sup_{0<\tau\le 1} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(\alpha\log\frac{1}{\tau};\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+\\
&= \frac{\Lambda_1}{\alpha}\sup_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+,
\end{aligned}
$$

which implies (3.2). □

We also denote $G_\kappa(s)=G_\kappa(s;1,1)$ for simplicity. The following lemma gives the behavior of the function $G_\kappa(s;\mu_1,\mu_2)$ as $s\to\infty$, which plays an essential role for proving Lemma 3.1. We shall use it also in Section 4 before proving it in Section 5.

Lemma 3.4. *Let $\kappa>0$.*

(i) *If either $\mu_1>1$, $\mu_2\in\mathbb{R}$, or $\mu_1=1$, $\mu_2>1$, then $G_\kappa(s;\mu_1,\mu_2)\to-\infty$ as $s\to\infty$. In particular, there exists $s_\kappa[\mu_1,\mu_2]\ge 0$ such that*

$$
G_\kappa(s_\kappa[\mu_1,\mu_2];\mu_1,\mu_2)=\max_{s\ge 0} G_\kappa(s;\mu_1,\mu_2). \tag{3.6}
$$

(ii) *There exist $\hat{s}_\kappa>0$ and $\hat{G}_\kappa\in\mathbb{R}$ such that*

$$
G_\kappa'(s)<0 \quad \text{for } s>\hat{s}_\kappa, \tag{3.7}
$$

and $G_\kappa(s) \to \hat{G}_\kappa$ as $s \to \infty$. In particular, there exists $s_\kappa[1,1] \geq 0$ such that (3.6) holds with $\mu_1 = \mu_2 = 1$.

(iii) *If either $\mu_1 < 1$, $\mu_2 \in \mathbb{R}$, or $\mu_1 = 1$, $\mu_2 < 1$, then $G_\kappa(s;\mu_1,\mu_2) \to \infty$ as $s \to \infty$.*

We now show Lemma 3.1 by using Proposition 3.3 and Lemma 3.4. We divide the assertion (i) in Lemma 3.1 into the following two assertions for the sake of convenience:

(i-1) *For any $\lambda_1 > \Lambda_1/\alpha$ and $\lambda_2 \geq 0$, it holds $F^*[\lambda_1,\lambda_2;\Omega] < \infty$;*
(i-2) *For any $\lambda_1 > \Lambda_1/\alpha$ and $\lambda_2 < 0$, it holds $F^*[\lambda_1,\lambda_2;\Omega] < \infty$.*

Proof of Lemma 3.1. (a) First we show the assertions (i-1) and (ii). We take $\mu_1 = \alpha\lambda_1/\Lambda_1$, $\mu_2 = \alpha\lambda_2/\Lambda_2$ and $s = \alpha\log(1/\tau)$. By virtue of Proposition 3.2, the assertions (i-1) and (ii) follow from Lemma 3.4 (i) and (ii), respectively.

(b) First we show the assertion (i-2). Since $\ell\circ\ell(s)/\ell(s) \to 0$ as $s \to \infty$, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\ell\circ\ell(s) \leq \varepsilon\ell(s) + C_\varepsilon \quad \text{for } s \geq 0.$$

By choosing $0 < \delta < \lambda_1 - \Lambda_1/\alpha$, we have from (a) that $\hat{F}^*[\lambda_1-\delta,0] < \infty$. Then

$$\begin{aligned}
&\sup_{u\in\hat{K}} F[u;\lambda_1,\lambda_2] \\
&= \sup_{u\in\hat{K}}\left(F[u;\lambda_1-\delta,0] - \lambda_2\left(\frac{\delta}{\lambda_2}\ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) + \ell\circ\ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right)\right)\right) \\
&\leq \hat{F}^*[\lambda_1-\delta,0] - \lambda_2 C_{-\delta/\lambda_2} \\
&< \infty,
\end{aligned}$$

and the assertion follows.

(c) Finally we show the assertions (iii) and (iv). In view of Proposition 3.2, it suffices to show that $\limsup_{\tau\searrow 0} F[u_\tau^\sharp;\lambda_1,\lambda_2] = \infty$, because $u_\tau^\sharp \in \hat{K}$ for all $0 < \tau \leq 1$. However, this follows immediately from Lemma 3.4 (iii) and (3.5). □

Thus we have proved Theorems 1.1 and 1.2.

Remark 3.5. As is mentioned in the introduction, the power $n/(n-1)$ on the left hand side of (1.6) is optimal in the sense that $q = n/(n-1)$ is the largest power for which

$$\|u\|_\infty^q \leq \lambda_1 \log(1 + \|u\|_{(\alpha)}) + C \tag{3.8}$$

can hold for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_n = 1$. Indeed, if $q > n/(n-1)$, then for any $\lambda_1 > 0$ and any constant C, (3.8) does not hold for some $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_n = 1$. On the contrary, if $1 \le q < n/(n-1)$, then for any $\lambda_1 > 0$, there exists a constant C such that (3.8) holds for all $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_n = 1$. To verify these facts, we have only to consider the behavior of the function

$$G_\kappa^q(s;\mu_1) = \left(\frac{(s+1)^n}{s+1/n}\right)^{q/n} - \mu_1 \ell\left(\frac{\kappa e^s}{(s+1/n)^{1/n}}\right) \quad \text{for } s \ge 0$$

as $s \to \infty$ instead of $G_\kappa(s;\mu_1,\mu_2)$.

Remark 3.6. As is mentioned in Remark 1.4, it is no more meaningful to consider an inequality with any weaker term. More precisely, we can prove the following facts. We shall omit the proof because one can prove them by a slight modification of the proof of Lemma 3.4.

(i) We choose a continuous function $\gamma\colon [0,\infty) \to [0,\infty)$ such that

$$\gamma(s) \to \infty, \ \frac{\gamma(s)}{\ell \circ \ell(s)} \to 0 \ \text{ as } s \to \infty,$$

$\max\{\gamma(st), \gamma(s+t)\} \le \gamma(s) + \gamma(t) + c$ for $s, t \ge 0$ for some constant $c \ge 0$, the functions

$$(0,\infty) \ni s \mapsto s^{n/(n-1)}\gamma\left(\frac{1}{s}\right) \in (0,\infty),\ (0,\infty) \ni s \mapsto s^{n/(n-1)}\gamma\circ\gamma\left(\frac{1}{s}\right) \in (0,\infty)$$

are both increasing, and consider the inequality

$$\|u\|_\infty^{n/(n-1)} \le \lambda_1 \ell(\|u\|_{(\alpha)}) + \lambda_2 \ell \circ \ell(\|u\|_{(\alpha)}) + \lambda\gamma(\|u\|_{(\alpha)}) + C$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_n = 1$. Then this inequality holds if and only if one of the following holds:

(I) $\lambda_1 > \Lambda_1/\alpha$ (and $\lambda_2, \lambda \in \mathbb{R}$);
(II-1) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 > \Lambda_2/\alpha$ (and $\lambda \in \mathbb{R}$);
(II-2) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$ and $\lambda \ge 0$.

(ii) Let $N \ge 3$ and consider the N-ple logarithmic inequality

$$\|u\|_\infty^{n/(n-1)} \le \sum_{j=1}^{N} \lambda_j \underbrace{\ell \circ \cdots \circ \ell}_{j}(\|u\|_{(\alpha)}) + C$$

for $u \in W_0^{1,n}(\Omega) \cap \dot{C}^{0,\alpha}(\Omega)$ with $\|\nabla u\|_n = 1$. Then this inequality holds if and only if one of the following holds:

(I) $\lambda_1 > \Lambda_1/\alpha$ (and $\lambda_2, \dots, \lambda_N \in \mathbb{R}$);

(II-1) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 > \Lambda_2/\alpha$ (and $\lambda_3, \dots, \lambda_N \in \mathbb{R}$);
(II-2′) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$, $\lambda_3 = \cdots = \lambda_{m-1} = 0$, $\lambda_m > 0$ for some $3 \le m \le N$ (and $\lambda_{m+1}, \dots, \lambda_N \in \mathbb{R}$);
(II-2″) $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$ and $\lambda_3 = \cdots = \lambda_N = 0$.

4 Existence of an extremal function

In this section, for fixed $\lambda_1, \lambda_2 \ge 0$ such that the inequality (1.6) holds, we consider the existence of an extremal function of (1.6) with the best constant C_0. Though it is difficult to ensure the existence of an extremal function for cases with general domains, we can find an extremal function in the case $\Omega = B_R$ with constants λ_1 and λ_2 in a suitable region. Our method is due to the argument described in the previous section.

For $R > 0$, define

$$u^\sharp_{\tau,R}(x) = u^\sharp_\tau\left(\frac{x}{R}\right),\ h_{\tau,R}(x) = h_\tau\left(\frac{x}{R}\right) \quad \text{for } x \in B_R.$$

We note that $u^\sharp_{\tau,R}$ is the minimizer of (M^n; $B_R, h_{\tau,R}$).

Lemma 4.1. *Let $R > 0$, and fix $\lambda_1, \lambda_2 \ge 0$ satisfying the assumption* (I) *or* (II) *in Theorem 1.1, i.e.,*

$$(\mathrm{I}')\ \lambda_1 > \frac{\Lambda_1}{\alpha} \text{ and } \lambda_2 \ge 0 \quad \text{or} \quad (\mathrm{II})\ \lambda_1 = \frac{\Lambda_1}{\alpha} \text{ and } \lambda_2 \ge \frac{\Lambda_2}{\alpha}.$$

(i) *If*

$$\sup_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}/R^\alpha}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right) \ge 0, \tag{4.1}$$

then there exists $0 < \tau_0 \le 1$ such that

$$\begin{aligned} F^*[\lambda_1, \lambda_2; B_R] &= F\left[\frac{u^\sharp_{\tau_0,R}}{\|\nabla u^\sharp_{\tau_0,R}\|_n}; \lambda_1, \lambda_2\right] \\ &= \frac{\Lambda_1}{\alpha} \max_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}/R^\alpha}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right). \end{aligned} \tag{4.2}$$

In particular, $u^\sharp_{\tau_0,R}/\|\nabla u^\sharp_{\tau_0,R}\|_n$ is an extremal function of (1.6) *with $\Omega = B_R$.*

(ii) *The best constant C_0 for the inequality* (1.6) *(with $\Omega = B_R$) is positive if and only if*

$$\sup_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}/R^\alpha}\left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2\right) > 0. \tag{4.3}$$

Because of Lemma 3.4 (i) and $\inf_{s\geq 0}\ell(\kappa e^s/(s+1/n)^{1/n}) > 0$, choosing a sufficiently large λ_1 forces (4.3) to fail for any fixed $\lambda_2 \geq 0$. In particular, we obtain the following corollary.

Corollary 4.2. *Let $n \geq 2$, $0 < \alpha \leq 1$, $R > 0$ and $\Omega = B_R$. If $\lambda_1 \geq \Lambda_1/\alpha$ is sufficiently large, then the best constant C_0 for the inequality (1.6) with $\lambda_2 = 0$ (and $\Omega = B_R$) is nonpositive. In particular,*

$$\|u\|_\infty^{n/(n-1)} \leq \lambda_1 \log(1 + \|u\|_{(\alpha)})$$

holds for all $u \in W_0^{1,n}(B_R) \cap \dot{C}^{0,\alpha}(B_R)$ with $\|\nabla u\|_n = 1$.

Here and below, we consider only in the case $R = 1$ for simplicity; one can argue similarly for a general $R > 0$. We need the following proposition to prove Lemma 4.1. We here have to introduce the rearrangement of a function. We denote by u^* the nonnegative symmetric decreasing rearrangement of u, i.e.,

$$u^*(x^*) = \inf\left\{t > 0;\, a[u](t) \leq \frac{\omega_{n-1}}{n}|x^*|^n\right\},$$

where $a[u](t) = |\{x \in \mathbb{R}^n;\, |u(x)| > t\}|$. We shall use the inequalities

$$\|u^*\|_\infty = \|u\|_\infty,\ \|\nabla u^*\|_n \leq \|\nabla u\|_n, \tag{4.4}$$

$$\|u^*\|_{(\alpha)} \leq \|u\|_{(\alpha)}. \tag{4.5}$$

While the inequalities (4.4) are well known, the inequality (4.5) seems to be little known. For the sake of completeness, we shall give the proof of (4.5) in Section 6 by using the Brunn-Minkowski inequality.

Proposition 4.3. *If $\lambda_1, \lambda_2 \geq 0$, then $F^*[\lambda_1, \lambda_2; B_1] \leq \hat{F}^*[\lambda_1, \lambda_2]_+$.*

Proof. Since $u^*/\|u^*\|_\infty \in \hat{K}$ for all $u \in W_0^{1,n}(B_1) \cap \dot{C}^{0,\alpha}(B_1) \setminus \{0\}$, it suffices to show that

$$F[u;\lambda_1,\lambda_2] \leq F\left[\frac{u^*}{\|u^*\|_\infty};\lambda_1,\lambda_2\right]_+ \quad \text{for } u \in W_0^{1,n}(B_1) \cap \dot{C}^{0,\alpha}(B_1) \setminus \{0\}. \tag{4.6}$$

Since the functions

$$(0,\infty) \ni s \mapsto s^{n/(n-1)}\ell\left(\frac{1}{s}\right) \in (0,\infty),\ (0,\infty) \ni s \mapsto s^{n/(n-1)}\ell\circ\ell\left(\frac{1}{s}\right) \in (0,\infty)$$

are both increasing, we have

$$\begin{aligned}
&\|\nabla u\|_n^{n/(n-1)} F[u;\lambda_1,\lambda_2] \\
&= \|u\|_\infty^{n/(n-1)} - \lambda_1 \|\nabla u\|_n^{n/(n-1)} \ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) - \lambda_2 \|\nabla u\|_n^{n/(n-1)} \ell\circ\ell\left(\frac{\|u\|_{(\alpha)}}{\|\nabla u\|_n}\right) \\
&\le \|u^*\|_\infty^{n/(n-1)} - \lambda_1 \|\nabla u^*\|_n^{n/(n-1)} \ell\left(\frac{\|u^*\|_{(\alpha)}}{\|\nabla u^*\|_n}\right) \\
&\quad - \lambda_2 \|\nabla u^*\|_n^{n/(n-1)} \ell\circ\ell\left(\frac{\|u^*\|_{(\alpha)}}{\|\nabla u^*\|_n}\right) \\
&= \|\nabla u^*\|_n^{n/(n-1)} F\left[\frac{u^*}{\|u^*\|_\infty};\lambda_1,\lambda_2\right] \\
&\le \|\nabla u\|_n^{n/(n-1)} F\left[\frac{u^*}{\|u^*\|_\infty};\lambda_1,\lambda_2\right]_+ \quad \text{for } u \in W_0^{1,n}(B_1)\cap \dot{C}^{0,\alpha}(B_1)\setminus\{0\},
\end{aligned}$$

which implies (4.6). □

Proof of Lemma 4.1. (i) By virtue of Lemma 3.4 (i)–(ii), the function $s \mapsto G_{(\Lambda_1/\alpha)^{1-1/n}}(s;\alpha\lambda_1/\Lambda_1,\alpha\lambda_2/\Lambda_2)$ is bounded from above and there exists $s_0 \ge 0$ such that

$$G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s_0;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right) = \sup_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right).$$

Define $0 < \tau_0 \le 1$ by

$$\tau_0 = \frac{1}{\exp(s_0/\alpha)}, \quad \text{i.e.,} \quad s_0 = \alpha \log\frac{1}{\tau_0}.$$

By applying (4.1), it holds

$$F\left[\frac{u^\sharp_{\tau_0}}{\|\nabla u^\sharp_{\tau_0}\|_n};\lambda_1,\lambda_2\right] = F\left[\frac{u^\sharp_{\tau_0}}{\|u^\sharp_{\tau_0}\|_\infty};\lambda_1,\lambda_2\right] = \hat{F}^*[\lambda_1,\lambda_2] \ge 0. \tag{4.7}$$

Indeed, in view of (3.2), we have

$$\begin{aligned}
\hat{F}^*[\lambda_1,\lambda_2] &\le \frac{\Lambda_1}{\alpha}\sup_{s\ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right)_+ \\
&= \frac{\Lambda_1}{\alpha} G_{(\Lambda_1/\alpha)^{1-1/n}}\left(s_0;\frac{\alpha}{\Lambda_1}\lambda_1,\frac{\alpha}{\Lambda_2}\lambda_2\right) \\
&= F\left[\frac{u^\sharp_{\tau_0}}{\|u^\sharp_{\tau_0}\|_\infty};\lambda_1,\lambda_2\right],
\end{aligned}$$

which implies (4.7) because $u^\sharp_{\tau_0}/\|u^\sharp_{\tau_0}\|_\infty \in \hat{K}$. By virtue of Proposition 4.3, we obtain (4.2).

(ii) Note that the best constant C_0 for the inequality (1.6) with $\Omega = B_1$ coincides with $F^*[\lambda_1, \lambda_2; B_1]$. If $F^*[\lambda_1, \lambda_2; B_1] > 0$, then we have from Proposition 4.3 and (3.2) that

$$0 < F^*[\lambda_1, \lambda_2; B_1] \le \hat{F}^*[\lambda_1, \lambda_2] \le \frac{\Lambda_1}{\alpha} \sup_{s \ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}} \left(s; \frac{\alpha}{\Lambda_1}\lambda_1, \frac{\alpha}{\Lambda_2}\lambda_2 \right)_+ ,$$

and (4.3) follows. Conversely, if (4.3) holds, then $F^*[\lambda_1, \lambda_2; B_1] > 0$ immediately follows from (i). □

As is mentioned in Remark 1.5 (ii), we shall examine the condition (4.1), which is a sufficient condition for the existence of an extremal function of (1.6), in the special case $R = 1$ and $\lambda_1 = \Lambda_1/\alpha$, $\lambda_2 = \Lambda_2/\alpha$. In fact, we can show the following proposition.

Proposition 4.4. *Let $R = 1$. Then one of the following holds:*

(i) There exists no $0 < \alpha \le 1$ such that (4.1) holds with $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$.
(ii) There exists $0 < \alpha_0 \le 1$ such that (4.1) holds with $\lambda_1 = \Lambda_1/\alpha$ and $\lambda_2 = \Lambda_2/\alpha$ if and only if $\alpha_0 \le \alpha \le 1$.

We give the proof of the proposition above once we accept the following lemma.

Lemma 4.5.(i) *It holds*

$$\sup_{s \ge 0} G_\kappa(s) \to -\infty \quad \text{as } \kappa \to \infty. \tag{4.8}$$

(ii) *If $\kappa > 0$ satisfies*

$$\sup_{s \ge 0} G_\kappa(s) < 0,$$

then there exists $\varepsilon_0 > 0$ such that

$$\sup_{s \ge 0} G_{\kappa - \varepsilon}(s) < 0 \quad \text{for } 0 < \varepsilon < \varepsilon_0.$$

Proof of Proposition 4.4. Define

$$\begin{aligned} A_0 &= \left\{ 0 < \alpha \le 1;\ (4.1) \text{ holds with } \lambda_1 = \frac{\Lambda_1}{\alpha} \text{ and } \lambda_2 = \frac{\Lambda_2}{\alpha} \right\} \\ &= \left\{ 0 < \alpha \le 1;\ \sup_{s \ge 0} G_{(\Lambda_1/\alpha)^{1-1/n}}(s) \ge 0 \right\}. \end{aligned}$$

We have to show that either $A_0 = \emptyset$ or $A_0 = [\alpha_0, 1]$ for some $0 < \alpha_0 \le 1$ holds. For this purpose, we have only to show the following.

(i) If $\tilde{\alpha} \in A_0$, then $\alpha \in A_0$ for all $\tilde{\alpha} < \alpha \le 1$.

(ii) If $\alpha \in A_0$ for all $\tilde{\alpha} < \alpha \leq 1$, then $\tilde{\alpha} \in A_0$.
(iii) $(0, 1] \setminus A_0$ is nonempty.

Since $G_\kappa(s)$ is decreasing with respect to κ, $G_{(\Lambda_1/\alpha)^{1-1/n}}(s)$ is increasing with respect to α for each $s \geq 0$, and hence the assertion (i) holds. The assertion (iii) immediately follows from Lemma 4.5 (i). Moreover, Lemma 4.5 (ii) claims that $(0, 1] \setminus A_0$ is open in $(0, 1]$, and the assertion (ii) follows. □

Remark 4.6. One can easily verify that Proposition 4.4 is valid also for $R \geq 1/e^{1-1/n}$ since $G_{(\Lambda_1/\alpha)^{1-1/n}/R^\alpha}(s)$ is increasing with respect to α for each $s \geq 0$.

Next we give the proof of Lemma 4.5.

Proof of Lemma 4.5. (i) Since $e^s \geq (1 + ns)^{1/n}$ for $s \geq 0$, it follows

$$\ell\left(\frac{e^s}{(s + 1/n)^{1/n}}\right) \leq \log\left(\left(1 + \frac{1}{n^{1/n}}\right)\frac{e^s}{(s + 1/n)^{1/n}}\right) \quad \text{for } s \geq 0.$$

Then we have

$$\begin{aligned}
G_\kappa(s) &= G_1(s) - \ell\left(\frac{\kappa e^s}{(s + 1/n)^{1/n}}\right) + \ell\left(\frac{e^s}{(s + 1/n)^{1/n}}\right) \\
&\quad - \frac{1}{n}\ell \circ \ell\left(\frac{\kappa e^s}{(s + 1/n)^{1/n}}\right) + \frac{1}{n}\ell \circ \ell\left(\frac{e^s}{(s + 1/n)^{1/n}}\right) \\
&\leq G_1(s) - \log\frac{\kappa e^s}{(s + 1/n)^{1/n}} + \log\left(\left(1 + \frac{1}{n^{1/n}}\right)\frac{e^s}{(s + 1/n)^{1/n}}\right) \\
&= G_1(s) + \ell\left(\frac{1}{n^{1/n}}\right) - \log\kappa \quad \text{for } s \geq 0,\ \kappa \geq 1,
\end{aligned}$$

which implies (4.8) because $G_1(s)$ is bounded.

(ii) Set

$$\eta(t) = \frac{1}{n}\left(\frac{t}{(1 + \ell(t))^2} - \frac{1}{1 + \ell(t)} - n\right) \quad \text{for } t \geq 0.$$

Then there exists $t_0 > 0$ such that

$$\eta(t) > 0 \quad \text{for } t > t_0.$$

Since

$$\frac{\partial}{\partial\kappa}[G'_\kappa(s)] = \frac{se^s}{(s + 1/n)^{1-1/n}(\kappa e^s + (s + 1/n)^{1/n})^2}\eta\left(\frac{\kappa e^s}{(s + 1/n)^{1/n}}\right) \quad \text{for } s > 0,$$

there exists $\tilde{s}_\kappa > 0$ such that

$$G'_\kappa(s) - G'_{\kappa-\varepsilon}(s) > 0 \text{ for } s > \tilde{s}_\kappa, \ 0 < \varepsilon < \frac{\kappa}{2}. \tag{4.9}$$

Indeed, if we choose $\tilde{s}_\kappa > 0$ such that $\kappa e^s/(2(s+1/n)^{1/n}) > t_0$ for $s > \tilde{s}_\kappa$, then (4.9) is satisfied. By virtue of Lemma 3.4 (ii) and (4.9), we have

$$G'_{\kappa-\varepsilon}(s) < G'_\kappa(s) < 0 \text{ for } s > \max\{\tilde{s}_\kappa, \hat{s}_\kappa\}, \ 0 < \varepsilon < \frac{\kappa}{2}.$$

In particular, we have

$$s_{\kappa-\varepsilon}[1,1] \leq \max\{\tilde{s}_\kappa, \hat{s}_\kappa\} \text{ for } 0 < \varepsilon < \frac{\kappa}{2}. \tag{4.10}$$

On the other hand, since

$$G_{\kappa-\varepsilon}(s) \to G_\kappa(s) \text{ as } \varepsilon \searrow 0 \text{ uniformly for } 0 \leq s \leq \max\{\tilde{s}_\kappa, \hat{s}_\kappa\},$$

there exists $0 < \varepsilon_0 < \kappa/2$ such that

$$|G_{\kappa-\varepsilon}(s) - G_\kappa(s)| < -\frac{1}{2}G_\kappa(s_\kappa[1,1]) \text{ for } 0 \leq s \leq \max\{\tilde{s}_\kappa, \hat{s}_\kappa\}, \ 0 < \varepsilon < \varepsilon_0.$$

Then we can show

$$G_{\kappa-\varepsilon}(s) < \frac{1}{2}G_\kappa(s_\kappa[1,1]) \text{ for } 0 \leq s \leq \max\{\tilde{s}_\kappa, \hat{s}_\kappa\}, \ 0 < \varepsilon < \varepsilon_0, \tag{4.11}$$

because

$$G_{\kappa-\varepsilon}(s) - G_\kappa(s_\kappa[1,1]) \leq |G_{\kappa-\varepsilon}(s) - G_\kappa(s)| < -\frac{1}{2}G_\kappa(s_\kappa[1,1])$$
$$\text{for } 0 \leq s \leq \max\{\tilde{s}_\kappa, \hat{s}_\kappa\}, \ 0 < \varepsilon < \varepsilon_0.$$

By using (4.10) and (4.11), we obtain

$$G_{\kappa-\varepsilon}(s) \leq G_{\kappa-\varepsilon}(s_{\kappa-\varepsilon}[1,1]) < \frac{1}{2}G_\kappa(s_\kappa[1,1]) < 0 \text{ for } s \geq 0, \ 0 < \varepsilon < \varepsilon_0,$$

and the assertion follows. □

Remark 4.7. Let us examine whether $\alpha \in A_0$ or not in the case $R = 1$. Since $G_{(\Lambda_1/\alpha)^{1-1/n}}(s)$ is increasing with respect to α, if the condition

$$\sup_{s\geq 0} G_{(\Lambda_1/\alpha)^{1-1/n}}(s) \geq 0 \tag{4.12}$$

holds for some $0 < \alpha < 1$, then it must hold

$$\sup_{s\geq 0} G_{\Lambda_1^{1-1/n}}(s) > 0. \tag{4.13}$$

Conversely, if (4.13) holds, then (4.12) is satisfied for α sufficiently close to 1. We invoke the following observation with the aid of computer calculations.

(i) If $n \leq 131$, then $G_{\Lambda_1^{1-1/n}}(s_1) > 0$ for some $s_1 > 0$, which implies that $A_0 = [\alpha_0, 1]$ for some $0 < \alpha_0 < 1$. Indeed, we can observe it by choosing $s_1 = 6$.

(ii) If $n \geq 132$, then (4.13) seems to fail, which implies $A_0 = \emptyset$. Furthermore, (4.1) also seems to fail with any $\lambda_1 \geq \Lambda_1/\alpha$, $\lambda_2 \geq \Lambda_2/\alpha$ and $0 < \alpha \leq 1$.

5 Elementary calculi

In this section, we prove a few lemmas of elementary calculi which we stated in Sections 2 and 3. For the definition of the function ℓ, see (3.1).

Proof of Lemma 2.8. We have to show that

$$H(\sigma;\rho) - H\left(\frac{1}{\rho+1};\rho\right) > 0 \text{ for } \frac{1}{\rho+1} < \sigma \leq 1,\ \rho > 0.$$

By the change of variables $s = \sigma(\rho+1) - 1$, $r = \rho - \sigma(\rho+1) + 1$, this is equivalent to

$$H\left(\frac{s+1}{r+s+1};r+s\right) - H\left(\frac{1}{r+s+1};r+s\right) > 0 \text{ for } r \geq 0,\ s > 0.$$

Define auxiliary functions H_1 and H_2 by

$$H_1(r;s) = (r+s+1)^n \left(H\left(\frac{s+1}{r+s+1};r+s\right) - H\left(\frac{1}{r+s+1};r+s\right)\right)$$
$$\text{for } r \geq 0,\ s > 0$$

and

$$H_2(s) = \frac{(s+1)^n - 1 - ns}{n} - (n-1)(s - \ell(s)) \text{ for } s > 0.$$

Since $H_2(0) = 0$ and

$$H_2'(s) = \frac{(s+1)^n - 1 - ns}{s+1} = \frac{1}{s+1}\sum_{j=2}^{n}\binom{n}{j}s^j > 0 \text{ for } s > 0,$$

it holds that $H_2(s) > 0$ for $s > 0$.

From the definition of H_1, we have

$$H_1(r;s) = \frac{(s+1)^n - 1 - ns}{n} - r + \frac{r^n}{(r+s-\ell(s))^{n-1}}$$
$$= \frac{(s+1)^n - 1 - ns}{n} - \frac{1}{(r+s-\ell(s))^{n-1}} \sum_{j=1}^{n-1} \binom{n-1}{j} (s-\ell(s))^j r^{n-j}$$
$$\text{for } r \geq 0,\ s > 0,$$

and hence

$$H_1(r;s) \to H_2(s) > 0 \ \text{ as } r \to \infty \ \text{ for } s > 0.$$

On the other hand, since

$$\frac{\partial H_1}{\partial r}(r;s) = -\frac{1}{(r+s-\ell(s))^n} \sum_{j=2}^{n} \binom{n}{j} (s-\ell(s))^j r^{n-j} < 0 \ \text{ for } r \geq 0,\ s > 0,$$

we conclude that $H_1(r;s) > 0$ for $r \geq 0$, $s > 0$. □

Proof of Lemma 3.4. (a) First we note that

$$s \leq \left(\frac{(s+1)^n}{s+1/n} \right)^{1/(n-1)} \leq s + 1 + \frac{1}{n} \ \text{ for } s \geq 0 \tag{5.1}$$

and

$$\log t \leq \ell(t) \leq \log(2t) \ \text{ for } t \geq 1. \tag{5.2}$$

We can choose $s_\kappa > 0$ such that

$$\left(s + \frac{1}{n} \right)^{1/n} \leq \kappa e^{s-1} \ \text{ for } s \geq s_\kappa,$$

and define an auxiliary function

$$g_\kappa(s) = \frac{s + 1/n}{s - (\log(s+1/n))/n + \log \kappa} \ \text{ for } s > s_\kappa.$$

Then it holds $g_\kappa(s) \to 1$ as $s \to \infty$.

(b) Set $\iota(\mu) = 1$ if $\mu \leq 0$, and $\iota(\mu) = 2$ if $\mu > 0$. By using (5.1) and (5.2), we have

$$G_\kappa(s;\mu_1,\mu_2)$$
$$\leq s + 1 + \frac{1}{n} - \mu_1 \log \frac{\kappa e^s}{(s+1/n)^{1/n}} - \frac{\mu_2}{n} \log \left(\iota(-\mu_2) \log \frac{\iota(-\mu_2)\kappa e^s}{(s+1/n)^{1/n}} \right)$$
$$= -(\mu_1 - 1)s - \frac{\mu_2 - \mu_1}{n} \log \left(s + \frac{1}{n} \right) + \frac{\mu_2}{n} \log \frac{g_{\iota(-\mu_2)\kappa}(s)}{\iota(-\mu_2)} - \mu_1 \log \kappa + 1 + \frac{1}{n}$$
$$\text{for } s > s_\kappa.$$

Hence, under the assumption of (i), it holds $G_\kappa(s;\mu_1,\mu_2) \to -\infty$ as $s \to \infty$.

(c) By using (5.1) and (5.2), we have

$$\begin{aligned}
&G_\kappa(s;\mu_1,\mu_2) \\
&\geq s - \mu_1 \log \frac{2\kappa e^s}{(s+1/n)^{1/n}} - \frac{\mu_2}{n} \log \left(\iota(\mu_2) \log \frac{\iota(\mu_2)\kappa e^s}{(s+1/n)^{1/n}} \right) \\
&= (1-\mu_1)s + \frac{\mu_1 - \mu_2}{n} \log \left(s + \frac{1}{n} \right) + \frac{\mu_2}{n} \log \frac{g_{\iota(\mu_2)\kappa}(s)}{\iota(\mu_2)} - \mu_1 \log(2\kappa) \\
&\qquad \text{for } s > s_\kappa.
\end{aligned}$$

Hence, under the assumption of (iii), it holds $G_\kappa(s;\mu_1,\mu_2) \to \infty$ as $s \to \infty$.

(d) By virtue of (b) and (c), G_κ is bounded. Hence, for the proof of (ii), it suffices to show that (3.7) holds for some $\hat{s}_\kappa > 0$. Set

$$\begin{aligned}
&\tilde{G}_\kappa(s) \\
&= \frac{1}{s+1} \left(1 + \ell \left(\frac{\kappa e^s}{(s+1/n)^{1/n}} \right) - \frac{1}{n} \frac{1}{((s+1)/(s+1/n))^{1/(n-1)} - 1} \right) \\
&= \frac{1}{n-1} \sum_{j=1}^{n-1} \left(1 - \left(\frac{s+1/n}{s+1} \right)^{j/(n-1)} \right) + \frac{1}{s+1} \log \left(\frac{\kappa}{(s+1/n)^{1/n}} + \frac{1}{e^s} \right).
\end{aligned}$$

Then we have

$$\begin{aligned}
-\frac{(s+1)^2}{\ell(s)-1} \tilde{G}'_\kappa(s) = \frac{1}{\ell(s)-1} \Bigg[&\frac{1}{n(n-1)} \sum_{j=1}^{n-1} j \left(\frac{s+1}{s+1/n} \right)^{1-j/(n-1)} \\
&+ \frac{1}{n} \frac{(s+1)(\kappa e^s + n(s+1/n)^{1+1/n})}{(s+1/n)(\kappa e^s + (s+1/n)^{1/n})} \\
&+ \log \left(\kappa + \frac{(s+1/n)^{1/n}}{e^s} \right) - \frac{1}{n} \log \left(s + \frac{1}{n} \right) \Bigg] \\
\to -\frac{1}{n} \ \text{ as } s \to \infty.&
\end{aligned}$$

Since $\tilde{G}_\kappa(s) \to 0$ as $s \to \infty$, it follows from de l'Hospital's rule that

$$\frac{s+1}{\ell(s)} \tilde{G}_\kappa(s) \to -\frac{1}{n} \quad \text{as } s \to \infty.$$

In particular, there exists $s_\kappa > 0$ such that

$$\tilde{G}_\kappa(s) < -\frac{1}{2n} \frac{\ell(s)}{s+1} \quad \text{for } s \geq s_\kappa,$$

which is equivalent to

$$
\begin{aligned}
&\left(\frac{s+1}{s+1/n}\right)^{1/(n-1)} - 1 - \frac{1}{n}\frac{1}{1+\ell(\kappa e^s/(s+1/n)^{1/n})} \\
&< -\frac{1}{2n}\frac{\ell(s)}{1+\ell(\kappa e^s/(s+1/n)^{1/n})}\left(\left(\frac{s+1}{s+1/n}\right)^{1/(n-1)} - 1\right) \quad \text{for } s \geq s_\kappa.
\end{aligned}
$$

Since

$$
s\left(\left(\frac{s+1}{s+1/n}\right)^{1/(n-1)} - 1\right) \to \frac{1}{n} \quad \text{as } s \to \infty,
$$

we have

$$
\begin{aligned}
&\frac{s(s+1/n)}{\ell(s)}G'_\kappa(s) \\
&= \frac{s^2}{\ell(s)}\left(\left(\frac{s+1}{s+1/n}\right)^{1/(n-1)} - 1 - \frac{1}{n}\frac{1}{1+\ell(\kappa e^s/(s+1/n)^{1/n})}\right) \\
&\quad + \frac{s^2(s+1/n)^{1/n}}{(\kappa e^s + (s+1/n)^{1/n})\ell(s)}\left(1 + \frac{1}{n}\frac{1}{1+\ell(\kappa e^s/(s+1/n)^{1/n})}\right) \\
&< -\frac{1}{2n}\frac{s^2}{1+\ell(\kappa e^s/(s+1/n)^{1/n})}\left(\left(\frac{s+1}{s+1/n}\right)^{1/(n-1)} - 1\right) \\
&\quad + \frac{s^2(s+1/n)^{1/n}}{(\kappa e^s + (s+1/n)^{1/n})\ell(s)}\left(1 + \frac{1}{n}\frac{1}{1+\ell(\kappa e^s/(s+1/n)^{1/n})}\right) \\
&\to -\frac{1}{2n^2} \quad \text{as } s \to \infty.
\end{aligned}
$$

Therefore, (3.7) holds for sufficiently large $\hat{s}_\kappa > 0$. □

6 Appendix

In this section, we give the proof of the inequality (4.5) for the sake of completeness, which was needed in proving the existence of an extremal function. We shall prove it for an arbitrary bounded domain Ω. We shall make use of the Brunn-Minkowski inequality in our proof.

Lemma 6.1. *Let $n \geq 1$, $0 < \alpha \leq 1$ and Ω be a bounded domain in $\mathbb{R}^n$. Then it holds*

$$
\|u^*\|_{(\alpha)} \leq \|u\|_{(\alpha)} \quad \textit{for } u \in \dot{C}^{0,\alpha}(\Omega).
$$

The proof is based on [2, Lemma 2.1] which provides the proof in the case $\alpha = 1$. To describe it in details, we introduce some notation. For a compact set K in $\mathbb{R}^n$, let $r^*[K]$ be the radius of the ball having the same volume as K;

$$r^*[K] = \left(\frac{n}{\omega_{n-1}}|K|\right)^{1/n}.$$

We denote by $\bar{B}_\rho(x)$ the closed ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ with radius $\rho > 0$, i.e., $\bar{B}_\rho(x) = \{z \in \mathbb{R}^n;\ |z - x| \leq \rho\}$. Let $i_\rho(K)$ be the set of all centers of all closed balls of radius $\rho > 0$ lying entirely in K, and $e_\rho(K)$ be the union of all closed balls of radius $\rho > 0$ whose centers lie in K;

$$i_\rho(K) = \{x \in K;\ \bar{B}_\rho(x) \subset K\},\ e_\rho(K) = \bigcup_{x \in K} \bar{B}_\rho(x) = K + \bar{B}_\rho,$$

which are called the *interior parallel set* and the *exterior parallel set* of K in distance ρ, respectively. Here, we denote $K+\tilde{K} = \{x+y \in \mathbb{R}^n;\ x \in K,\ y \in \tilde{K}\}$. We prove Lemma 6.1 by using the following lemma (see also [2, Chapter I, §1.3]).

Lemma 6.2. *Let K be a compact set in $\mathbb{R}^n$.*

(i) *For $\rho > 0$, it holds $r^*[e_\rho(K)] \geq r^*[K] + \rho$.*
(ii) *If $r^*[K] \geq \rho > 0$, then it holds $r^*[i_\rho(K)] \leq r^*[K] - \rho$.*

Proof. (i) We use the Brunn-Minkowski inequality

$$|K + \tilde{K}|^{1/n} \geq |K|^{1/n} + |\tilde{K}|^{1/n} \tag{6.1}$$

for compact sets $K, \tilde{K}$ in $\mathbb{R}^n$; see [16] and [17] for instance. Then the assertion is a special case of (6.1); substitute $\tilde{K} = \bar{B}_\rho$.

(ii) The assertion is an immediate consequence of (i) in view of the inclusion $e_\rho(i_\rho(K)) \subset K$, which implies $r^*[e_\rho(i_\rho(K))] \leq r^*[K]$. □

We are now in a position to prove Lemma 6.1. The proof is based on [2, Chapter II, §1.1].

Proof of Lemma 6.1. It will be understood tacitly that u is defined on $\bar{\Omega}$ by the continuous extension. We use the notation such as $\{f > a\}_{\bar{\Omega}} = \{x \in \bar{\Omega};\ f(x) > a\}$. Fix μ, ν and $x^*, y^* \in \bar{B}_{r^*[\bar{\Omega}]}$ such that

$$u^*(r^*[\bar{\Omega}]) \leq \nu = u^*(y^*) < \mu = u^*(x^*) \leq u^*(0). \tag{6.2}$$

Let $0 < \varepsilon < (\mu - \nu)/4$. Since

$$|x^*| \leq r^*\left[\{u^* \geq \mu\}_{\bar{B}_{r^*[\bar{\Omega}]}}\right] < r^*\left[\{u^* \geq \nu + \varepsilon\}_{\bar{B}_{r^*[\bar{\Omega}]}}\right] \leq |y^*|$$

and

$$r^*[\{|u|\geq\nu+\varepsilon\}_{\bar\Omega}]=r^*\Big[\{u^*\geq\nu+\varepsilon\}_{\bar B_{r^*[\bar\Omega]}}\Big],\ r^*[\{|u|\geq\mu\}_{\bar\Omega}]=r^*\Big[\{u^*\geq\mu\}_{\bar B_{r^*[\bar\Omega]}}\Big],$$

we have

$$0<r^*[\{|u|\geq\nu+\varepsilon\}_{\bar\Omega}]-r^*[\{|u|\geq\mu\}_{\bar\Omega}]\leq|x^*-y^*|. \tag{6.3}$$

Since $\bar\Omega$ is compact, there exist $x_{\mu,\nu+2\varepsilon}\in\{|u|=\mu\}_{\bar\Omega}$, $y_{\mu,\nu+2\varepsilon}\in\{|u|=\nu+2\varepsilon\}_{\bar\Omega}$ such that

$$|x_{\mu,\nu+2\varepsilon}-y_{\mu,\nu+2\varepsilon}|=\operatorname{dist}(\{|u|\geq\mu\}_{\bar\Omega},\{|u|\leq\nu+2\varepsilon\}_{\bar\Omega}).$$

We denote $\gamma_{\mu,\nu+2\varepsilon}=|x_{\mu,\nu+2\varepsilon}-y_{\mu,\nu+2\varepsilon}|$. Then it follows

$$\gamma_{\mu,\nu+2\varepsilon}\geq\operatorname{dist}\left(\{|u|\geq\mu\}_{\bar\Omega},\left\{|u|\leq\frac{\mu+\nu}{2}\right\}_{\bar\Omega}\right)>0. \tag{6.4}$$

Let $0<\delta<\gamma_{\mu,\nu+2\varepsilon}$. Then we have

$$\{|u|\geq\mu\}_{\bar\Omega}\subset i_{\gamma_{\mu,\nu+2\varepsilon}-\delta}(\{|u|\geq\nu+\varepsilon\}_{\bar\Omega}). \tag{6.5}$$

Indeed, if $x\in\{|u|\geq\mu\}_{\bar\Omega}$, then

$$\gamma_{\mu,\nu+2\varepsilon}\leq\operatorname{dist}(x,\{|u|\leq\nu+2\varepsilon\}_{\bar\Omega})\leq\operatorname{dist}(x,\{|u|<\nu+\varepsilon\}_{\bar\Omega}),$$

and hence

$$\bar B_{\gamma_{\mu,\nu+2\varepsilon}-\delta}(x)\cap\{|u|<\nu+\varepsilon\}_{\bar\Omega}\subset\bar B_{\operatorname{dist}(x,\{|u|<\nu+\varepsilon\}_{\bar\Omega})-\delta}(x)\cap\{|u|<\nu+\varepsilon\}_{\bar\Omega}=\emptyset,$$

which implies (6.5). Therefore,

$$r^*[\{|u|\geq\mu\}_{\bar\Omega}]\leq r^*\Big[i_{\gamma_{\mu,\nu+2\varepsilon}-\delta}(\{|u|\geq\nu+\varepsilon\}_{\bar\Omega})\Big]. \tag{6.6}$$

Using Lemma 6.2 (ii), (6.3) and (6.6), we have

$$\begin{aligned}\gamma_{\mu,\nu+2\varepsilon}-\delta&\leq r^*[\{|u|\geq\nu+\varepsilon\}_{\bar\Omega}]-r^*\Big[i_{\gamma_{\mu,\nu+2\varepsilon}-\delta}(\{|u|\geq\nu+\varepsilon\}_{\bar\Omega})\Big]\\&\leq r^*[\{|u|\geq\nu+\varepsilon\}_{\bar\Omega}]-r^*[\{|u|\geq\mu\}_{\bar\Omega}]\\&\leq|x^*-y^*|,\end{aligned}$$

and hence

$$\gamma_{\mu,\nu+2\varepsilon}\leq|x^*-y^*| \tag{6.7}$$

because δ is arbitrary. With (6.2) in mind, we calculate the Hölder seminorm of u^* by applying (6.4) and (6.7):

$$\begin{aligned}\frac{|u^*(x^*)-u^*(y^*)|}{|x^*-y^*|^\alpha}&=\frac{|u(x_{\mu,\nu+2\varepsilon})|-|u(y_{\mu,\nu+2\varepsilon})|+2\varepsilon}{|x^*-y^*|^\alpha}\\&\leq\frac{|u(x_{\mu,\nu+2\varepsilon})-u(y_{\mu,\nu+2\varepsilon})|+2\varepsilon}{\gamma^\alpha_{\mu,\nu+2\varepsilon}}\\&\leq\|u\|_{(\alpha)}+\frac{2\varepsilon}{(\operatorname{dist}(\{|u|\geq\mu\}_{\bar\Omega},\{|u|\leq(\mu+\nu)/2\}_{\bar\Omega}))^\alpha}.\end{aligned}$$

Therefore, we obtain the desired conclusion because ε is arbitrary. □

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