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# “A DynamicModel of Conflict and Appropriation”

Wolfgang Eggert Jun-ichi Itaya Kazuo Mino

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# A Dynamic Model of Conflict and Appropriation*

Wolfgang Eggert[†] Jun-ichi Itaya[‡] Kazuo Mino[§]

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**Abstract**

This paper conducts the analysis of conflict and appropriation by extending the static contest models such as Hirshleifer (1991, 1995) and Skaperdas (1992) to a continuous-time, differential game setting. This paper shows that there is a unique Markov perfect equilibrium (MPE) strategy, which may be linear or nonlinear depending on the structural parameters of the model, when strategies are defined over the entire state space. We show that 'partial cooperation' can be seen as a long-run response to conflict. In particular, we find that a decrease in the effectiveness of appropriation, the depreciation rate of a common-pool stock which is subject to appropriation or the rate of time preferences or an increase in the 'degree of noise' improves the degree of 'partial cooperation' and thus the welfare of an anarchic society in the long run.

*Keywords*: Conflict, Cooperation, Differential Game, Markov Perfect Equilibrium, Nonlinear Markov strategy

*JEL classifications*: D 74, L 11

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[†]University of Freiburg , Bertholdstr. 17, 79098 Freiburg, Germany and ifo Institute for Economic Research at the University of Munich. Tel: +49-0761-203-2358; E-mail: Wolfgang.Eggert@vwl.uni-freiburg.de

[‡]Corresponding author: Graduate School of Economics and Business Admministration, Hokkaido University, Sapporo, 060-0809, Japan and ifo Institute for Economic Research at the University of Munich. Tel:+81-11-706-2858; Fax:+81-11-706-4947; E-mail: itaya@econ.hokudai.ac.jp

[§]Institute of Economic Research, Kyoto University, Yoshida Honmachi, Sakyo-ku, Kyoto 606-8501, Japan. Tel:+81-75-753-5114; E-mail: mino@lier.kyoto-u.ac.jp

# 1 Introduction

Conflict and appropriation are increasingly gaining attention among economists as a powerful force driving human interactions. In reality economic agents not only engage in purely economic activities like production, consumption or exchange, but also *sometimes* allocate resources to conflict as well as to appropriation activities in order to capture what others have produced or to secure certain rents. There is a relatively small but growing literature in political economics initiated by Hirshleifer (1991, 1995), Skaperdas (1992) and Grossman and Kim (1995) which allows for the possibility of conflict and appropriation in economic interactions. Their models share four common features. First, they postulate that conflict arises from the choice of rational and self-interested agents. Second, a well-defined and enforced property right over, at least, some goods do not exist. Third, the agents are assumed to be myopic in a way that they maximize only the current payoff. Fourth, their models are static. This paper conducts the analysis of conflict and appropriation by extending their static models to a dynamic one.

Hirshleifer (1995) takes an initial step towards a dynamic approach by recognizing successive iterations of the one-shot game, and focuses on the convergent point of such iterations (he calls such a fixed point "a steady state"). Nevertheless, as Maxwell and Reuveny (2005, p.31) correctly point out, "However, this approach is not fully dynamic: it does not specify equations of motion for any variables, time is not a variable in the model, and the condition for dynamic stability is not derived based on standard dynamic analysis".

In response to such long-term desires, there have been several papers which attempt to construct a dynamic variation of the one-shot conflicting game analyzed by the above-mentioned authors. Garfinkel (1990) examines a dynamic model in which agents make choices between productive and fighting activities. She uses a repeated game setting where threats and punishments are available. Existence of cooperative, disarmament equilibria can be established using Folk Theorem arguments. Skaperdas and Syropoulos (1996) discuss a two-period model of conflict in which time-dependence is introduced by the assumption that second period resources of each agent are increasing in first-period's payoff. As a result, "the shadow of the future" may impede the possibilities for cooperation. In other words, competing agents engage more in appropriation in order to capture a bigger share of today's pie. The equilibrium solution concept we employ in this paper allows us to identify possible cooperative outcomes as a result of decentralized decision-making by rational and forward-looking agents, without having to rely on the Folk Theorem of repeated games or enforceable commitments.[1] Moreover, since the one-shot game is repeated every period due to the nature of the repeated game, it would be unsatisfactory to describe true dynamic situations which are not "stationary". More recently, Maxwell and Reuveny (2005) construct a conflict model with two competing groups in which each group's population and a stock of common (natural) resources both change over time. Since three non-linear differential equations characterizing the dynamic paths of these stock variables do not allow for an analytical solution, they resort to numerical simulations. These exercises reveal that *mild* appropriation activity depresses the use of natural resources for production, thus possibly creating a Pareto improvement compared to cooperative situations where there is no appropriation activity, and, moreover, tends to reduce the volatility of those stocks through the transition. Although their model generates interesting insights, they still assume that agents are myopic. The authors in the literature have called for a full dynamic and multi-period model of the Skaperdas-Hirshleifer based literature which incorporates the behavior of non-myopic agents who are taking into account the consequences of their future actions.[2] More recently, Hafer (2006) develops a large-population infinite-horizon dynamic game in which players are randomly matched in each period to play the war of attrition. She shows that although the distribution of types among the winners and losers changes with each

[1] According to the so-called Folk Theorem, if players are sufficiently patient in infinitely repeated games, any outcome that is feasible and individual rational could be realized as an equilibrium outcome. Such multiplicity of equilibria would lose the predictability of the equilibrium outcome. Although even in the literature on differential games, cooperative behavior has been investigated based on non-Markovian trigger strategies (see, e.g., Benhabib and Radner, 1992; Dockner et al., 2000, Chap.6), we do not adopt this approach here.

[2] More recently, there is another class of dynamic conflict models that include, e.g., Gradstein (2004) and Gonzalez (2007), and Tornell and Lane (1999). There are several important differences between the models in their papers and ours. First, in their models a flow of the output produced each period is subject to predation, while in our model a stock variable is subject to predation. Secondly and more importantly, those papers investigate the relationship between conflict and economic growth in the standard growth model based explicitly on the investment and saving decisions of a large number of agents. Hence, their models are mostly concerned with the macroeconomic consequences, such as growth effects of insecure property rights. Since our model is a straightforward dynamic extension of Grossman, Hershleifer and Skaperdas which allows for interaction among a small number of agents, it enables us to directly compare our results with those of their static conflict models and thus to highlight the strategic role of appropriation among those few agents in the intertemporal context.

round of conflict, in a steady state there is no conflict. Although the motivation is the same as ours, she has to use the model of the war of attrition between two players because she has assumed that there is only one, indivisible parcel of land. This assumption significantly distinguishes their model from the models of Skaperdas and Hirshleifer in which a prize is divisible and the war of attrition never takes place.

We develop a forward-looking agent-based infinite horizon, general-equilibrium model to study the dynamic evolution of self-enforcing property rights. There are various ways of extending one-shot, static models of Skaperdas and Hirshleifer to a dynamic setting. Following their models, we first assume that the initial resource endowment is fixed over time. This assumption would be defended either by interpreting the initial resource endowment as a time or labor supply, or by assuming the fixed population in order to keep the model tractable. The relevant state variable in our dynamic model is a durable stock which accumulates through time according to the production process using collective efforts of all parties involved. This durable stock is exhaustible or rival in the sense that one agent's use of the stock *does* diminish its availability to other agents, and is also open to appropriation by rivals due to the lack of well-defined or enforceable property rights. Hence each of the agents is tempted by the immediate benefit attainable from capturing the stock. Natural (renewable) resources such as fishes and forest, and land in primitive historical societies are examples of such durable stocks or disputed wealth. In order to acquire land, people develop land through cooperative efforts, while they can also seize it from others. The stock of knowledge is another example.

We model the incentives of agents to exert effort in an attempt to challenge the claims of others. All agents who succumb to the temptation reduce their help in production of the common-pool stock to increase their efforts to convert claims on the common stock into effective property rights. More specifically, agents derive a payoff (or utility) from owning the stock of durable goods and, at every moment in time, choose how to allocate an endowment between appropriation of the common-pool stock (creating property rights) and participating in the production process to accumulate the common-pool stock. The production and appropriation decisions made independently and noncooperatively by each of the agents end up determining the evolution of the open-accessible stock. We use a tractable version of a differential game formulation of this model of conflict between several agents who attempt to appropriate a common-pool durable stock over an infinite horizon. The solution concept employed is a Markov perfect equilibrium (MPE), restricting strategies to be functions of the current payoff-relevant state variable. The key to determining which describe equilibrium outcomes is subgame perfection over the entire state space $[0, \infty)$, and not for its subset.[3]

The results obtained in this paper are summarized as follows. First, there uniquely exists either a linear, singular MPE or a nonlinear, non-singular MPE strategy. More precisely, depending on the structural parameter values of the model, either of the strategies, which can be extended by corner solutions to the entire state space, is qualified as MPE strategies. This uniqueness property of MPE strategies stems from the requirement that the domain of a state variable must be defined over the entire state space. This requirement plays an essential role in refining equilibria as well as associated steady states without appealing strict concave objectives unlike the static contest models. Second, both solutions commonly reveal that initially poor countries will exhibit an increase in appropriation as the aggregate stock of durable good gets larger until a steady state is reached. On the other hand, in economies with an affluent endowment of natural resources the "marginal gain (or utility)" of appropriation is higher and thus agents substitute appropriation for production for a while until the state variable reaches a threshold level. From that threshold onwards, agents choose to engage in production activity to some extent until a steady state is reached where the output of production is only just sufficient to replace the stock of durable goods. This result relates to the observation that rent-seeking activities in rich countries may result in deindustrialization as suggested by the literature on the resource curse (e.g., Sachs and Warner, 1999).[4]

Fourth, in the long run (=steady state) property rights may be "partially" enforced in the sense that appropriation and productive activities coexist, so that neither a totally peaceful (disarmed) equilibrium nor a full-fighting equilibrium emerges as a long run stable outcome.

[3] Nevertheless, there is a literature that has not required strategies to have the standard game-theoretical meaning. This has odd theoretical foundations—at best—either requiring endogenously defined domains (see Tsutsui and Mino, 1990), or not requiring that strategies be defined for every possible subgames (see Itaya and Shimomura, 2001; Rubio and Casino, 2002). In this paper strategies are defined in the standard game-theoretic sense; that is, strategies should be defined over "the entire state space" which may possibly be joined with corner strategies.

[4] There is the evidence that resource abundance in the definition used by Sachs and Warner (1999) is associated with civil war (e.g., Collier and Hoeffler, 2004).

Moreover, the less the productivity of conflict technology, the more patient the contenders, the less the rate of depreciation, or the less sensitive to appropriative effort the conflict technology, the less each contender is to undertake appropriation, and thus the greater are the degree of partially cooperation as well as the welfare of an anarchic society.

The organization of the paper is as follows. In the next section we formulate the model and state all assumptions. In Section 3 we characterize Markov Perfect Equilibrium strategies. In Section 4 we conduct comparative static analysis with respect to several principle structural parameters, and then designs institutions or policies which make the non-cooperative solution closer to a first-best (i.e., cooperative) solution. Section 5 concludes the paper. Some mathematical proofs will be given in the appendices.

# 2 The Model

## 2.1 Analytical Framework

Consider an infinite horizon economy populated by $n \geq 2$ agents (or contenders) who strategically interact. Each of the contenders derives utility from the consumption or services of a common-pool asset, such as land territories, natural resources, or the tangible or intangible stock of durables. We want our model to capture the role of productive and aggressive activities with the understanding that aggressive investment causes an inward shift of the aggregate production possibility frontier. Accordingly, we use a setup where appropriation and production are two substitutable investment choices. Specifically, let contender $i$ decide at each point in time, $t$, how much resources to devote for appropriation $a^i(t) \geq 0$ and production $l^i(t) \geq 0$. The resource (e.g., time) constraint of $i$ is:

$$a^i(t) + l^i(t) = 1, \tag{1}$$

where the time-invariant endowment is normalized at 1, and is *not* subject to appropriation.[5] The time arguments have been suppressed in this and all subsequent equations except when it is strictly necessary.

The common-pool stock is subject to appropriation. The stock is generated by accumulation of output. At each point in time, $t$, output is produced with a linear production technology:[6]

$$Y\left(l^1(t), \ldots, l^n(t)\right) = \sum_{j=1}^{n} l^j(t), \tag{2}$$

which captures the idea that higher productive efforts by contenders cause an outward shift of the production possibility frontier for the economy as a whole. The output of production can be stored to augment the common-pool stock. However, storage entails costs such that the stock $Z$ evolves according to

$$\dot{Z}(t) = Y\left(l^1(t), \ldots, l^n(t)\right) - \delta Z(t), \tag{3}$$

where $\delta \in (0, 1)$ is the rate at which output will depreciate if stored for future consumption, and $\dot{Z}$ denotes a change of $Z(t)$ over time.

A main ingredient of the model is the conflict technology which, for any given values of $a^1, \ldots, a^n$, determines each contender's probability of winning sole possession in obtaining the stock $Z(t)$ in a given period. To model this probability for contender $i$, a natural assumption is that the probability is increasing in $i$'s aggressive investment, i.e., the fraction of time $i$ devotes to aggression, but decreasing in the sum of aggressive investment of all contenders. To represent the relative success of $i$ in the contest, therefore, we will use the following conflict

[5] In the existing literature on conflict and appropriation the assumption that each agent has some essential property rights is standard. Individual's labor supply is such an example. Maxwell and Reuveny (2005) further assume that the amount of labor supply is growing over time as a result of the growth of population. However, to avoid unnecessary complications, we assume that the population of agents remains constant through time.

[6] The production function may be generalized to the CES production function. Nevertheless, such generalization does not essentially alter the result without complications.

technology which is slightly more general than the Tullock (1980) contest success function:[7]

$$p^i\left(a^1(t),\ldots,a_n(t)\right)=\frac{\left[a^i(t)+\eta\right]^r}{\left[a^i(t)+\eta\right]^r+\sum_{j\neq i}^{n}\left[a^j(t)+\eta\right]^r}\quad \text{for } a^i\in[0,1], \tag{4}$$

where the parameter $r$ captures the effectiveness of aggression and $\eta > 0$.[8] The role of a positive constant number $\eta$ is to prevent $p_i(.)$ from being discontinuous at point $(0,\ldots,0)$. As will be seen, if the model is to allow the possibility that the equilibrium values of all $a_i$'s equal zero, then the function $p_i(.)$ is either discontinuous or undefined on point $(0,\ldots,0)$ when $\eta = 0$. This property holds true in the most static rent-seeking models based on the Tullock (1980) success function (i.e., setting $\eta = 0$ in (4)). One may interpret $\eta$ as either the *prior* common winning probability of each contender before any aggressive activity is undertaken (see Cochon, 2000), or the "degree of noise" which captures the extent of which pure luck as opposed to appropriate efforts determines success in the contest (see Amegashie, 2006). It should be also noted that the contest success function (4) may also be interpreted as a sharing rule or ownership of assets or outputs generated by the "productive assets" among contenders proportional to their choices of $a_i$.

The second restriction placed on (4) is as follows:

$$\frac{\partial^2 p^i}{\partial a^{i2}}=r\left(n-1\right)\frac{n\left(r-1\right)-2r}{n^3\left[a^i(t)+\eta\right]^2}<0\ \text{ for }\begin{cases} n=2\wedge r>0, \\ n>2\wedge 0<r<n/(n-2), \end{cases} \tag{5}$$

which is obtained by differentiating (4) with respect to $a^i$ twice and then imposing symmetry. Condition $0 < r < n/(n-2)$ ensures the inequality in (5), implying that the r.h.s. of (10) is concave in $a^i \in [0,1]$ and thus the second-order condition for an optimal choice of $a^i$ holds.[9] Nevertheless, we impose a slightly more stringent condition as follows:

[7] Although the contest success function (4) satisfies axioms A1-A5 in Skaperdas (1996), it is not homogeneous of degree one in $a_i$ (i.e., Axiom A6 of Skaperdas).

[8] This specification has been also used by several authors in the rent-seeking literature including Neary (1997), Dasgupta and Nti (1998), Amegashie (2006), and Rai and Sarin (2009). We are grateful to an anonymous referee for bring this contest success function to our attention.

[9] Tullock (1980) assumes that condition $r < n/(n-1)$ satisfies the second-order condition in his $n$-agent contest game. Although Hirshleifer (1991, 1995) and Gonzalez (2007) assume that $r < 1$ in their two-agent games, this assumption is implied by the above second-order condition for Tullock's model.

**Assumption 1**: $0 < r < 1/(n-1)$.

Assumption 1 guarantees not only condition $0 < r < n/(n-2)$, but also the existence of Markov perfect Nash equilibrium strategies which will be defined later.

Consider the following symmetric, stationary differential game. At each moment in time, $t$, each of the contenders chooses controls, $a^i(t)$ and $l^i(t)$, from its feasible set to maximize the discounted value of total expected payoffs, discounted at rate $\rho \in \mathbb{R}_{++}$, over the infinite horizon $[0, \infty)$. If all opponents of contender $i$ use Markovian strategies $a^j(t) = \phi^j(Z(t))$, $j \neq i$, then $i$ solves the following optimal control problem denoted by $\Gamma(Z_0, 0)$:

$$\max_{\{a^i(t)\}} J^{i,\phi^{-i}}(a^i(.)) \equiv \max_{\{a^i(t)\}} \int_0^\infty p^i\left(a^i(t), \phi^{-i}(Z(t))\right) \beta Z(t) e^{-\rho t} dt \tag{6}$$

$$\text{s.t. } \dot{Z}(t) = \sum_{j=1,\ j\neq i}^{n} \left[1 - \phi^{-j}(Z(t))\right] + \left[1 - a^i(t)\right] - \delta Z(t), \tag{7}$$

$$Z(0) = Z_0 > 0, \tag{8}$$

$$a^i(t) \in [0, 1] \quad \text{for} \quad \forall t \in [0, \infty), \tag{9}$$

where $p^i\left(a^i, \phi^{-i}(Z)\right) \equiv p^i(\phi^1(Z), \ldots, \phi^{i-1}(Z), a^i, \phi^{i+1}(Z), \ldots \phi^n(Z))$ and $Z_0$ is the initial stock. The first and second arguments of $\Gamma(Z_0, 0)$ refer to the initial level of the common stock and the time at which the game starts, respectively. The game is symmetric and stationary since the instantaneous payoff functions and feasible sets are identical among agents, and the equation of motion (7) is common to all contenders, and since these components of the model are all not explicitly dependent on time.

The instantaneous expected payoff (or income) to contender $i$ is given by $p^i(a^1, \ldots, a^n)\beta Z$. We may view $\beta Z(t)$ as a linear utility function to capture amenities from the stocks of land, natural resources such as forests, animals and so on. Alternatively, it can be viewed that the flow of $\beta Z(t)$ represents the income, return, or output generated by the "productive assets" $Z$, where $\beta$ represents a productivity parameter for harvest or production efficiency. For example, land produces harvest: fishing grounds produce fishes: forests, which may be utilized for leisure activities, yield clean air and timber production: financial assets yield monetary returns: the stock of knowledge (intellectual property) produces new goods (patent revenues). Maxwell and Reuveny (2009) have interpreted the variable $Z$ as the pool of potential voters which produces political power or the number of supporters $\beta Z$.

## 2.2 Solution Concept

We solve the differential game $\Gamma(Z_0, 0)$ using the notion of a stationary MPE, which is appealing because, in ruling out all direct strategic interactions, it allows use of optimal control tools. To state this concept of equilibrium in a mathematically rigorous way, we reproduce a series of definitions given by Dockner et al. (2000) with notational modifications:

**Definition 1** *(Dockner et al., 2000, Definition 3.1). A control path* $a^i : [0, \infty) \mapsto \mathbb{R}_+$ *is* ***feasible*** *for the game* $\Gamma(Z_0, 0)$ *if the initial value problem defined by* $(7) - (9)$ *has a unique, absolutely continuous solution* $Z(.)$ *such that the constraints* $Z(t) \in \mathbb{R}_+$ *and* $a^i(t) \in [0, 1]$ *hold for all* $t$ *and the integral in* $(6)$ *is well defined.*

As the game $\Gamma(Z_0, 0)$ is stationary, we can focus on equilibria supported by stationary strategies. For analytical simplicity, we further restrict ourselves to stationary Markov strategies throughout the paper.

**Definition 2** *(Dockner et al.,2000, p.97). A stationary Markov strategy is a mapping* $\phi^i : \mathbb{R}_+ \mapsto [0, 1]$*, so that the time path of the control is* $a^i(t) = \phi^i(Z(t))$*.*

Hence, stationary Markov strategies are functions only of the current state. Then we can define:

**Definition 3** *The* $n$*-tuple of functions* $(\phi^1, \phi^2, ..., \phi^n)$ *is a stationary Markov Nash equilibrium if for each* $i \in \{1, 2, .., n\}$ *an optimal control path* $a^i(t)$ *of the problem* $\Gamma(Z_0, 0)$ *exists and is given by the stationary Markov strategy* $a^i(t) = \phi^i(Z(t))$*.*

We use the further strengthening of Markov Nash equilibrium, that is, subgame perfectness, to characterize an equilibrium path:

**Definition 4** *(Dockner et al., 2000, Definition 4.4). The* $n$*-tuple of functions* $(\phi^1, \phi^2, ..., \phi^n)$ *is a Markov Nash equilibrium of the game* $\Gamma(Z_0, t)$*. The Markov Nash equilibrium is Markov* *perfect equilibrium (MPE) if for each* $(Z,t) \in \mathbb{R}_+ \times [0,\infty)$*, the subgame* $\Gamma(Z,t)$ *admits a Markov Nash equilibrium* $(\psi^1, \psi^2, ..., \psi^n)$ *such that* $\psi^i(\hat{Z}, s) = \phi^i(\hat{Z}, s)$ *for all* $i \in \{1, 2, ..., n\}$ *and all* $(\hat{Z}, s) \in \mathbb{R}_+ \times [t, \infty)$*.*

Since the subgame $\Gamma(Z,t)$ is stationary, $\Gamma(Z,0) = \Gamma(Z,t)$ and thus all stationary Markov Nash equilibria are MPE. We make the following assumption:

**Assumption 2**: The value function of contender $i$, $V^i(Z) = \max_{\{a^i(t)\}} J^{i,\phi^{-i}}(a^i(.))$: $R_+ \to R$, is locally Lipschitz continuous.

When contender $i$'s value function is differentiable almost everywhere,[10] it solves the Hamilton-Jacobi-Bellman equation:

$$\rho V^i(Z) = \max_{a^i \in [\eta, 1]} \left[ p^i\left(a^i, \phi^{-i}\right) \beta Z + V_Z^i(Z) \left\{ \sum_{j=1,\ j \neq i}^{n} \left(1 - \phi^j\right) + \left(1 - a^i\right) - \delta Z \right\} \right], \tag{10}$$

where $V_Z^i(Z)$ represents the derivative of $V^i(Z)$ with respect to $Z$.

The first-order necessary condition for $i$'s choice of appropriation is given by

$$\frac{\partial p^i}{\partial a^i} \beta Z - V_Z^i(Z) \begin{cases} = 0 \implies a^i \in [0,1], \\ > 0 \implies a^i = 1, \\ < 0 \implies a^i = 0. \end{cases} \tag{11}$$

According to (11), each contender, when choosing $a^i$, trades the marginal increase in expected payoff from an increase in appropriation against the marginal loss in the discounted value of the future stream of payoffs which results from a reduction of productive effort. If the payoff gain from an increase in $a^i$ is larger than the payoff loss implied by the decrease in $l^i$ for all levels of $a^i \in [0,1]$, then $i$ will rationally devote all resources to appropriation. In contrast, $i$ chooses $a^i = 0$ in cases where the discounted marginal gain from productive investment exceeds the instantaneous marginal gain from aggressive behavior for all levels of $a^i \in [0,1]$.

[10] It follows from Rademacher's Theorem that the value function $V^i(Z)$ is differentiable almost everywhere. Even if it is not differentiable at some point, we can apply generalized HJB function coupled with the generalized gradient of $V^i(Z)$ (see Dockner et al. (2000), Chapter 3).

## 2.3 Candidate Markov Perfect Equilibrium Strategies

Since we have started our analysis assuming identical contenders and since the state equation (7) is symmetric with respect to their controls, a natural focus is placed on symmetric equilibria. The symmetry assumption allows us to drop the subscript $i$ in the subsequent discussion, and we will suppress this index unless strictly necessary for expositional clarity.

At an interior solution of $\phi(Z)$ we may apply the envelope theorem to characterize $\phi'(Z)$. Using the symmetry assumption, we obtain the following (see Appendix A of the paper for derivation):

$$\phi'(Z) = \frac{[\phi(Z)+\eta]\left\{\dfrac{n[\phi(Z)+\eta]}{r(n-1)Z} + \dfrac{[1-\phi(Z)]n}{Z} - (\rho+2\delta)\right\}}{(1+\eta)n - \delta Z}. \tag{12}$$

We will draw the representatives of Markov strategies in a control and state space in order to characterize qualitative solutions to the nonlinear differential equation (12). To this end we first identify the steady state locus where $\dot{Z}=0$ in (3), called $C_1$ in the following. Let $C_2$ denote the loci where $\phi'(Z)$ goes to plus/minus infinity, and by $C_3$ the loci where $\phi'(Z)$ equals zero in the $(Z,a)$ space:

$$\begin{aligned} C_1 &:= \{(Z,a) : \dot{Z} = (1-\phi(Z))\,n - \delta Z = 0\}, \\ C_2 &:= \{(Z,a) : \phi'(Z) \to \pm\infty\}, \\ C_3 &:= \{(Z,a) : \phi'(Z) = 0\}. \end{aligned} \tag{13}$$

The steady-state line $C_1$ is a downward-sloping, straight line in the $(Z,a)$ space. It intersects the vertical axis at point $(0,1)$ and the horizontal axis at point $(n/\delta,0)$. Turn to $C_2$. Setting the denominator in (12) equal to zero, we obtain a vertical line at point $(Z_E,0)$ where $Z_E = (1+\eta)n/\delta$ (which we call "the non-invertibility ($NI$) locus" following Rowat, 2007). The locus $C_3$ is obtained by setting the numerator in (12) equal to zero. Solving for $a$ gives the following locus:

$$a = \frac{r(n-1)}{1-r(n-1)}\frac{\rho+2\delta}{n}Z - \frac{r(n-1)+\eta}{1-r(n-1)}, \tag{14}$$

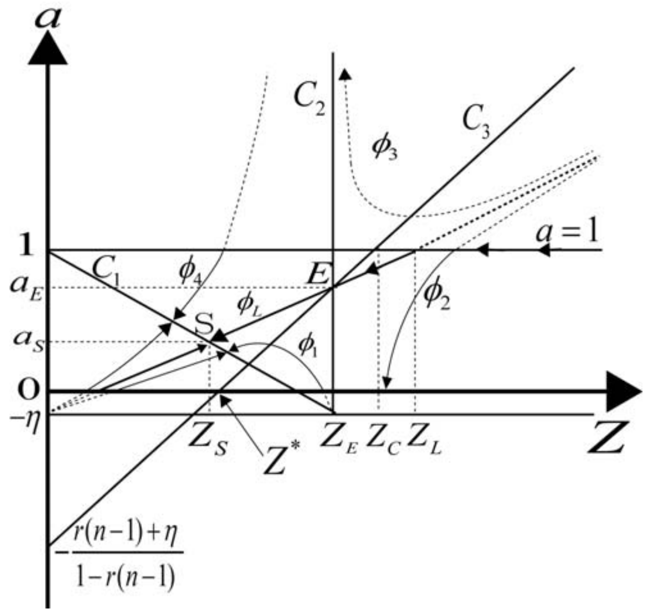


Figure 1: Markov strategies when $r\,(n-1)\,(\rho+\delta) \leq [1-r\,(n-1)]\,\delta$.

which, due to **Assumption 1,** shows that the straight line $C_3$ has a positive slope and a negative intercept on the vertical axis, as illustrated in Figs. 1 and 2. As a result, the point of intersection between $C_2$ and $C_3$, labelled $E$, will be situated in the nonnegative quadrant of the $(Z, a)$ plane:

$$(Z_E, a_E) = \left(\frac{(1+\eta)n}{\delta}, \frac{r\,(n-1)\,(\rho+\delta)(1+\eta)}{[1-r\,(n-1)]\,\delta} - \eta\right), \tag{15}$$

which is called "a singular point". Note, however, that point $E$ may be located below or above the resource constraint (1), since the value of $a_E$ may or may not be less than 1. Depending on this value we can draw two diagrams such as Figs. 1 and 2. Moreover, it follows from (3) that any strategy $\phi\,(Z)$ above $C_1$ implies that $Z$ declines in time, while any strategy $\phi\,(Z)$ below $C_1$ entails an increase of $Z$ over time.

Collecting the arguments, we can illustrate an uncountable number of the curves corresponding to the (interior) solutions satisfying the HJB equation (10) in Figs. 1 and 2. These figures display representatives of those integral curves that are divided into five types of the families of strategies. Arrows on the *families* of integral curves $\phi_j$, $j = 1, \ldots, 4$, and $\phi_L$ illustrate the evolution of $Z$ over time.

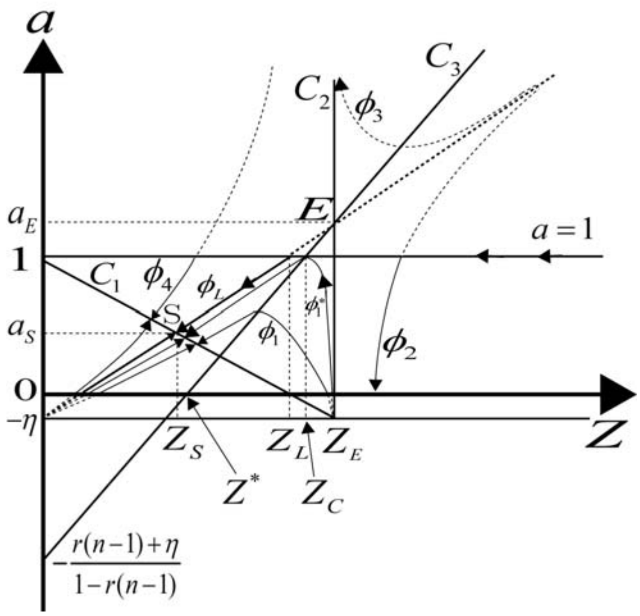


Figure 2: Markov strategies when $r\,(n-1)\,(\rho+\delta) > [1-r\,(n-1)]\,\delta$.

Furthermore, by direct integration of (12) and manipulating we can obtain a general solution to the differential equation (12)

$$\phi\,(Z) + \eta = \frac{r\,(n-1)\,(\rho+\delta)\,Z\,[(1+\eta)n - \delta Z]^{\frac{\rho+\delta}{\delta}}}{n\,[1-r\,(n-1)]\,[(1+\eta)n-\delta Z]^{\frac{\rho+\delta}{\delta}} + r\,(n-1)\,(\rho+\delta)\,c_1}, \tag{16}$$

where $c_1$ represents an arbitrary constant of integration and may take a positive, zero or negative value. When $c_1 = 0$, (16) simplifies to[11]

$$\phi_L\,(Z) = \frac{r\,(n-1)\,(\rho+\delta)}{[1-r\,(n-1)]\,n} Z - \eta. \tag{17}$$

It is seen from Figs. 1 and 2 that the left branch of the linear strategy $\phi_L\,(Z)$ to the steady state line $C_1$ starts from point $(0, -\eta)$, and then reaches point $S$ on $C_1$, while its right branch

[11] Mino (1983), and Long and Shimomura (1998) show that in the class of differential games the value function is homogeneous of degree $\alpha$ in terms of a state variable and the policy functions satisfying the corresponding HJB equation contains a linear function of a state variable whenever the instantaneous objective function is homogeneous of degree $\alpha$ and the constraints are homogeneous of degree one in terms of state and control variables. The emergence of the linear strategy as a solution for the present model is consistent with their finding.

starts from any $Z \in (Z_S, \infty)$, then reaching point $S$ also, where

$$(Z_S, a_S) = \left( \frac{(1+\eta)n\left[1 - r\left(n-1\right)\right]}{r\left(n-1\right)\rho + \delta}, \frac{r\left(n-1\right)\left[\rho + (1+\eta)\delta\right] - \eta\delta}{r\left(n-1\right)\rho + \delta} \right). \tag{18}$$

Note, moreover, that the right branch of $\phi_L$ passes through the singular point $E$.

Inspection of (16) further reveals that the $\phi_1$ (resp., $\phi_3$) *family* of strategies represents the solution curve of (16) when $c_1 > 0$ and $(1+\eta)n > \delta Z$ (resp., $(1+\eta)n < \delta Z$), while the $\phi_4$ (resp., $\phi_2$) *family* of strategies represents the solution curve of (16) when $c_1 < 0$ and $(1+\eta)n > \delta Z$ (resp., $(1+\eta)n < \delta Z$). Moreover, all members of the left branch of the $\phi_4$ *family* start from point $(0, -\eta)$ as well, while no members of its right branch cross the non-invertibility locus $C_2$. All members of the left branch of the $\phi_1$ *family* also start from point $(0, -\eta)$ and then reaches points on the steady state line $C_1$, while all members of its right branch start from point $(Z_E, -\eta)$, and then reach the same point on $C_1$. On the other hand, although the members of the $\phi_2$ and $\phi_3$ families start from any initial value $Z_0 > Z_E$, all members of the $\phi_2$ *family* approach the horizontal axis, while all members of the $\phi_3$ *family* go to plus infinity as $Z$ approaches $Z_E$, as illustrated in Figs. 1 and 2.

As implied by Definition 4, candidate strategies must be mapped from any element of the entire state space of $Z$, that is, $R_+$. In other words, strategies should cover the entire state space, i.e., $[0, \infty)$. At first glance this requirement seems to eliminate all interior strategies $\phi_j$, $j = 1, \ldots, 4$, and $\phi_L$ due to the presence of the control constraint $[0, 1]$ as well as the $NI$ locus $C_2$. Nevertheless, those strategies could potentially be continuously extended by the cornered strategy $\phi = 1$ along the resource constraint (1), and/or by the non-aggressive strategy $\phi = 0$, which has been suggested by Rowat (2007). Both potential extensions are triggered by the cornered strategies when the equality in (11) does not apply. We use the hat to indicate those potential extensions so that $\hat{\phi}_j = \min\left\{1, \max\left\{0, \phi_j\right\}\right\}$ where $j = 1, .., 4$, and $L$.

# 3 Refining Candidate Strategies

Unfortunately, not all the extended strategies are qualified as MPE strategies defined in Definition 4. Dockner et al. (2000) and Rowat (2007) have provided sufficiency conditions for the existence of MPE. So we have to test whether those potentially extended strategies satisfy the sufficiency conditions in Theorem 3 of Rowat (2007).

To do this, we first need the following lemma:

**Lemma 1**

*(i) The value function along the cornered strategy $\phi = 0$ is not bounded if $\phi = 0$ possess a constant of integration $c_2 > 0$ in the value function $V(Z) = [\beta(n + \rho Z)/\rho(\rho + \delta)n] + c_2[n - \delta Z]^{-\frac{\rho}{\delta}}$; and*

*(ii) it is impossible to extend either the interior strategy $\phi_1$ or $\phi_2$ by $\phi = 0$ at every value of $Z$ that is strictly greater than $n\eta/r(n-1)(\rho+\delta)$, where this threshold value is located between the two intersections of the horizontal axis with the linear strategy $\phi_L$, and that with $C_3$.*

The proof is given in Appendix B. Although the strategy $\phi = 0$ alone cannot form MPE strategies, it might be possible to be joined with some of the interior strategies identified in the previous subsection to cover the whole range of the domain of the state variable $Z$.

To check whether it is possible or not, we have to inspect Fig. 1 (i.e., $r(n-1)(\rho+\delta) \leq [1 - r(n-1)]\delta$) and Fig. 2 (i.e., $r(n-1)(\rho+\delta) > [1 - r(n-1)]\delta$) separately. With the help of these diagrams, we have the following lemma:

**Lemma 2** *Consider the differential game $\Gamma(Z_0, 0)$.*

*(i)If $r(n-1)(\rho+\delta) \leq [1 - r(n-1)]\delta$, then none of the members of the $\hat{\phi}_1$ and $\hat{\phi}_2$ families in Fig. 1 can form MPE strategies; and*

*(ii)if $r(n-1)(\rho+\delta) > [1 - r(n-1)]\delta$, then none of the members of the $\hat{\phi}_2$ family in Fig. 2 can form MPE strategies.*

**Proof.** It follows from Lemma 1 that by choosing a constant of integration $c_2 \in \left[-n^{\frac{\rho}{\delta}}\beta/\rho(\rho+\delta), 0\right]$ in Fig. 3 of Appendix B, $\phi = 0$ can be connected with the $\phi_1$ or $\phi_4$ family of strategies at the associated value of $Z \in [0, Z^*]$ in Fig. 1. However, it is not possible to extend $\phi_1$ by $\phi = 0$

in Fig. 1. This is because although all members of the $\phi_1$ family have to be connected with $\phi = 0$ in the interval between the intersection of $C_1$ with the horizontal axis and $Z_E$, $\phi = 0$ is not qualified as an optimal solution at any point in that interval due to Lemma 1. Moreover, it follows from Lemma 1 that it is also impossible to extend the $\phi_2$ family by $\phi = 0$ in Figs. 1 and 2. Taken together, the families $\hat{\phi}_1$ and $\hat{\phi}_2$ both violate condition (i) of Theorem 3 of Rowat (2007) in which the $n$-tuple of the solution paths is feasible for $\Gamma(Z_0, 0)$. ■

Nevertheless, it is seen from Fig. 2 that it might be possible to extend some of the $\phi_1$ family of strategies by another corner strategy $\phi(Z) = 1$, labelled $\phi = 1$. Although the interior strategy $\phi_4$ cannot intersect the $NI$ locus $C_2$, it may be also possible to extend $\phi_4$ by $\phi = 1$. To check these possibilities, we need the following lemma (its proof is given in Appendix B):

**Lemma 3**

*(i) The value function along the cornered strategy $\phi = 1$ is not upper bounded if $\phi = 1$ possess a constant of integration $c_3 > 0$ in the value function $V(Z) = \beta \left[Z + Z^{-\frac{\rho}{\delta}} n(\rho + \delta) c_3\right] / n(\rho + \delta)$; and*

*(ii) it is impossible to extend either the interior strategy $\phi_1$, $\phi_3$ or $\phi_4$ by $\phi = 1$ at every value of $Z$ which is strictly less than $Z_C \equiv (1 + \eta)n/r(n-1)(\rho + 2\rho)$.*

Moreover, it follows from Figs. 3 and 4 in Appendix B that $\phi = 1$ eventually becomes a solution for larger values of $Z$ (more precisely, which are larger than the l.h.s. of (B.7)). This seems to be quite intuitive because when the existing stock $Z$ is abundant (i.e., larger than this threshold value), contenders have a stronger incentive to capture the stock from others through appropriation activity rather than to engage in production activity, thus devoting to full-fighting $\phi = 1$.

**Lemma 4** *None of the members of the $\hat{\phi}_3$ and $\hat{\phi}_4$ families can form MPE strategies.*

**Proof.** Although in Fig. 1 all members of the $\phi_4$ family intersect the constraint $a = 1$ (i.e., $\phi = 1$), none of them can intersect the $NI$ locus $C_2$. On the other hand, since it follows from Lemma 3 that $\phi = 1$ remains an optimal solution only when $Z > Z_C$, it is seen from Fig. 1 that it is impossible to extend $\phi_4$ by $\phi = 1$.

In Fig. 2, all members of the $\phi_4$ family intersect the constraint $a = 1$ (i.e., $\phi = 1$) at the points whose values of $Z$ are strictly less than $Z_L$. Since $Z_C > Z_L$ in Fig. 2, it is also impossible to continuously extend $\phi_4$ by $\phi = 1$.

Since any members of the $\phi_3$ family in Fig. 2 intersect neither $C_2$ nor the constraint $a = 1$, any extension of $\phi_3$ is not possible. In contrast, in Fig. 1 there exist some members of the $\phi_3$ family which intersect the constraint $a = 1$ at the points whose values of $Z$ are strictly larger than $Z_C$. In this case, they have to cross $\phi = 1$ again at a value of $Z$ which is less than $Z_C$. However, it follows from Lemma 3 that at this intersection $\phi = 1$ is no longer an optimal solution. To sum up, any members of the families $\phi_3$ and $\phi_4$ cannot be continuously extended by $\phi = 1$, thus violating condition (i) of Theorem 3 of Rowat (2007). ■

In spite of those eliminations, there still remain the linear strategy $\phi_L$ and some of the $\phi_1$ family of strategies as candidate MPE ones. These strategies may be extended by $\phi = 1$. This conjecture will be confirmed in our main proposition as follows:

**Proposition 1** *Consider the differential game* $\Gamma(Z_0, 0)$.
*(i) If* $r\,(n-1)\,(\rho+\delta) \leq [1 - r\,(n-1)]\,\delta$, *then only the* ***linear*** *strategy extended by* $\phi(Z) = 0$ *and* $\phi(Z) = 1$ *forms a MPE strategy; and*
*(ii) if* $r\,(n-1)\,(\rho+\delta) > [1 - r\,(n-1)]\,\delta$, *then only the* ***non-linear*** *strategy of the* $\phi_1$ *family extend by* $\phi(Z) = 0$ *and* $\phi(Z) = 1$ *at* $Z_C \equiv (1+\eta)n/r(n-1)(\rho+2\rho)$ *forms a MPE strategy.*

There are several remarks in order. First, Proposition 1 implies not only that the linear (singular) and nonlinear (non-singular) MPE strategies qualified above, labelled $\hat{\phi}_L$ and $\hat{\phi}_1^*$, are *asymptotically stable* in the sense that from any arbitrary initial value of $Z$ they can reach finite steady states in the long run. Proposition 1 also says that there always exists either of the two extended strategies, depending on the parameter values of the model. It is then best understood that the uniqueness of MPE strategies arises from the stringent requirement that the domain of a state variable should be defined over the entire state space, i.e., $[0, \infty)$. Although the emergence of a unique steady state in our model is similar to the results of Hirshleifer (1991, 1995) and Skaperdas (1992) using a static contest model but also from those of Maxwell and Reuveny (2005) using a dynamic model with myopic agents in which the unique one-shot Nash equilibrium is repeated every period, the reasons for the uniqueness of equilibrium are quite different between their models and ours; that is, the uniqueness property of their model is due to strict concave objectives, while our uniqueness property arises from the defining characteristic for the concept of subgame perfect equilibrium.

Secondly, although "partial cooperation" in the steady state is also found in the static models of Hirshleifer and Skaperdas, different degrees of "partial cooperation" depend on which strategy is qualified for MPE strategies in our differential game. If the nonlinear MPE strategy is qualified, a more efficient steady state will be realized compared to the linear one. This feature does not emerge in the corresponding static contest models, because there is no distinction between linear and nonlinear strategies in their models. Notably, the more effective the conflict technology, the less patient the contenders, the larger the number of contenders or the greater the rate of depreciation, the less likely that the linear strategy is chosen as a MPE one; instead, the more likely that a more efficient nonlinear strategy is qualified as a MPE one.

Thirdly, another important aspect of Proposition 1 is that an optimal choice of appropriation activity varies according to the size of the common stock $Z$ (=the prize) that changes over time, which stands in contrast with the results based on the above-mentioned static contest models where the size of the prize is constant through time. When $Z = 0$, there is no incentive for each contender to engage in appropriation activity because they get nothing from this activity, so that all contenders choose the non-aggressive strategy $\phi = 0$. As a result, the common stock of $Z$ is accumulating over time. When the initial stock level is relatively low, investment in aggressive behavior monotonically increases toward the steady state point $S$ over time. In other words, the contenders will become greedier, as the common-pool stock $Z$ gets larger over time, because the marginal gain of appropriation will be higher. These features apparently have not been addressed by Hirshleifer and Skaperdas because due to the static nature of their models there does not exist a state variable. The primary driving force is that the best-reply strategies irrespective of linear or non-linear ones display strategic complements in such a way that when a contender increases his or her appropriation effort, the other contenders also increase their appropriation efforts.

In the long run, every contender chooses to contribute to the production of the common-pool stock $Z$ to some extent. In other words, (implicit) "partial cooperation" can be seen as a best response to the risk of appropriation. In affluent economies, on the other hand, where the initial level of the common-pool stock is sufficiently large, investment in aggression reaches the maximum possible level (i.e., $\phi = 1$) in finite time. In other words, since there is initially a large amount of the common-pool stock in affluent economies, a full-fighting strategy will be rationally and inevitably chosen during the transition to the steady state until the stock decreases to a certain level.

# 4 Comparative Static Analysis

## 4.1 Steady State Effects

In this section we discuss the effects of a change in the model parameters on the transition path of the linear strategy $\phi_L$ as well as on the associated long-run equilibrium point $S$, since the other nonlinear MPE strategy displays the same comparative statics properties. Consider first the effects of a change in the productivity (or effectiveness) of conflict technology. In the model, a change in the productivity is captured by a change in $r$. The shift of point $S$ can be calculated by differentiating (18) with respect to the parameter $r$, respectively:

$$\begin{aligned}\frac{da_S}{dr} &= (1+\eta)(n-1)\,\delta\,(\rho+\delta)\,\Delta^{-2} > 0,\\ \frac{dZ_S}{dr} &= -(1+\eta)n\,(n-1)\,(\rho+\delta)\,\Delta^{-2} < 0,\end{aligned}$$

where $\Delta \equiv r\,(n-1)\,\rho + \delta > 0$. Although an increase in $r$ does not affect $C_1$, this increase strengthens the intensity of appropriation associated with every level of the common-pool stock $Z$ during the transition path, thus making the linear strategy $\phi_L$ steeper. Since the productivity of appropriation becomes more effective with higher $r$, all competing agents engage in more aggressive behavior in the hope of capturing more resources. This finding is quite intuitive, and is also consistent with the static conflict models of Hirshleifer (1991, 1995).

As an increase in the number of contenders augments the aggregate endowment in proportion to $n$, more resources will be available to production and appropriation activities, which is captured by an outward shift of the aggregate resource constraint $C_1$. On the other hand, the larger number of contenders will enhance the intensity of contest among contenders, thereby intensifying each contender's aggressive behavior and thus making the linear strategy $\phi_L$ steeper. Hence, these two effects together intensify individual appropriation, while the long run effect on the common-pool stock $Z$ is ambiguous since the increase in endowment interacts with the intensified aggressive behavior of contenders:[12]

$$
\begin{aligned}
\frac{da_S}{dn} &= (1+\eta)r\,(\rho+\delta)\,\delta\Delta^{-2} > 0, \\
\frac{dZ_S}{dn} &= (1+\eta)\left[(1-r\,(2n-1))(-r\rho+\delta)-n^2r^2\right]\Delta^{-2} \gtreqless 0.
\end{aligned}
$$

A higher depreciation rate causes a reduction in the level of the common-pool stock $Z$ available to contenders, thereby discouraging appropriation. This *resource-extraction effect* causes a clockwise turn of $C_1$ around point $(0,1)$ (i.e., the aggregate resource constraint $C_1$ moves inward toward the origin). At the same time, a higher $\delta$ implies that the cost of reproducing the common-pool stock increases more than the cost of aggressive behavior. Since this strengthens an incentive for aggressive behavior, $\phi_L$ gets steeper. Although these two effects on appropriation operate in opposite directions, the following result indicates that the former effect will outweigh the latter effect in the long run:

$$
\begin{aligned}
\frac{da_S}{d\delta} &= -\left[(1+\eta)r\,(n-1)\,\rho\,[1-r\,(n-1)]+\eta\delta\right]\Delta^{-2} < 0, \\
\frac{dZ_S}{d\delta} &= -(1+\eta)n\,[1-r\,(n-1)]\,\Delta^{-2} < 0.
\end{aligned}
$$

A decrease of the subjective rate of time preference makes $\phi_L$ steeper, but it has no effect on $C_1$. Hence we obtain the following long run effects:

$$
\begin{aligned}
\frac{da_S}{d\rho} &= (1+\eta)r\,(n-1)\,\delta\,[1-r\,(n-1)]\,\Delta^{-2} > 0, \\
\frac{dZ_S}{d\rho} &= -(1+\eta)r\,(n-1)\,n\,[1-r\,(n-1)]\,\Delta^{-2} < 0.
\end{aligned}
$$

[12] This effect has been also found in Result 4B of Hirshleifer (1995).

The economic explanation is that as contenders become more patient (i.e., smaller $\rho$), they put more weight on future stocks of durable goods rather than the current one, and thus tend to spend more resources on production activity rather than on aggressive investment. This result has apparently not been addressed by Hirshleifer (1991, 1995) and Skaperdas (1992), who have used the static conflict models. It stands in contrast to Skaperdas and Syropoulos' (1996) result in which the higher is the valuation of the future (i.e., smaller $\rho$), the stronger is the intensity of fighting. The reason for this difference is that in their two-period model contender's first-period expenditure on appropriation increases his or her second-period payoff. Rather, our result is similar to Garfinkel's (1990) Folk Theorem type result in repeated games where a lower discount factor (i.e., a smaller $\rho$) makes it easier to sustain cooperative outcomes. An interpretation of our result is that long-sighted contenders become less aggressive because they are more concerned about the future, i.e., the future benefits resulting from cooperation.

Finally, increasing the degree of noise mitigates the intensity of long-run contests:

$$\begin{aligned}\frac{da_S}{d\eta} &= -n\left[1-r\left(n-1\right)\right]\Delta^{-1}<0,\\ \frac{dZ_S}{d\eta} &= \left[1-r\left(n-1\right)\right]\delta\Delta^{-1}>0,\end{aligned}$$

which accords with intuition. That is, a larger extent of pure luck (i.e., $\eta$) discourages an incentive of aggressive behavior, because its increases lowers the marginal increase in the probability of wining with respect to $i$'s aggressive effort.

We may then summarize the discussion in the following proposition:

**Proposition 2**

*(i) An increase in the effectiveness of aggression leads to a higher level of aggression and to a lower level of the common-pool stock (i.e., $da_S/dr>0$ and $dZ_S/dr<0$);*

*(ii) an increase in the number of contenders leads to a higher level of aggression, but the effect on the common-pool stock is ambiguous (i.e., $da_S/dn>0$ and $dZ_S/dn \gtreqless 0$);*

*(iii) an increase in the depreciation rate leads to lower levels of aggression and of the common-pool stock (i.e., $da_S/d\delta<0$ and $dZ_S/d\delta<0$);*

*(iv)a decrease in the subjective rate of time preference (i.e., contenders become more pa-*

*tient) leads to a lower level of aggression and to a higher level of the common-pool stock (i.e., $da_S/d\rho > 0$ and $dZ_S/d\delta < 0$); and*

*(v)an increase in the degree of noise leads to a lower level of aggression and to a higher level of the common-pool stock (i.e., $da_S/d\eta < 0$ and $dZ_S/d\eta > 0$).*

## 4.2 First-Best Solution

In this subsection, we will characterize the explicit cooperative (first-best) solution as a benchmark steady state in the following. Assume an outside enforcer or centralized agency has the power to enforce every contender to execute its command. The cooperative strategy is one for which a centralized agency chooses the infinite-horizon planning profile of strategy $a \in R_+^n$ at the outset of the game so as to maximize $\int_0^\infty Ze^{-\rho t}dt$ subject to $\dot{Z} = n - \sum_{j=1}^n a^j - \delta Z$ where $a^j(t) \in [0, 1]$ for all $j$. Clearly, this optimization gives rise to a totally peaceful solution, that is, $a^j(t) = 0$ for $t \in [0, \infty)$ and all $j$, which leads to the most efficient long-run outcome $(0, n/\delta)$. The result is understood by noting that an enforced peaceful resolution completely eliminates socially wasteful aggressive activity if a central agency is strong enough to directly control the allocation between production and appropriation. Based on this observation, we can see that making the steady state level of $Z$ closer to the Pareto efficient one leads to the higher total discounted value of expected payoffs.

These results should be interpreted against the insight from Section 2 that a socially efficient steady state is not self-enforcing for it does not usually constitute a subgame perfect (Nash) equilibrium. Nevertheless, the results are suggestive in the sense that even weak governments which cannot fully control private agents might attempt to deter the development of the conflict technology, to increase the "noise degree" in the conflict technology, to reduce the depreciation rate of common-pool assets or to induce people to have longer sight. Such structural or institutional reforms could reduce the likelihood of aggression and thus avoid socially waste of resources, leading to peaceful and more efficient outcomes in the long run. The nuclear nonproliferation treaty which deters the development of nuclear weapons (i.e., aggressive conflicting technology) would be socially desirable in a way that makes the long run outcome resulting from the non-cooperative equilibrium behavior closer to the peaceful and efficient one.

Even in an anarchic situation where every contender follows the Markov perfect equilibrium behavior described in Sections 2 and 3, a decrease in either the productivity of aggressiveness, the depreciation rate, or the subjective rate of time preference moves the resulting long run equilibrium closer to the first-best one.

Another example is patent law, which aims at enforcing property rights on investment return and thus makes anarchic situations closer to the first-best one by restricting socially wasteful activities. Patent law potentially prevents a rapid fall in the expected return from new innovation, which would be a consequence of imitation by rivals. The increase in the return on investment caused by secure property rights may be approximately captured by the effect of a lower depreciation rate in our model.

# 5 Conclusions

The first message of this paper is that completely aggressive behavior is not necessarily a rational strategy for a contender in anarchic situations. Rather, every contender will individually and voluntarily choose "partial cooperation", in which each contender devotes its individual resource both to productive and appropriation activities at the same time, even though contenders act fully rational and are guided by their self-interest. The primary driving force is the **durability** of the common-pool stock in conjunction with the forward looking behavior of contenders. These intrinsically dynamic ingredients induce every contender to behave partially cooperatively, even without punishments and threats, unlike Garfinkel (1990). In other words, either if the stock depreciates completely each period or if contenders have myopic foresight, they are less motivated to follow a cooperative behavior in producing a commonly-accessible good.

The second major finding is that even if nonlinear Markov strategies are available, there is a unique MPE strategy. This result is in sharp contrast with the results of Dockner and van Long (1993), and Rowat (2007) which provide multiplicity of equilibrium strategies and uncountable many long run equilibria including the better outcomes supported by the nonlinear MPE strategies. However, it remains an open question as to the extent to which this uniqueness result of our model is model-specific or robust under different contest success, production or/and objective functions.

The model presented in this paper should be developed further in several directions. In particular, introducing asymmetry among agents would enable us to compare the results of the present model with those static models which do incorporate asymmetric agents. The "paradox of power" (Hirshleifer, 1991) – the relatively less well-endowed agents improve their position compared with their better-endowed counterparts – may be generated in such an asymmetric dynamic conflict model. Another interesting research agenda is to investigate non-Markovian equilibria supported by history-dependent strategies such as trigger ones in the present conflict model, which might support multiple and more efficient, peaceful equilibria (see Benhabib and Radner, 1992).

# Appendix A: Derivation of Equation (12)

In this appendix we show how to derive (12) in the text. Assuming an interior solution, we solve (11) for each contender to get the optimal strategy $a^i = \phi^i(Z)$. By substituting this optimal strategy into (10), the HJB equation (10) associated with $i$ is transformed into

$$\rho V^i(Z) = p^i(\phi^i(Z), \phi^{-i}(Z))\beta Z + V_Z^i(Z)\left[\sum_{j=1}^{n}\left(1-\phi^j(Z)\right) - \delta Z\right]. \tag{A.1}$$

By differentiating (A.1) with respect to $Z$ and applying the envelope theorem to the resulting expression, we obtain

$$\begin{aligned}\rho V_Z^i(Z) &= \sum_{j=1}^{n}\frac{\partial p^i}{\partial \phi^j}\phi^{j\prime}(Z)\beta Z + p^i(.)\beta + V_{ZZ}^i(Z)\left[\sum_{j=1}^{n}\left(1-\phi^j(Z)\right)-\delta Z\right] \\ &\quad + V_Z^i(Z)\left[-\sum_{j=1}^{n}\phi^{j\prime}(Z)-\delta\right].\end{aligned} \tag{A.2}$$

We also differentiate the interior first-order condition in (11) to get

$$V_{ZZ}^{i}\left(Z\right)=\sum_{j=1}^{n}\frac{\partial^{2}p^{i}}{\partial\phi^{i}\partial\phi^{j}}\phi^{j\prime}\left(Z\right)\beta Z+\frac{\partial p^{i}}{\partial\phi^{i}}\beta. \tag{A.3}$$

Substituting (11) and (A.3) into $V_{Z}^{i}\left(Z\right)$ and $V_{ZZ}^{i}\left(Z\right)$ in (A.2), respectively, and using symmetry, we obtain

$$\begin{gathered}0=(n-1)\left[\frac{\partial p^{i}}{\partial\phi_{k}}\beta Z-\frac{\partial p^{i}}{\partial\phi^{i}}\beta Z\right]\phi'\left(Z\right)+p(.)\beta+\\ \left[\frac{\partial^{2}p^{i}}{\partial\phi^{i2}}\phi^{i\prime}\left(Z\right)+(n-1)\frac{\partial^{2}p^{i}}{\partial\phi^{k}\partial\phi^{i}}\phi_{Z}^{k}(Z)\right]\beta Z\left[n\left(1-\phi(Z)\right)-\delta Z\right]\\ +\frac{\partial p^{i}}{\partial\phi^{i}}\beta\left[n\left(1-\phi(Z)\right)-\delta Z\right]-(\delta+\rho)\frac{\partial p^{i}}{\partial\phi^{i}}\beta Z,\ k\neq i.\end{gathered} \tag{A.4}$$

Since the assumption of symmetry allows us to make use of the following expressions:

$$\begin{aligned}p_{i}&=\frac{1}{n},\ \frac{\partial p^{i}}{\partial\phi^{i}}=\frac{r\left(n-1\right)}{n^{2}[\phi(Z)+\eta]},\ \frac{\partial p^{i}}{\partial\phi^{k}}=-\frac{r}{n^{2}[\phi(Z)+\eta]},\\ \frac{\partial^{2}p^{i}}{\partial\phi^{i2}}&=r\left(n-1\right)\frac{n\left(r-1\right)-2r}{n^{3}[\phi(Z)+\eta]^{2}},\ \frac{\partial^{2}p^{i}}{\partial\phi^{k}\partial\phi^{i}}=\frac{r^{2}\left(-n+2\right)}{n^{3}[\phi(Z)+\eta]^{2}},\end{aligned} \tag{A.5}$$

we substitute those expressions into (A.4) to obtain

$$\begin{gathered}0=\frac{n-1}{[\phi(Z)+\eta]n^{2}}\left[-r-r\left(n-1\right)\right]\beta Z\phi'(Z)+\frac{\beta}{n}+\\ \frac{r\left(n-1\right)}{n^{3}[\phi(Z)+\eta]^{2}}\left[r\left(-n+2\right)+n\left(r-1\right)-2r\right]\beta Z\left[n\left(1-\phi(Z)\right)-\delta Z\right]\phi'(Z)\\ +\frac{r\left(n-1\right)}{n^{2}[\phi(Z)+\eta]}\beta\left[n\left(1-a(Z)\right)-\delta Z\right]-(\delta+\rho)\frac{r\left(n-1\right)}{n^{2}[\phi(Z)+\eta]}\beta Z.\end{gathered} \tag{A.6}$$

Further rearranging (A.6) gives rise to (12) in the text.

# Appendix B: Proofs of Lemma 1, 2 and Proposition 1

To prove the existence of MPE we apply sufficiency conditions stated in Theorem 3 of Rowat (2007). Although it may be appropriate to distinguish between the value functions associated the *candidate* MPE strategies and qualified MPE strategies, it is omitted for the sake of notational simplicity.

**Proof of Lemma 1.** When all contenders play $\phi = 0$, the HJB equation (10) becomes

$$\rho V(Z) = \frac{1}{n}\beta Z + V_Z(Z)[n - \delta Z]. \tag{B.1}$$

Integrating (B.1) yields the equation given in (i) of Lemma 1, which we reproduce here for readers' convenience:

$$V(Z) = \frac{\beta(n + \rho Z)}{\rho(\rho + \delta)n} + c_2 [n - \delta Z]^{-\frac{\rho}{\delta}}. \tag{B.2}$$

Note that when $c_2 \neq 0$, $\lim_{Z \to n/\delta} V(Z) = \pm\infty$. This contradicts the bounded value function, which will be shown below. Since it is clearly seen from (7) and the bounded control constraint on $a_i(t)$ (i.e., the time-invariant interval $[0, 1]$ for $\forall t \in [0, \infty)$) that the cornered strategy $\phi = 0$ brings about the convergence of $Z(t)$ toward a finite and constant value of $Z$ on the steady state line $C_1$, so does the instantaneous objective function in (6). With a positive discount factor (i.e., $\rho > 0$), therefore, the value function in (6) is bounded (since the its lower boundedness is trivially satisfied).

Following Rowat (2007), by choosing $c_2$ appropriately, it may be possible to extend the interior strategy $\phi_1$, $\phi_2$ or $\phi_4$ by $\phi = 0$. To see this, setting $\phi = 0$ for $\partial p^i/\partial a^i$ in (A.5) to obtain $(\partial p^i/\partial a^i)\beta Z \equiv r(n-1)\beta Z/n^2\eta$, the auxiliary condition $(\partial p^i/\partial a^i)\beta Z \leq V_i'(Z)$ allows the solution $\phi = 0$ to hold for values of $Z$ satisfying

$$c_2 \geq \frac{\beta(n - \delta Z)^{\frac{\rho+\delta}{\delta}}}{\rho} \frac{r(n-1)}{n^2\eta}\left[Z - \frac{n\eta}{(\rho+\delta)r(n-1)}\right] \text{ if } (n - \delta Z)^{\frac{\rho+\delta}{\delta}} > 0, \tag{B.3}$$

whereas the opposite inequality holds if $(n - \delta Z)^{\frac{\rho+\delta}{\delta}} < 0$. Consider a first case (i.e., (B.3)). Let the r.h.s. of (B.3) denote $\Psi(Z)$ with $\Psi(0) = -\beta n^{\frac{\rho}{\delta}}/\rho(\rho+\delta) < 0$. Differentiating $\Psi(Z)$

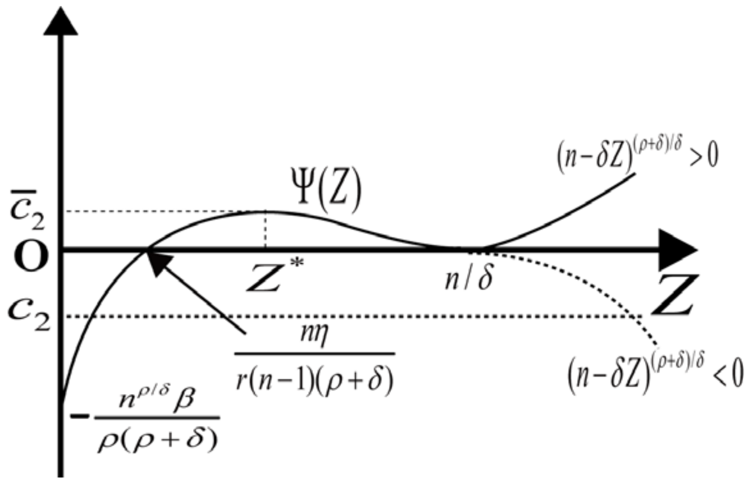


Figure 3: Transitions between the corner and interior solutions

with respect to $Z$ yields

$$\frac{d\Psi(Z)}{dZ} = \frac{\beta(n-\delta Z)^{\frac{\rho}{\delta}}}{\rho}\frac{r(n-1)(\rho+2\delta)}{n^2\eta}\left\{-Z + \frac{n\left[\eta + r(n-1)\right]}{r(n-1)(\rho+2\delta)}\right\}. \tag{B.4}$$

Let $Z^*$ denote the value of $Z$ at which the curly braces on the r.h.s. of (B.4) equals zero and thus the function $\Psi(Z)$ achieves its local maximum, where

$$Z^* = \frac{n\left[\eta + r(n-1)\right]}{r(n-1)(\rho+2\delta)}.$$

Note also that $Z^*$ equals the value of $Z$ at the intersection of $C_3$ with the horizontal axis in Figs. 1 and 2, and $Z^* < n/\delta$. Taken together, we can draw the graph of $\Psi(Z)$ as a *real line* in Fig.3.

Inspection of Fig.3 reveals the following facts. First, if $c_2 > 0$ is chosen, the value function in (B.2) becomes unbounded as $Z \to \delta/n$ because (B.3) is eventually satisfied and so $\phi = 0$ remains a solution at $Z = \delta/n$.[13] Secondly, when $c_2 < -n^{\frac{\rho}{\delta}}\beta/\rho(\rho+\delta)$ is chosen, $\phi = 0$ is disqualified as an optimal solution for every value of $Z \in [0, \infty)$, so that either the interior strategy $\phi_1$, $\phi_2$ or $\phi_4$ must be chosen; however, none of them can cross the $NI$ locus $C_2$. As

[13]In the second case (i.e., $(n-\delta Z)^{\frac{\rho+\delta}{\delta}} < 0$), the graph of $\Psi(Z)$ is the same as that under $(n-\delta Z)^{\frac{\rho+\delta}{\delta}} > 0$ when $Z < n/\delta$, while it is declining in $Z$ when $Z > n/\delta$ (the latter part of $\Psi(Z)$ is illustrated as a dotted line in Fig.3). Inspection of Fig.3 reveals that when $c_2 > 0$, the corner strategy $\phi = 0$ still remains a solution in the neighborhood of $Z = n/\delta$, which leads to the unbounded value function.

a result, only when we choose $c_2 \in \left[-n^{\frac{\rho}{\delta}}\beta/\rho(\rho+\delta), 0\right]$ in Fig.3, $\phi = 0$ can be switched to a member of the family $\phi_4$ or $\phi_1$ at some value of $Z \in [0, \eta n/r\,(n-1)\,(\rho+\delta)]$ associated with their intersection. ■

**Proof of Lemma 3.** When all contenders play $\phi = 1$, the HJB equation (10) becomes

$$\rho V(Z) = \frac{1}{n}\beta Z + V_Z(Z)(-\delta Z). \tag{B.5}$$

Integrating (B.5) yields the equation given in (i) of Lemma 3, which we reproduce here for readers' convenience:

$$V(Z) = \beta \frac{Z + Z^{-\frac{\rho}{\delta}} n(\rho+\delta)\, c_3}{n(\rho+\delta)}. \tag{B.6}$$

Note that when $c_3 \neq 0$, $\lim_{Z\to 0} V(Z) = \infty$, which contradicts the bounded value function. This can readily be shown by the fact that the feasible path of $Z(t)$ always converges from any $Z \in [0,\infty)$ to a finite and constant value of $Z$ on the steady state line $C_1$, as in the proof of Lemma 1.

As before, we have to choose $c_3$ appropriately in order to extend either the interior strategy $\phi_2$, $\phi_3$ or $\phi_4$ by $\phi = 0$. To do this, setting $\phi = 1$ for $\partial p^i/\partial a^i$ in (A.5) to obtain $(\partial p^i/\partial a^i)\beta Z \equiv r\,(n-1)\,\beta Z/n^2(1+\eta)$, the auxiliary condition $(\partial p^i/\partial a^i)\,\beta Z \geq V_Z^i(Z)$ allows the solution $\phi = 1$ to hold for values of $Z$ satisfying

$$\frac{\delta}{\rho n}\left[\frac{1}{\rho+\delta} Z^{\frac{\rho+\delta}{\delta}} - \frac{r\,(n-1)}{n(1+\eta)} Z^{\frac{\rho+2\delta}{\delta}}\right] \leq c_3. \tag{B.7}$$

For expositional purposes, we denote by $\Phi(Z)$ the l.h.s. of (B.7). As the exponent of the first term on the l.h.s. of (B.7) is smaller than that of the second, it dominates for smaller values of $Z$, while for larger $Z$ the second term dominates the first one. Hence, the graph of $\Phi(Z)$ has the maximum at $Z_C = n(1+\eta)/r(n-1)(\rho+2\delta)$ (which also corresponds to the value of $Z$ at the intersection of $C_3$ with the resource constraint $a = 1$ in Figs. 1 and 2). Put together, we can draw the inverted U-shape graph of $\Phi(Z)$ as illustrated in Figs.4 and 5. When $c_3 > \bar{c}_3$ is chosen, it follows from (B.6) that $V(Z) \to +\infty$ as $Z \to 0$, since $\Phi(Z) = 1$ satisfies (B.7) for every value of $Z \in [0,\infty)$, thus eliminating this case. ■

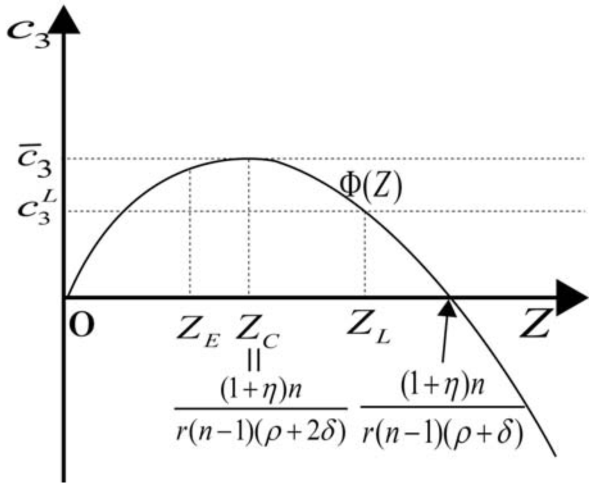


Figure 4: Transitions between the corner and interior solutions when $Z_L > Z_C$.

**Proof of Proposition 1.** First, we first consider Fig. 1. If any value of $c_3$ is less than $c_3^L$ in Fig.4, $\phi = 1$ has to be connected with the interior strategy $\phi_2$ which always crosses the horizontal axis at a value of $Z$ lying on the right side of $Z_E$. However, an extension by $\phi = 0$ is not possible because $\phi = 0$ is not an optimal solution at that intersection point due to (ii) in Lemma 1. If $c_3$ is chosen such that $c_3 \in \left(c_3^L, \bar{c}_3\right]$ in Fig.4, $\phi = 1$ has to be connected with $\phi_3$ in Fig. 1. However, this extension cannot cover the whole domain of $Z$ because the family $\phi_3$ of strategies never crosses the $NI$ locus $C_2$.

When $c_3$ precisely equals $c_3^L$, the cornered strategy $\phi = 1$ is connected with the linear strategy $\phi_L$, which can intersect the $NI$ locus $C_2$. Moreover, since $\phi_L$ intersects with the horizontal axis at a value of $Z$ which is less than that at the intersection of $C_3$ and the horizontal axis. Because of Lemma 1, there exists $c_2 \in \left[-n^{\rho/\delta}\beta/\rho(\rho+\delta), 0\right]$ such that $\phi = 1$ can be connected with $\phi_L$ at the value of $Z$ associated with the chosen $c_2$. As a result, the extended strategy $\hat{\phi}_L(Z)$ can continuously cover the whole range of the domain.

Next, consider Fig. 2. If $c_3$ is chosen such that $c_3 < \bar{c}_3$ in Fig.5, $\phi = 1$ has to be connected with an interior strategy at a value of $Z > Z_C$. However, an extension by $\phi_2$ is not possible for the same reason stated above. Only when $c_3 = \bar{c}_3$ is chosen, there exists a unique member of the family $\phi_1$, labeled $\phi_1^*$, that can be connected with $\phi = 1$ at $Z_C$ where the interior strategy $\phi_1^*$ is tangent to the constraint $a = 1$ as illustrated in Fig. 2. This is because $\phi = 1$ still remains an optimal solution at $Z_C$ due to (ii) in Lemma 3. However, other members of the family $\phi_1$

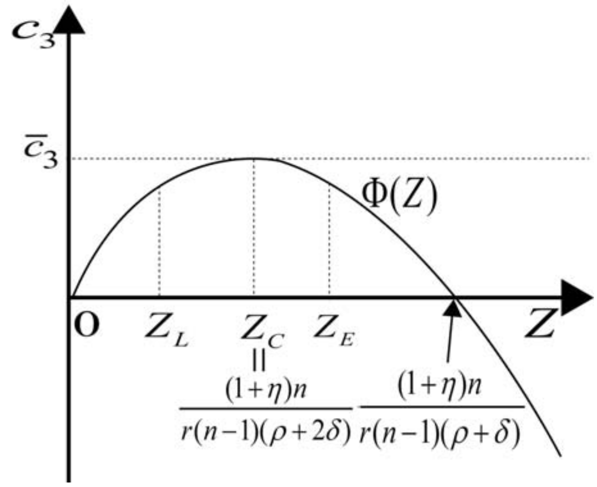


Figure 5: Transitions between the corner and interior solutions when $Z_L < Z_C$.

cannot be extended by $\phi = 1$. The reason is as follows. To connect $\phi = 1$ with the members of the family $\phi_1$ lying on the left side of $\phi_1^*$, we have to choose $c_3$ less than $\bar{c}_3$ in Fig.5, which leads $\phi = 1$ to connect with either the family $\phi_2$ or the members of the family $\phi_1$ lying on the right side of $\phi_1^*$ in Fig.2; however, either extension is not possible.

In addition, the strategy $\phi_1^*$ can be extended by $\phi = 0$, since $\phi = 0$ remains a solution at the intersection of $\phi_1^*$ with the horizontal axis due to (ii) in Lemma 1. Taken together, the unique strategy $\phi_1^*$ can be continuously extended by $\phi = 0$ and $\phi = 1$ to the whole space of the domain of $Z$.

Therefore, the only two extensions, $\hat{\phi}_L$ and $\hat{\phi}_1^*$, satisfy condition (i) of Theorem 3 of Rowat (2007). Although we further need to confirm that both strategies satisfy conditions (ii)-(iv) in Theorem 3 of Rowat, we will not repeat it here, since the rest of the proof proceeds by exactly following his proof.

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