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When Market Impact Causes Gradual Liquidation? : From the Theoretical View of Mathematical Finance *

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1 Introduction

Liquidity (or illiquidity) is one of the important topics in mathematical finance theory. For an optimal investment problem, Merton studied in his classical works [27] and [28] how a trader (or an investor) manages his/her portfolio in the ideal market where there are no liquidity problems. This is called the Merton problem or the Merton theory. After that, as generalization of the Merton problem, many papers have been studying a trader's optimal policy in the illiquid market. Of course the real market is more or less illiquid, so it is meaningful to consider such a problem.

The word "illiquidity" has several meanings and is used for wide sense in mathematical finance. One may use for the uncertainty if a trader completes the transaction or not. One may use for the effect of transaction costs. Market impact (MI) is also one of liquidity problems and this is the main interest of this paper. Here MI means the effect of a trader's investment behavior on security prices.

When we consider such liquidity problems, the optimal policy of Merton is no longer optimal and the different policy becomes optimal. We regard that illiquidity causes difference of the optimal policies against the Merton policy. In this paper, we set the ideal model as the benchmark model (as for an optimal investment problem, we take the Merton problem) and consider what factors of illiquidity bring about for a trader's policy.

Another important optimization problem in mathematical finance is "an optimal execution problem" which is especially focused recently. This problem treats the case that a trader has a certain volume of a security and he/she tries to sell them until the time horizon (here the word "execution" means both buying and selling, but in this paper we consider only the selling. So our problem is also called "an optimal liquidation problem.") Considering this situation, the most essential factors of illiquidity is the effect of MI. [19] formulates an optimal execution problem with MI and derives the continuous-time model as a stochastic control problem. The model of this paper is based on [19] and [21].

One of the purpose of this paper is to find how MI affects a trader's policy by comparing the optimal execution strategies in the fully liquid and illiquid market. We set our benchmark model as a maximization problem of expected proceeds by the trader's execution in the liquid market where there is no MI (so a trader is risk-neutral.) In this case, a trader's optimal strategy is "block liquidation", that is, he/she should sell up all the shares of the security at once. But such

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a liquidation causes big MI, following crucial decrease of proceeds in the real market. So a trader may execute the security by taking time. We call such an execution "gradual liquidation."

When does MI cause gradual liquidation? This is the title of this paper and the gradual liquidation is optimal in an adequate model of optimal execution. This paper gives some situations of them and suggests the method of measuring MI cost. We define total MI cost as the damage rate for the proceeds of the execution, and we show that the forms of total MI cost functions are consistent with several empirical studies.

The rest of this paper is organized as follows. In Section 2, we introduce an optimal investment problem. We review the Merton problem and we present various types of liquidity problems and compare them with the classical theory. In Section 3 we introduce briefly our model of an optimal execution problem. In Section 4, we give some examples where MI causes gradual liquidation. Section 5 gives the concepts of total MI cost and compares the forms of MI functions by using examples in Section 4. We conclude this paper in Section 6.

Almost all proofs are omitted, but you can refer to [19] and [21].

2 Optimal Investment Problems and Effects of Iliquidity

First of all, we notice our assumption to make the point of this study simple. We always consider the market which consists of only two financial assets. The one is a risk-free asset (namely cash) and the other is a risky asset (namely a security.) The price of cash is always equal to 1, which means that a risk-free rate is equal to zero. The price of a security fluctuates stochastically.

We prepare a (complete) stochastic basis (a filtered probability space which satisfies usual conditions, see [18]) $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ for $T > 0$. Here T is called the time horizon and for brevity we assume $T = 1$.

We consider a single trader (or an investor) who has an initial endowment $w_0 \geq 0$ as cash at $t = 0$ (initial time.) He/She buys and sells the security and tries to maximize his/her expected utility of the terminal wealth. This is mathematically described as the following stochastic control problem

$$V_0(w_0) = \sup_{(\varphi_t)_{0\leq t\leq T}} \mathrm{E}[u(W_T)], \tag{2.1}$$

where $u : \mathbb{R} \longrightarrow \mathbb{R}$ is a trader's utility function and W_T means the wealth of a trader at the time $t = T(= 1)$. Here the stochastic process $(W_t)_{0\leq t\leq T}$ is given by the following stochastic differential equation (SDE)

$$dW_t = \varphi_t dS_t, \quad W_0 = w_0,$$

where $(S_t)_{0\leq t\leq T}$ denotes the price process of a security. Our control variable $(\varphi_t)_t$ means security holdings, that is his/her shares of the security held.

We assume that the fluctuation of the security price is given as

$$dS_t = S_t(\mu dt + \sigma dB_t), \quad S_0 = s_0 \tag{2.2}$$

for some $\mu \in \mathbb{R}$, $\sigma > 0$ and $s_0 > 0$. If $(S_t)_t$ satisfies the equation (2.2), we say that $(S_t)_t$ follows a geometric Brownian motion or is in Black-Scholes model.

To consider (2.1), it is useful to replace a control variable $(\varphi_t)_t$ with $(\pi_t)_t$, where $\pi_t = \dfrac{\varphi_t S_t}{W_t}$. π_t means the portfolio ratio or weight of investment of a security.

In the Merton theory, we assume that a trader has log-utility function $u(w) = \log w$ (we also consider a case of power utility functions, but we omit such the case in this paper.) Then, under suitable mathematical assumptions, we can derive the optimal portfolio ratio for a trader as follows.

Theorem 1 *It holds that* $V_0(w_0) = \log w_0 + \dfrac{\mu^2}{2\sigma^2}$ *and the optimal strategy is given by*

$$\pi_t^* = \pi^{\text{Merton}} = \frac{\pi}{\sigma^2}. \tag{2.3}$$

This theorem means that a trader should keep the portfolio ratio equal to the constant π^{Merton}, which is called the Merton proportion. So he/she needs to buy or sell continuously. This is the summary of the Merton problem.

Now we consider the liquidity problem. First we consider the effect of transaction costs. The Merton strategy (that is to keep the portfolio ratio with π^{Merton}) needs the continuous trading and makes infinite cost. So a trader bankrupts immediately when there are transaction costs.

What strategy is optimal when we consider transaction costs? The answer is given by [8] when there is a proportional transaction cost. We see that the optimal strategy is to keep the portfolio ratio within some range $\mathrm{NT} \subset [0, 1]$. The set NT is called "no-transaction region." If the portfolio ratio is in NT, a trader does no action. When there is no transaction cost, NT is equal to $\{\pi^{\text{Merton}}\}$, and he/she needs to repeat the trading continuously. If a transaction cost is proportional with respect to a trading amount, a trader's optimization problem is given as the solution of a singular control problem. If a transaction cost has a fixed part (which is independent of a trading amount,) the corresponding problem is characterized by an impulse control problem. For more details, see [24], [30] and [34].

Next we consider a case of random trading time. [26] consider a case where transactions are fulfilled at the time when a certain Poisson process $(N_t)_t$ jumps. Let λ be an intensity parameter of $(N_t)_t$. Then it holds that an optimal policy is to try to keep the portfolio ratio with

$$\pi_t^* = \pi^{\text{Merton}} + O(\lambda^{-1})$$

for sufficient large λ, where O denotes order notation (Landau's symbol.) For other related studies, see [11], [25] and [32].

Although it is not liquidity problem, let us introduce the studies of insider models. We take up the study of [6] here. In this model, instead of considering illiquidity, we assume that a trader has an additional information about the future price of the security. In this case, a trader's optimal strategy is given as the following form

$$\pi_t^* = \pi^{\text{Merton}} + \mu_t^G.$$

The term μ_t^G means the effect of the additional information. For instance, if a trader knows that the security price (may) goes up, then he/she wants to increase the ratio of an investing amount more than π^{Merton}. Here μ_t^G is called "an information drift."

These effects are arranged like Figure 1. By considering liquidity problems (and insider models,) optimal strategies are no longer the Merton strategy, and the difference between them represents the effects of illiquidity (and additional informations). In this situation, the Merton problem and its solution (the Merton proportion) are set as our benchmark model.

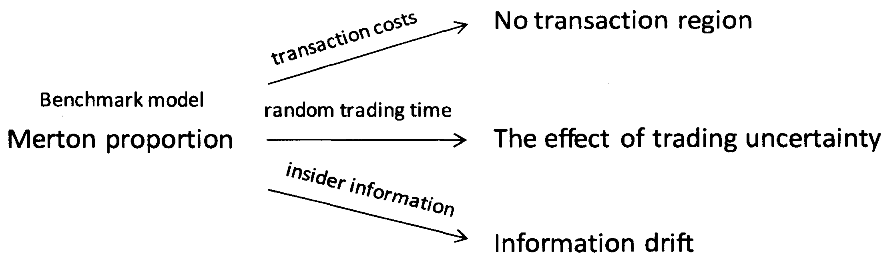

Figure 1: Effects of illiquidity (and additional information)

3 Mathematical Model of an Optimal Execution Problem

Now we consider an optimal execution problem, which is also important in real trading operations. We consider a case that a trader has $\Phi_0(>0)$ shares of a security at the initial time $t=0$ and he/she tries to sell (execute or liquidate) them until the time horizon $t=T=1$. A trader's problem is represented as a maximization problem of the expected utility of terminal cash holdings (which is the proceeds of the execution.) The optimization problem of this type is studied by [1], [4], [7], [13], [23], [29], [35] and so on.

The purpose of this section is to construct a mathematical model of such a problem in illiquid market. When considering the optimal execution, the most important and essential factor of illiquidity is the effect of market impact (MI). So we should construct the model of an optimal execution problem with MI.

To construct a model, the discrete-time model is significant to describe realistic phenomena exactly, but sometimes it is hard to get a the clean model by complex noises. On the other hand the continuous-time model often makes problems clear, but the superficial construction of continuous-time models may overlook the essence of the problems. So first we consider a discrete-time model of an optimal execution problem with MI and then derive a continuous-time model as their limit. In this paper we introduce only the notation of the continuous-time model except the definition of MI function. For concrete procedure of this is given in [19], [20] and the generalized model are treated in [16] and [17]. Please refer to them for details.

First we define the MI function in discrete-time model. We consider a case where a trader tries to sell ψ shares of a security and where the security price before the transaction is equal to

s. Then the security price changes to $se^{-g_n(\psi)}$ and he/she gets cash amount of $\psi se^{-g_n(\psi)}$ as the proceeds of his/her execution after the price down, where the function $g_n(\psi)$ is non-decreasing, continuously differentiable and satisfies $g_n(0) = 0$. $g_n(\psi)$ means the decrease of the log-price of the security by the trader's execution. We call g_n the (one-shot) MI function in the discrete-time model.

To derive a continuous-time model, we need some technical condition for g_n. Let $h : [0,\infty) \longrightarrow [0,\infty)$ be a non-decreasing continuous function. We introduce the following condition.

[A] $\lim_{n\to\infty} \sup_{\psi\in[0,\Phi_0]} \left|\frac{d}{d\psi}g_n(\psi) - h(n\psi)\right| = 0.$

Let $g(\zeta) = \int_0^\zeta h(\zeta')d\zeta'$ for $\zeta \in [0,\infty)$. Then the function g is the limit of g_n in the following sense

$$\sup_{\psi\in(0,\Phi_0]} \left|\frac{g_n(\psi)}{\psi} - \frac{g(n\psi)}{n\psi}\right| \longrightarrow 0, \quad n \to \infty.$$

Then we can regard g as the MI function in the continuous-time model.

Now we define the value function in the continuous-time model. Let $b, \sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be Lipschitz continuous and linear growth functions. For $t \in [0,1]$ and $\varphi \in [0,\Phi_0]$ we denote by $\mathcal{A}_t^{\mathrm{SO}}(\varphi)$ the set of $(\mathcal{F}_r)_{0\le r\le t}$-progressively measurable process $(\zeta_r)_{0\le r\le t}$ such that $\zeta_r \ge 0$ for each $r \in [0,t]$, $\int_0^t \zeta_r dr = \varphi$ almost surely and $\sup_{r,\omega}\zeta_r(\omega) < \infty$. For $t \in [0,1], (w,\varphi,s) \in \mathbb{R}\times[0,\Phi_0]\times(0,\infty)$ and the continuous, non-decreasing and polynomial growth function $U : \mathbb{R} \longrightarrow \mathbb{R}$, we define $V_t(w,\varphi,s;U)$ by

$$V_t(w,\varphi,s;U) = \sup_{(\zeta_r)_r\in\mathcal{A}_t^{\mathrm{SO}}(\varphi)} \mathrm{E}[U(W_t)] \tag{3.1}$$

subject to

$$dW_r = \zeta_r S_r dr, \quad dS_r = \hat{\sigma}(S_r)dB_r + \hat{b}(S_r)dr - g(\zeta_r)S_r dr \tag{3.2}$$

and $(W_0, S_0) = (w, s)$, where $\hat{\sigma}(s) = s\sigma(\log s), \hat{b}(s) = s\Big\{b(\log s) + \frac{1}{2}\sigma(\log s)^2\Big\}$ for $s > 0$ and $\hat{\sigma}(0) = \hat{b}(0) = 0$. When $s > 0$, we obviously see that the process $X_r = \log S_r$ is well-defined and satisfies

$$dX_r = \sigma(X_r)dB_r + b(X_r)dr - g(\zeta_r)dr.$$

We remark that $V_0(w,\varphi,s;U) = U(w)$.

Here we interpret the definition of the function $V_t(w,\varphi,s;U)$. The controlled processes $(W_r)_r$ and $(S_r)_r$ correspond to the amount of cash holdings and the security price, respectively. The value of ζ_r means the instant sales (or the execution speed) at time r. We call this process $(\zeta_r)_r \in \mathcal{A}_t^{\mathrm{SO}}(\varphi)$ an (admissible) execution strategy for a trader who has φ shares of the security. The notation SO stands for "sell-out." We consider a case where a trader must sell up all the shares of the security φ until the time horizon. This condition is expressed like $\varphi_t = 0$, where $\varphi_r = \varphi - \int_0^r \zeta_v dv$ is the process of the amount of security holdings.

The (continuous-time) value function $V_t(w, \varphi, s; U)$ means the maximum of expected utility of terminal cash holdings. The function U means the trader's utility function of cash holdings W_t. In the related papers [16], [17], [19] and [20], the utility function also depends on the security price, but this paper treats only the case where U is independent of it.

The function $V_t(w, \varphi, s; U)$ corresponds to an optimal execution problem with MI in the continuous-time model, and in fact it is derived as the limit of the value functions in the discrete-time model. The arguments of the limit transition and the properties of value functions are in [19] and [20]. These papers assume that the functions b and σ are bounded. But we can weaken this assumptions such that b, σ, $\hat{b}$ and $\hat{\sigma}$ are all linear growth. This arguments are treated in [21]. Anyway our optimization problem is given as the following stochastic control problem.

$$\begin{aligned} V_1(0, \Phi_0, s_0; U) &= \sup_{(\zeta_r)_r \in \mathcal{A}_1^{\mathrm{SO}}(\Phi_0)} \mathrm{E}[U(W_1)] \\ &\text{s.t. } S_t = s_0 + \int_0^t \hat{\sigma}(S_r) dB_r + \int_0^t \hat{b}(S_r) dr - \int_0^t g(\zeta_r) S_r dr, \quad W_1 = \int_0^1 \zeta_r S_r dr. \end{aligned}$$

We fix s_0 and simply denote $\hat{V}(\Phi_0) = V_1(0, \Phi_0, s_0; U)$ especially when $U(w) = w$.

4 Block Liquidation v.s. Gradual Liquidation

In this section we give the case studies of the effect of MI through some examples. As mentioned below, an optimal strategy in our benchmark model is a block liquidation. But in the trading operations a trader sells a security gradually to avoid the MI cost. So it is meaningful to give the factors which invite the gradual liquidation theoretically.

4.1 Benchmark Model : Black-Scholes model without MI

First we consider the ideal market model without MI (i.e. $g(\zeta) \equiv 0$) as our benchmark. As in the Merton problem, we assume that the security price follows the geometric Brownian motion (2.2). We also assume that $\tilde{\mu} > 0$, where $\tilde{\mu} = -\mu - \sigma^2/2 > 0$. So the security price process has negative drift and the expected price decreases until the time horizon.

The Merton theory assumes that a trader is risk-averse and has the log-utility function (or the power utility function.) But we consider a risk-neutral trader in our benchmark model, because it is not so easy to consider an optimization problem for a risk-averse trader and we because can describe the very effects of MI when considering the risk-neutral trader. We set the utility function as $U(w) = w$.

Apparently, the optimal strategy for a trader in this case is the initial block liquidation, that is, he/she should sell up all the shares Φ_0 immediately. The corresponding execution strategy cannot be described in our settings straightforwardly, but we can construct the nearly optimal strategy $(\zeta_r^\delta)_r$ as

$$\zeta_r^\delta = \frac{\Phi_0}{\delta} 1_{[0,\delta]}(r). \tag{4.1}$$

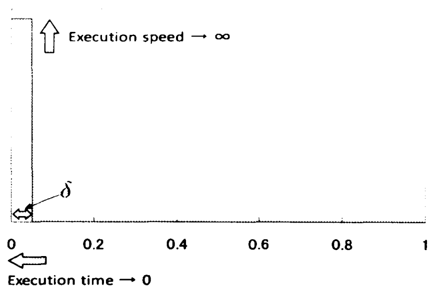

Figure 2: The execution strategy of almost block liquidation $(\zeta_r^\delta)_r$. The horizontal axis is the time r.

Figure 2 gives the image of this strategy. This means that a trader sells Φ_0 shares of the security by dividing infinitely in infinitely short time near the initial time. We call such a strategy "an (initial) almost block liquidation."

By executing in this way, a trader gets cash amount of $\Phi_0 s_0$. This is the ideal proceeds when there is no MI.

4.2 Black-Scholes model with linear MI

Now we consider the case with MI. The fluctuation of the security price is the same as in Section 4.1. So the security price is given as

$$S_t = s_0 \exp\left(-\mu t + \sigma B_t - \alpha \int_0^t g(\zeta_r)dr\right). \tag{4.2}$$

In this section we treat the linear MI function $g(\zeta) = \alpha\zeta$ for some constant $\alpha > 0$. We also set $U(w) = w$. Then we have the following theorem.

Theorem 2 *It holds that* $\hat{V}(\Phi_0) = \dfrac{1 - e^{-\alpha\Phi_0}}{\alpha} s_0$ *and a nearly optimal strategy is given as* (4.1).

The proof is in [19]. This theorem implies that the optimal strategy in this case is (almost) the same as our benchmark model. So, although the proceeds decreases by the effect of MI, we cannot see the influential effect in this example and linear MI does not cause gradual liquidation.

4.3 Black-Scholes model with quadratic MI

Next we consider the strictly convex MI function. The fluctuation of the security price is also given by (4.2) and the utility function is $U(w) = w$. For MI function, we give the quadratic form $g(\zeta) = \alpha\zeta^2$ for some constant $\alpha > 0$.

Then Theorem 7 and Theorem 9 in [19] imply the following theorem.

Theorem 3
(i) *If* $\dfrac{\operatorname{arctanh}\sqrt{1 - e^{-2\tilde{\mu}}}}{\sqrt{\alpha\tilde{\mu}}} \le \Phi_0$, *then* $\hat{V}(\Phi_0) = \dfrac{s_0\sqrt{1 - e^{-2\tilde{\mu}}}}{2\sqrt{\alpha\tilde{\mu}}}$ *and the nearly optimal strategy is*

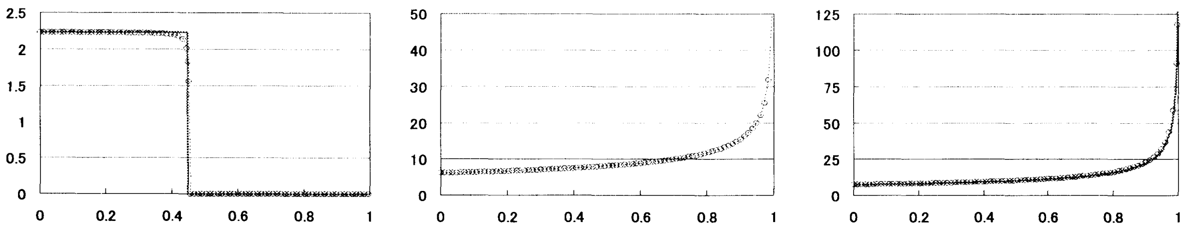

Figure 3: The forms of optimal execution strategies $(\zeta_r)_r$. Circle-marked dotted line means the numerical solution, and solid line means the analytical solution. Horizontal axis is the time r. The left graph : $\Phi_0 = 1$. The middle graph : $\Phi_0 = 10$. The right graph : $\Phi_0 = 100$.

given by

$$\zeta_r^\delta = \sqrt{\frac{\tilde{\mu}}{\alpha(1-e^{-2\tilde{\mu}(1-r)})}} 1_{[0,1-\delta]}(r) + \frac{\Phi_0 - \varphi_{1-\delta}^*}{\delta} 1_{[1-\delta,1]}(r),$$

where $\varphi_{1-\delta}^* = \dfrac{\operatorname{arctanh}\sqrt{1-e^{-2\tilde{\mu}\delta}} - \operatorname{arctanh}\sqrt{1-e^{-2\tilde{\mu}}}}{\sqrt{\alpha\tilde{\mu}}}$.

(ii) *If* $\Phi_0 \leq \sqrt{\dfrac{\tilde{\mu}}{\alpha}}$, *then* $\hat{V}(\Phi_0) = \dfrac{s_0}{2\sqrt{\alpha\tilde{\mu}}}(1 - e^{-2\sqrt{\alpha\tilde{\mu}}\Phi_0})$ *and the optimal strategy is given by* $\zeta_r = \sqrt{\dfrac{\tilde{\mu}}{\alpha}} 1_{[0,\Phi_0\sqrt{\alpha/\tilde{\mu}}]}(r)$.

This theorem implies that the form of optimal strategies and value functions vary according to the amount of security holdings Φ_0. If a trader has a little amount of securities, then we have the case (ii) and the optimal strategy is to sell up the entire shares of the security until the time $\Phi_0\sqrt{\alpha/\tilde{\mu}}$. If he/she has large amount, then we have the case (i). Figure 3 (respectively, Figure 4) shows the forms of the optimal strategies (respectively, the processes of the amount of the security holdings) with parameters $s_0 = 1, \alpha = 0.01, \tilde{\mu} = 0.05$ and $\Phi_0 = 1, 10, 100$. When $\Phi_0 = 10$, we cannot apply Theorem 3. But we can also solve the optimization problem numerically. These results show that forms of the solutions entirely change according to the amount of the initial security holdings Φ_0. We also remark that in these cases the optimal strategies are gradual liquidation (and terminal almost block liquidation.) So we can conclude that strictly convex MI causes gradual liquidation.

4.4 Geometric OU model with linear MI

We consider the case of linear MI again. And we introduce the price recovery effect. For details of these discussions in this section, please refer to [21].

[4], [14] and other papers pointed out that MI is decomposed into permanent and temporary impact. To simply describe the phenomenon of temporary impact, we assume that the process of the security price is mean-reverting. More concretely, we set $b(x) = \beta(F - x)$ and $\sigma(x) \equiv \sigma$ for some $\beta, \sigma \geq 0$ and $F \in \mathbb{R}$. Then we can write off the explicit form of $(S_r)_r$ as

$$S_r = \exp\left(e^{-\beta r}\log s_0 + (1-e^{-\beta r})F - e^{-\beta r}\int_0^r e^{\beta v} g(\zeta_v)dv + \sigma e^{-\beta r}\int_0^r e^{\beta v} dB_v\right).$$

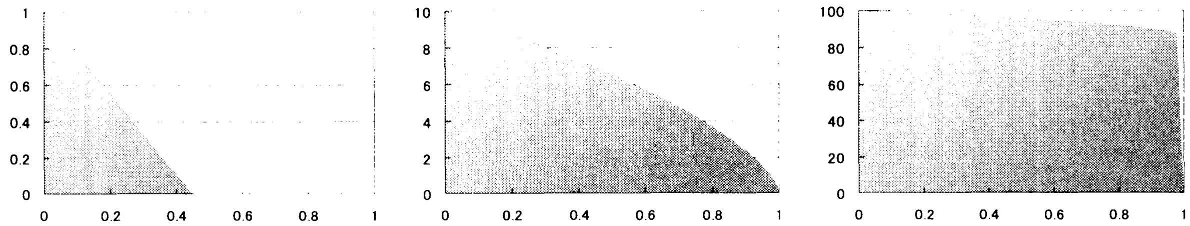

Figure 4: The forms of the amount of security holdings $(\varphi_r)_r$ corresponding with optimal strategies. Horizontal axis is the time r. The left graph : $\Phi_0 = 1$. The middle graph : $\Phi_0 = 10$. The right graph : $\Phi_0 = 100$.

We notice that when there is no MI, the log-price of the security $X_r = \log S_r$ follows an Ornstein-Uhlenbeck process and the optimal strategy is also the initial block liquidation under an additional assumption like $s_0 > e^{F+\sigma^2/2}$. We set $g(\zeta) = \alpha\zeta$.

First we consider the simple case of $\sigma = 0$. Let $z = \log s_0 - F$. We define the function $C(x)$, $x \in \mathbb{R}$, by

$$C(x) = \exp(\alpha(t\beta + 1)x - \alpha\varphi - \beta z) + \alpha x - z - 1.$$

$C(x)$ is strictly increasing and $C(\pm\infty) = \pm\infty$. To make a situation simple we assume $z > 0$. Then the equation $C(x) = 0$ has the positive unique solution. We have the following theorem.

Theorem 4 *Assume $\varphi > z/\alpha$. Then it holds that*

$$\hat{V}(\Phi_0) = \frac{1 - e^{-\alpha(p^* + q^*)}}{\alpha} s_0 + s_0 e^{-\alpha p^*} \zeta^*,$$

where p^* is the solution of $C(p^*) = 0$,

$$\zeta^* = \beta(p^* - z/\alpha), \quad q^* = \Phi_0 - p^* - t\zeta^*.$$

We notice that the assumption $\varphi > z/\alpha > 0$ implies $0 < p^*, \zeta^*, q^* < \varphi$ and the nearly optimal strategy is given as

$$\zeta_r^\delta = \frac{p^*}{\delta} 1_{[0,\delta]}(r) + \zeta^* + \frac{q^*}{\delta} 1_{[1-\delta,1]}(r).$$

This strategy consists of three terms. The first term in the right-hand side corresponds to "initial (almost) block liquidation." A trader should sell p^* shares of the security at the initial time by dividing infinitely to avoid to decrease proceeds. The second term means "gradual liquidation." A trader execute the selling at the same speed ζ^* until the time horizon. By doing so, a trader can keep the security price constant. Then the trader completes the liquidation by selling the rest shares by "terminal (almost) block liquidation" as the third term. So the (nearly) optimal strategy is a mixture of both block liquidation and gradual liquidation, and we'd like to point out especially that the gradual liquidation is also necessary in this case. Figure 5 expresses the image of this strategy.

This result is quite similar to [1] and [29], despite the fact that there is a little difference between their model and ours. We consider the geometric OU process for a security price and MI is linear. On the other hand [1] and [29] assumed that the process of a security price follows arithmetic Brownian motion (or a martingale) and there is an exponential resilience for MI. The relation between mean-reverting property of OU process and exponential resilience induce the similarity of results.

Next we consider the general case of $\sigma \geq 0$. In this case we can also get the explicit solution of the optimization problem (but forms of the solutions are more complicated.) First we define the function $H(\lambda)$ on $[0, \infty)$ by

$$H(\lambda) = \alpha \exp\left(\alpha\beta \int_0^1 P^{-1}\left(\exp(e^{-2\beta r}y)\lambda/\alpha\right) dr - \alpha\Phi_0 + z - y(1 - e^{-2\beta})\right) - \lambda,$$

where $z = \log s_0 - F > 0$, $y = \sigma^2/(4\beta)$, $P(x) = e^{-\alpha x}(1 - \alpha x)$ and P^{-1} is its inverse function. We remark that P^{-1} is well-defined and strictly decreasin on $[0, \infty]$. Moreover we assume the following condition

$$\Phi_0 > \frac{1}{\alpha}(1 + \beta + y + z). \tag{4.3}$$

We see that H is non-increasing on $[0, \infty)$ and (4.3) implies

$$H\left(\alpha \exp\left(-e^{-2\beta}y\right)\right) < 0 < H(0).$$

Then the equation $H(x) = 0$ has a unique solution $\lambda^* = \lambda^*(t, \varphi) \in \left(0, \alpha \exp\left(-e^{-2\beta}y\right)\right)$.

Theorem 5 *Assume* (4.3). *Then it holds that*

$$\hat{V}(\Phi_0) = \frac{s_0}{\alpha}\left(1 - \exp\left(-\alpha\Phi_0 + \alpha\beta\int_0^1 \xi_r^* dr - y(1 + e^{-2\beta})\right)\right) + \beta\int_0^1 (\xi_r^* + \frac{2y}{\alpha}e^{-2\beta r})e^{F - \alpha\xi_r^*} dr,$$

where $\xi_r^* = P^{-1}(\exp(e^{-2\beta r}y)\lambda^*/\alpha)$.

In this case a nearly optimal strategy is expressed like

$$\zeta_r^\delta = \frac{p^*}{\delta}1_{[0,\delta]}(r) + \zeta_r^* + \frac{q^*}{\delta}1_{[1-\delta,1]}(r), \tag{4.4}$$

where $p^* = \xi_0^* + z/\alpha$ and

$$\begin{aligned} \zeta_r^* &= \beta\xi_r^* - \frac{2\beta\lambda^* e^{-2\beta r} y \exp(\alpha\xi_r^*)}{\alpha^2(\alpha\xi_r^* - 2)} + \frac{2\beta y}{\alpha}e^{-2\beta r}, \\ q^* &= \Phi_0 - \beta\int_0^1 \xi_r^* dr - \xi_1^* - \frac{z}{\alpha} + \frac{y}{\alpha}(1 + e^{-2\beta}). \end{aligned}$$

Figure 6 is the image of this strategy. Both Theorem 4 and Theorem 5 imply that the (nearly) optimal strategy is given as "initial almost block liquidation" plus "gradual liquidation" plus "terminal almost block liquidation." So we can find that the price recovery effect causes gradual liquidation.

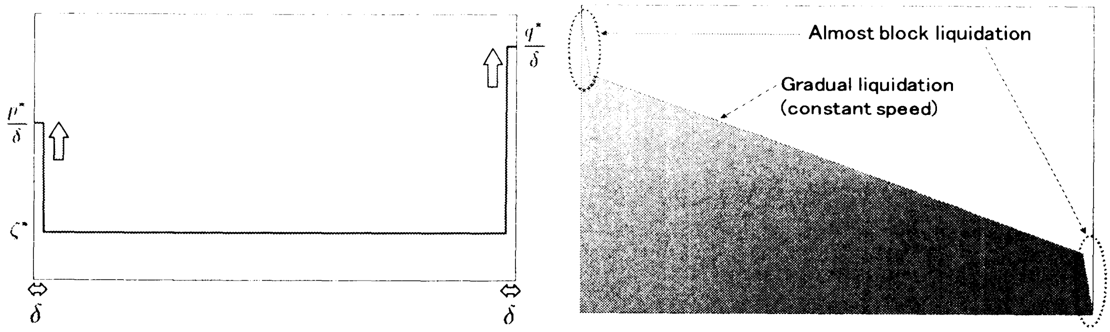

Figure 5: The forms of a nearly optimal strategy $(\zeta_r^\delta)_r$ (the left graph) and the corresponding process of the amount of the security holdings (the right graph) when $\sigma = 0$. Horizontal axis is the time r.

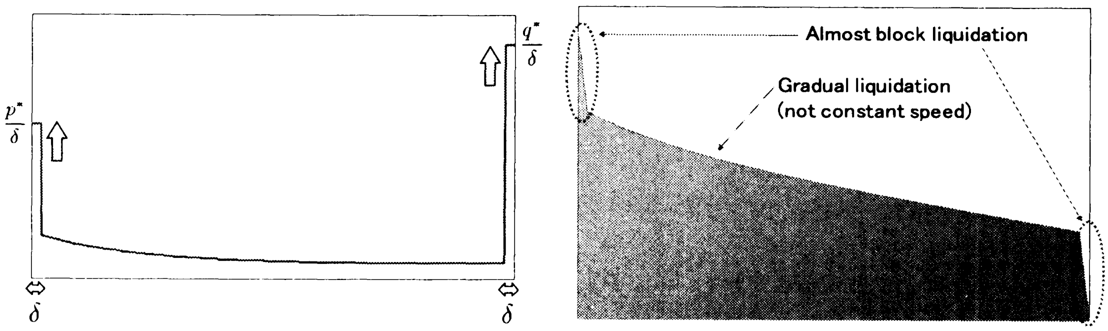

Figure 6: The forms of a nearly optimal strategy $(\zeta_r^\delta)_r$ (the left graph) and the corresponding process of the amount of the security holdings (the right graph) when $\sigma > 0$. Horizontal axis is the time r.

4.5 Risk-Averse Utility with Linear MI

In this section we present some preceding studies of an optimal execution problem of a risk-averse trader. [33] studied such a problem with linear MI when a security price follows an arithmetic Brownian motion and investigated the relations between the measure of absolute risk aversion and the form of an optimal strategy. In particular, they derived the explicit form of the optimal strategy when the utility function is CARA type. [13] studied the case where a trader has CRRA utility function and a security price follows a geometric Brownian motion. They formulated the optimal execution problem with linear MI as a singular control problem and calculated the solution numerically. In each cases the optimal strategy becomes gradual liquidation. So we see that the risk-averse utility causes gradual liquidation, and not always by MI.

5 Measurement of MI Costs

Until now we have studied how to execute Φ_0 shares of a security by considering MI. Next question is "how much is the MI cost of the execution?" One of the possible method of measurement of it is to compare the decreased proceeds with the ideal proceeds without MI. By Section 4.1, the ideal proceeds is equal to $\Phi_0 s_0$ when the expected price of the security goes down as time passes. So, in the spirit of implementation shortfall method ([31]), we define the total MI cost of risk-neutral trader by

$$\mathrm{MI}(\Phi_0) = -\log(\hat{V}(\Phi_0)/(s_0\Phi_0)).$$

Using this formulation, the decreased proceeds is expressed like $\hat{V}(\Phi_0) = \mathrm{E}[W_1^*] = s_0\Phi_0 e^{-\mathrm{MI}(\Phi_0)}$, where $(W_r^*)_r$ is a process of cash holdings induced by the optimal execution strategy. Comparing to this, we call the function g_n a one-shot MI function. This is because the decreased proceeds by a single order of selling Φ_0 shares is equal to $s_0\Phi_0 e^{-g_n(\Phi_0)}$ and the value of $g_n(\Phi_0)$ is its cost.

Now we draw the form of the total MI function $\mathrm{MI}(\Phi_0)$ via execution volumes Φ_0 in our examples. The comparison of the forms of total MI functions are found in Figure 7.

First we consider the case of Section 4.2. In this case we have $\mathrm{MI}(\Phi_0) = -\log\dfrac{1-e^{-\alpha\Phi_0}}{\alpha\Phi_0}$. This is approximated by $(\alpha/2)\Phi_0$ when α and Φ_0 are not so large. So the total MI cost forms almost linear with respect to Φ_0.

Section 4.3 gives us interesting findings. Although one-shot MI function is a quadratic function and increasing rapidly according to the increase of Φ_0, the total MI cost function forms S-shape. This means that we can control the total MI cost by appropriate gradual liquidation.

Moreover we can also control the total MI cost function in the case of Section 4.4 by the mixture of (almost) block liquidation and gradual liquidation and the form of the total MI cost function is concave.

These investigations suggest that the concavity (or S-shape) of MI function is theoretically brought by an appropriate execution strategy even if MI function of a single trade is not concave. These results are consistent with the findings of several empirical studies (for instance, [5] and [14]). And these also claim that to select an adequate execution policy is significant to control MI cost.

6 Concluding Remarks

In this paper we studied effects of MI to the execution strategy of a trader. We focused on the cases where a trader needs gradual liquidation to avoid MI cost and found that strict convexity of (one-shot) MI function, price recovery effect of a security price (and risk-averse utility function) cause gradual liquidation. These are theoretical findings from the view of mathematical finance, suggesting the appropriate formulation of MI models to satisfy these conditions.

What conditions are necessary for constructing mathematical models of an optimal execution? One of them is the property of "no-price manipulation" (see [2], [3], [12] and [15].) This is similar and deeply corresponding to the concept of arbitrage, which is very significant in the

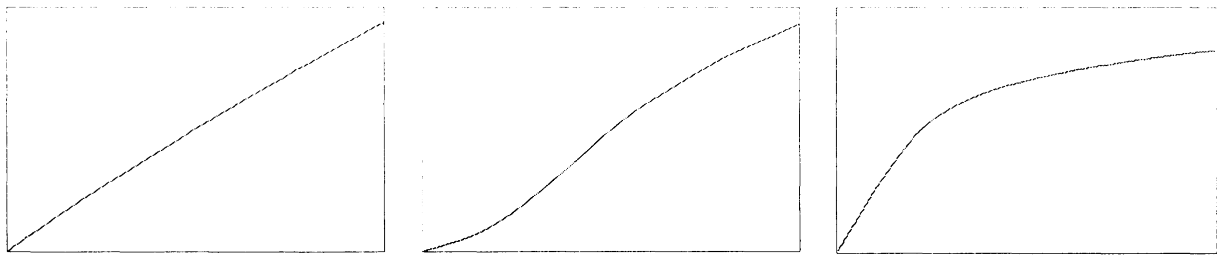

Figure 7: The forms of total MI cost function MI(Φ_0). Horizontal axis is the initial security holdings Φ_0. The left graph corresponds to Section 4.2, the middle graph to Section 4.3, and the right graph to 4.4.

theory of mathematical finance. Price manipulation does not occur in the proper model. Our model assumes that transactions of a trader are only to sell, so we do not challenge the problem of price manipulation. It will be important to consider such a problem also in our model.

We need (linearity or) convexity of one-shot MI functions to derive the continuous-time model and to get characteristic results, but we do not know a proper shape of it. To investigate this from the view of an empirical study is another future subject.

It is also to be developed further to generalize our model to treat time-varying MI and bid-ask spread, which cannot be ignored in trading operations.

Standard theoretical model of an optimal execution problem from the view of mathematical finance have not been established yet. Thus to construct a mathematical model theoretically is the significant task and it is meaningful to describe phenomena found in trading operations like gradual liquidation.

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