Vanishing S-curvature of Randers spaces

Shin-ichi OHTA*

Department of Mathematics, Faculty of Science, Kyoto University,
Kyoto 606-8502, JAPAN (e-mail: sohta@math.kyoto-u.ac.jp)

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Abstract

We give a necessary and sufficient condition on a Randers space for the existence of a measure for which Shen's **S**-curvature vanishes everywhere. Moreover, if it exists, such a measure coincides with the Busemann-Hausdorff measure up to a constant multiplication.

Keywords: Randers spaces, **S**-curvature, Ricci curvature

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1 Introduction

This short article is concerned with a characterization of Randers spaces admitting measures with vanishing **S**-curvature. A Randers space (due to Randers [Ra]) is a special kind of Finsler manifold (M, F) whose Finsler structure $F : TM \longrightarrow [0, \infty)$ is written as $F(v) = \alpha(v) + \beta(v)$, where α is a norm induced from a Riemannian metric on M and β is a one-form on M. Randers spaces are important in applications and reasonable for concrete calculations. See [AIM] and [BCS, Chapter 11] for more on Randers spaces.

We equip a Finsler manifold (M, F) with an arbitrary smooth measure m. Then the **S**-curvature $\mathbf{S}(v) \in \mathbb{R}$ of $v \in TM$ introduced by Shen (see [Sh, §7.3]) measures the difference between m and the volume measure of the Riemannian structure induced from the tangent vector field of the geodesic η with $\dot{\eta}(0) = v$ (see §2.2 for the precise definition). The author's recent work [Oh], [OS] on the weighted Ricci curvature (in connection with optimal transport theory) shed new light on the importance of this quantity.

A natural and important question arising from the theory of weighted Ricci curvature is: when does (M, F) admit a measure m with $\mathbf{S} \equiv 0$? If such a measure exists, then we can choose it as a good reference measure. Our main result provides a complete answer to this question for Randers spaces.

Theorem 1.1 *A Randers space (M, F) admits a measure m with $\mathbf{S} \equiv 0$ if and only if β is a Killing form of constant length. Moreover, then m coincides with the Busemann-Hausdorff measure up to a constant multiplication.*

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It has been observed by Shen [Sh, Example 7.3.1] that a Randers space with the Busemann-Hausdorff measure satisfies $\mathbf{S} \equiv 0$ if β is a Killing form of constant length. Our theorem asserts that his condition on β is also necessary for the existence of m with $\mathbf{S} \equiv 0$, and then it immediately follows that m must be a constant multiplication of the Busemann-Hausdorff measure.

On the one hand, Shen's result (the "if" part of Theorem 1.1) ensures that there is a rich class of non-Riemannian Randers spaces satisfying $\mathbf{S} \equiv 0$. On the other hand, the "only if" part says that the class of general Randers spaces is much wider and many Randers spaces have no measures with $\mathbf{S} \equiv 0$. This means that there are no canonical (reference) measures on such Finsler manifolds (in respect of the weighted Ricci curvature). Therefore, for a general Finsler manifold, it is natural to start with an arbitrary measure, as was discussed in [Oh] and [OS].

2 Preliminaries for Finsler geometry

We first review the basics of Finsler geometry. Standard references are [BCS] and [Sh]. We will follow the notations in [BCS] with a little change (e.g., we use v^i instead of y^i).

2.1 Finsler structures

Let M be a connected n-dimensional C^∞-manifold with $n \geq 2$, and $\pi : TM \longrightarrow M$ be the natural projection. Given a local coordinate $(x^i)_{i=1}^n : U \longrightarrow \mathbb{R}^n$ of an open set $U \subset M$, we will always denote by $(x^i; v^i)_{i=1}^n$ the local coordinate of $\pi^{-1}(U)$ given by $v = \sum_i v^i (\partial/\partial x^i)|_{\pi(v)}$.

A C^∞-*Finsler structure* is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following conditions:

(I) F is C^∞ on $TM \setminus \{0\}$;

(II) $F(cv) = cF(v)$ for all $v \in TM$ and $c \geq 0$;

(III) The $n \times n$ matrix

$$g_{ij}(v) := \frac{1}{2} \frac{\partial(F^2)}{\partial v^i \partial v^j}(v)$$

is positive-definite for all $v \in TM \setminus \{0\}$.

The positive-definite matrix $(g_{ij}(v))$ defines a Riemannian structure g_v of T_xM through

$$g_v\Bigg(\sum_i a^i \frac{\partial}{\partial x^i}, \sum_j b^j \frac{\partial}{\partial x^j} \Bigg) := \sum_{i,j} g_{ij}(v) a^i b^j. \tag{2.1}$$

Note that $g_v(v, v) = F(v)^2$. This inner product g_v is regarded as the best approximation of $F|_{T_xM}$ in the direction v. Indeed, the unit sphere of g_v is tangent to that of $F|_{T_xM}$ at $v/F(v)$ up to the second order. If (M, F) is Riemannian, then g_v always coincides with the original Riemannian metric. As usual, (g^{ij}) will stand for the inverse matrix of (g_{ij}).

We define the *Cartan tensor*

$$A_{ijk}(v) := \frac{F(v)}{2}\frac{\partial g_{ij}}{\partial v^k}(v)$$

for $v \in TM \setminus \{0\}$, and remark that $A_{ijk} \equiv 0$ holds if and only if (M,F) is Riemannian. We also define the *formal Christoffel symbol*

$$\gamma^i{}_{jk}(v) := \frac{1}{2}\sum_l g^{il}(v)\left\{\frac{\partial g_{lj}}{\partial x^k}(v) + \frac{\partial g_{kl}}{\partial x^j}(v) - \frac{\partial g_{jk}}{\partial x^l}(v)\right\}$$

for $v \in TM \setminus \{0\}$. Then the geodesic equation is written as $\ddot{\eta} + G(\dot{\eta}) = 0$ with the *geodesic spray coefficients*

$$G^i(v) := \sum_{j,k} \gamma^i{}_{jk}(v) v^j v^k$$

for $v \in TM$ ($G^i(0) := 0$ by convention). Using these, we further define the *nonlinear connection*

$$N^i{}_j(v) := \sum_k \left\{\gamma^i{}_{jk}(v)v^k - \frac{1}{F(v)}A^i{}_{jk}(v)G^k(v)\right\}$$

for $v \in TM$ ($N^i{}_j(0) := 0$ by convention), where $A^i{}_{jk}(v) := \sum_l g^{il}(v)A_{ljk}(v)$. Note that (see [BCS, Exercise 2.3.3])

$$N^i{}_j(v) = \frac{1}{2}\frac{\partial G^i}{\partial v^j}(v).$$

2.2 S-curvature and weighted Ricci curvature

We choose an arbitrary positive C^∞-measure m on a Finsler manifold (M,F). Fix a unit vector $v \in F^{-1}(1)$ and let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic with $\dot{\eta}(0) = v$. Along η, the tangent vector field $\dot{\eta}$ defines the Riemannian metric $g_{\dot{\eta}}$ via (2.1). Denoting the volume form of $g_{\dot{\eta}}$ by $\mathrm{vol}_{\dot{\eta}}$, we decompose m into $m(dx) = e^{-\Psi(\dot{\eta})}\,\mathrm{vol}_{\dot{\eta}}(dx)$ along η. Then we define the $\mathbf{S}$-*curvature* of v by

$$\mathbf{S}(v) := \frac{d(\Psi \circ \dot{\eta})}{dt}(0).$$

We extend this definition to all $w = cv$ with $c \geq 0$ by $\mathbf{S}(w) := c\mathbf{S}(v)$. Clearly $\mathbf{S} \equiv 0$ holds on Riemannian manifolds with the volume measure.

The *weighted Ricci curvature* is defined in a similar manner as follows:

(i) $\mathrm{Ric}_n(v) := \mathrm{Ric}(v) + (\Psi \circ \eta)''(0)$ if $\mathbf{S}(v) = 0$, $\mathrm{Ric}_n(v) := -\infty$ otherwise;

(ii) $\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + (\Psi \circ \eta)''(0) - \mathbf{S}(v)^2/(N-n)$ for $N \in (n, \infty)$;

(iii) $\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + (\Psi \circ \eta)''(0)$.

Here $\mathrm{Ric}(v)$ is the usual (unweighted) Ricci curvature of v. The author [Oh] shows that bounding Ric_N from below by $K \in \mathbb{R}$ is equivalent to the curvature-dimension condition $\mathsf{CD}(K,N)$, and then there are many analytic and geometric applications. Observe that the bound $\mathrm{Ric}_n \geq K > -\infty$ makes sense only when the $\mathbf{S}$-curvature vanishes everywhere.

Therefore the class of such special triples (M, F, m) deserves a particular interest. We remark that, if there are two measures m_1, m_2 on (M, F) satisfying $\mathbf{S} \equiv 0$, then $m_1 = c \cdot m_2$ holds for some positive constant c.

We rewrite $\mathbf{S}(v)$ according to [Sh, §7.3] for ease of later calculation. Recall that η is the geodesic with $\dot\eta(0) = v$. Fix a local coordinate $(x^i)_{i=1}^n$ containing η and represent m along η as

$$m(dx) = \sigma(\eta)\, dx^1 dx^2 \cdots dx^n = \frac{\sigma(\eta)}{\sqrt{\det(g_{\dot\eta})}} \operatorname{vol}_{\dot\eta}(dx).$$

We have by definition

$$\mathbf{S}(v) = \frac{d}{dt}\Big|_{t=0} \log\left(\frac{\sqrt{\det(g_{\dot\eta(t)})}}{\sigma(\eta(t))}\right) = \frac{1}{2\det(g_v)} \frac{d}{dt}\Big|_{t=0} \big[\det(g_{\dot\eta(t)})\big] - \sum_i \frac{v^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x).$$

Since η solves the geodesic equation $\ddot\eta + G(\dot\eta) = 0$, the first term is equal to

$$\begin{aligned} &\frac{1}{2} \sum_{i,j,k} \left\{ g^{ij}(v) \frac{\partial g_{ij}}{\partial x^k}(v) v^k + g^{ij}(v) \frac{\partial g_{ij}}{\partial v^k}(v) \ddot\eta^k(0) \right\} \\ &= \sum_{i,k} \left\{ \gamma^i{}_{ik}(v) v^k - \frac{1}{F(v)} A^i{}_{ik}(v) G^k(v) \right\} = \sum_i N^i{}_i(v). \end{aligned}$$

Thus we obtain

$$\mathbf{S}(v) = \sum_i \left\{ N^i{}_i(v) - \frac{v^i}{\sigma(x)} \frac{\partial \sigma}{\partial x^i}(x) \right\}. \tag{2.2}$$

Observe that $\mathbf{S}(cv) = c\mathbf{S}(v)$ indeed holds for $c \geq 0$ in this form.

2.3 Busemann-Hausdorff measure and Berwald spaces

Different from the Riemannian case, there are several constructive measures on a Finsler manifold, each of them is canonical in some sense and coincides with the volume measure for Riemannian manifolds. Among them, here we treat only the Busemann-Hausdorff measure which is actually the Hausdorff measure associated with the suitable distance structure if F is symmetric in the sense that $F(-v) = F(v)$ holds for all $v \in TM$.

Roughly speaking, the Busemann-Hausdorff measure is the measure such that the volume of the unit ball of each tangent space equals the volume of the unit ball in $\mathbb{R}^n$. Precisely, using a basis $w_1, w_2, \ldots, w_n \in T_xM$ and its dual basis $\theta^1, \theta^2, \ldots, \theta^n \in T_x^*M$, the *Busemann-Hausdorff measure* $m_{BH}(dx) = \sigma_{BH}(x)\, \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n$ is defined as

$$\frac{\omega_n}{\sigma_{BH}(x)} = \operatorname{vol}_n\left(\left\{ (c^i) \in \mathbb{R}^n \,\Big|\, F\Big(\sum_i c^i w_i\Big) < 1 \right\}\right),$$

where vol_n is the Lebesgue measure and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Let (M, F) be a *Berwald space* (see [BCS, Chapter 10] for the precise definition). Then it is well known that $\mathbf{S} \equiv 0$ holds for the Busemann-Hausdorff measure (see [Sh, Proposition 7.3.1]). In fact, along any geodesic $\eta : [0, l] \longrightarrow M$, the parallel transport $T_{0,t} : T_{\eta(0)}M \longrightarrow T_{\eta(t)}M$ with respect to $g_{\dot\eta}$ preserves F. Therefore choosing parallel vector fields along η as a basis yields that σ_{BH} is constant on η, which yields $\mathbf{S} \equiv 0$.

3 Proof of Theorem 1.1

Let (M, F) be a Randers space, i.e., $F(v) = \alpha(v) + \beta(v)$ such that α is a norm induced from a Riemannian metric and that β is a one-form. In a local coordinate $(x^i)_{i=1}^n$, we can write

$$\alpha(v) = \sqrt{\sum_{i,j} a_{ij}(x)v^i v^j}, \qquad \beta(v) = \sum_i b_i(x)v^i$$

for $v \in T_xM$. The *length* of β at x is defined by $\|\beta\|(x) := \sqrt{\sum_{i,j} a^{ij}(x)b_i(x)b_j(x)}$, which is necessarily less than 1 in order to guarantee $F > 0$ on $TM \setminus \{0\}$.

We denote the Christoffel symbol of (a_{ij}) by $\tilde{\gamma}^i{}_{jk}$. We also define

$$b^i(x) := \sum_j a^{ij}(x)b_j(x), \qquad b_{i|j}(x) := \frac{\partial b_i}{\partial x^j}(x) - \sum_k b_k(x)\tilde{\gamma}^k{}_{ij}(x).$$

Note that $b_{i|j}$ is the coefficient of the covariant derivative $\tilde{\nabla}$ of β with respect to α, namely $\tilde{\nabla}_{\partial/\partial x^j}\beta = \sum_i b_{i|j}dx^i$. We find by calculation that

$$\frac{\partial(\|\beta\|^2)}{\partial x^i}(x) = 2\sum_j b_{j|i}(x)b^j(x). \tag{3.1}$$

We say that β is a *Killing form* if $b_{i|j} + b_{j|i} \equiv 0$ holds on M. The geodesic spray coefficients of F are given by (see [BCS, (11.3.11)])

$$\begin{aligned} G^i(v) &= \sum_{j,k} \gamma^i{}_{jk}(v)v^j v^k \\ &= \sum_{j,k} \Big[\tilde{\gamma}^i{}_{jk}(x)v^j v^k + b_{j|k}(x)\big(a^{ij}(x)v^k - a^{ik}(x)v^j\big)\alpha(v) \\ &\quad + b_{j|k}(x)\frac{v^i}{F(v)}\big\{v^j v^k + \big(b^k(x)v^j - b^j(x)v^k\big)\alpha(v)\big\} \Big] \\ &=: \sum_{j,k} \tilde{\gamma}^i{}_{jk}(x)v^j v^k + X^i(v) + Y^i(v). \end{aligned} \tag{3.2}$$

If $\mathbf{S} \equiv 0$ on T_xM, then we deduce from (2.2) that $\sum_i N^i{}_i(v)$ is linear in $v \in T_xM$. We shall see that only this infinitesimal constraint is enough to imply the condition on β stated in Theorem 1.1. To see this, we calculate $2N^i{}_i = \partial G^i/\partial v^i$ using (3.2). As the first term $\sum_{j,k} \tilde{\gamma}^i{}_{jk}(x)v^j v^k$ comes from a Riemannian structure, it suffices to consider only the linearly of $\sum_i \{\partial X^i/\partial v^i(v) + \partial Y^i/\partial v^i(v)\}$. For the sake of simplicity, we will omit evaluations at x and v in the following calculations.

We first obtain

$$\begin{aligned} \sum_i \frac{\partial X^i}{\partial v^i} &= \sum_{i,j}(b_{j|i} - b_{i|j})a^{ij}\alpha + \sum_{i,j,k,l} b_{j|k}(a^{ij}v^k - a^{ik}v^j)\frac{a_{il}v^l}{\alpha} \\ &= \sum_{i,j} b_{j|i}(a^{ij} - a^{ji})\alpha + \sum_{j,k} b_{j|k}(v^k v^j - v^j v^k)\alpha^{-1} = 0. \end{aligned}$$

As Euler's theorem [BCS, Theorem 1.2.1] ensures

$$\sum_i \frac{\partial}{\partial v^i}\left(\frac{v^i}{F}\right) = \frac{1}{F^2}\sum_i \left(F - v^i \frac{\partial F}{\partial v^i}\right) = \frac{n-1}{F},$$

we next observe

$$\begin{aligned}
\sum_i \frac{\partial Y^i}{\partial v^i} &= \sum_{i,j} \frac{v^i}{F}\left\{(b_{i|j}+b_{j|i})v^j + (b_{i|j}-b_{j|i})b^j\alpha + \sum_{k,l} b_{j|k}(b^k v^j - b^j v^k)\frac{a_{il}v^l}{\alpha}\right\} \\
&\quad + \frac{n-1}{F}\sum_{j,k} b_{j|k}\{v^j v^k + (b^k v^j - b^j v^k)\alpha\} \\
&= \frac{n+1}{2}\sum_{i,j}(b_{i|j}+b_{j|i})\frac{v^i v^j}{F} + (n+1)\sum_{i,j}(b_{i|j}-b_{j|i})b^j \frac{\alpha v^i}{F}.
\end{aligned}$$

By comparing the evaluations at v and $-v$, the coefficients $b_{i|j} + b_{j|i}$ in the first term must vanish for all i, j, and hence β is a Killing form. For the second term, we find that $(\alpha/F)\sum_j (b_{i|j} - b_{j|i})b^j$ must be constant on each T_xM. If α/F is not constant on some T_xM (i.e., $\|\beta\|(x) \neq 0$), then it holds that $\sum_j (b_{i|j} - b_{j|i})b^j = 0$. Since β is a Killing form, we deduce from (3.1) that

$$0 = \sum_j (b_{i|j} - b_{j|i})b^j = -2\sum_j b_{j|i}b^j = -\frac{\partial(\|\beta\|^2)}{\partial x^i}.$$

Therefore β has a constant length as required, for $\|\beta\| \neq 0$ is an open condition. If α/F is constant on some T_xM, then the above argument yields that $\beta \equiv 0$ on M. This completes the proof of the "only if" part of Theorem 1.1.

For the "if" part, it is sufficient to show that the Busemann-Hausdorff measure satisfies $\mathbf{S} \equiv 0$, that can be found in [Sh, Example 7.3.1]. We briefly repeat his discussion for completeness. We first observe from [Sh, (2.10)] that

$$m_{BH}(dx) = \left(1 - \|\beta\|(x)^2\right)^{(n+1)/2}\sqrt{\det(a_{ij}(x))}\, dx^1 \cdots dx^n =: \sigma_{BH}(x)\, dx^1 \cdots dx^n.$$

Since β has a constant length, we have

$$\sum_k \frac{v^k}{\sigma_{BH}(x)}\frac{\partial \sigma_{BH}}{\partial x^k}(x) = \frac{1}{2}\sum_{i,j,k} v^k a^{ij}(x)\frac{\partial a_{ij}}{\partial x^k}(x) = \sum_{i,j} \tilde{\gamma}^i{}_{ij}(x)v^j.$$

Therefore we conclude, by (2.2),

$$\mathbf{S}(v) = \frac{1}{2}\sum_{i,j,k}\frac{\partial}{\partial v^i}\left[\tilde{\gamma}^i{}_{jk}(x)v^j v^k\right] - \sum_k \frac{v^k}{\sigma_{BH}(x)}\frac{\partial \sigma_{BH}}{\partial x^k}(x) = 0.$$

□

We finally remark related known results and several consequences of Theorem 1.1.

Remark 3.1 (a) A Randers space is a Berwald space if and only if β is *parallel* in the sense that $b_{i|j} \equiv 0$ for all i, j (see [BCS, Theorem 11.5.1]). Thanks to [Sh, Example 7.3.2], we know that a Killing form of constant length is not necessarily parallel.

(b) In [De], Deng gives a characterization of vanishing **S**-curvature for homogeneous Randers spaces endowed with the Busemann-Hausdorff measure.

(c) It is easy to construct a Randers space whose β does not have a constant length. Hence many Finsler manifolds do not admit measures with $\mathbf{S} \equiv 0$ (in other words, with $\mathrm{Ric}_n \geq K > -\infty$).

(d) Another consequence of Theorem 1.1 is that only (constant multiplications of) the Busemann-Hausdorff measures can satisfy $\mathbf{S} \equiv 0$ on Randers spaces. Then a natural question is the following:

Question *Is there a Finsler manifold* (M, F) *on which some measure* m *other than* (*a constant multiplication of*) *the Busemann-Hausdorff measure satisfies* $\mathbf{S} \equiv 0$*? If yes, what kind of measure is* m*?*

If such a measure exists, then it is more natural than the Busemann-Hausdorff measure in respect of the weighted Ricci curvature.

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