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AN OPTIMIZATION PROBLEM FOR A PRODUCTION SYSTEM WITH REAL OPTION APPROACH

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ABSTRACT. In this paper, we consider an optimization problem for a production system in consideration of the uncertainty of demand change, applying the real option called *option to transfer*. Furthermore, we suggest a risk minimization model for a production system using value-at-risk and conditional value-at-risk, and analyze the sensitivity of the model.

1. INTRODUCTION

In a production system on the manufacturing industry, the manufacturers make a decision about quantities of production considering demand and supply. In general, it is hard to estimate accurately the uncertainty of demand change. As an issue of a production system according to the uncertainty of demand change, it can be occurred the trade-off between excess inventory and chance loss. So, it is important for the manufacturers to evaluate a production project to decide quantities of products considering the balance between excess inventory and chance loss.

Recently, the real option valuation (ROV, for short) method is paid attention as one of effective valuation methods for a project. For example, L. E. Brandao and J. S. Dyer analyze about a decision making in discrete time with the ROV method in [2]. H. T. J. Smit and L. A. Ankum consider about real option with game-theoretic approach under competition in [7]. Robert S. Pindyck considers about irreversibility, uncertainty, and investment with the ROV method in [5].

In this paper, we consider an optimization problem for a production system with real option approach. Furthermore, we suggest a risk minimization model using value-at-risk (VaR, for short) and conditional VaR (CVaR, for short) as downside risk measure. R. T. Rockafellar and S. Uryasev consider about optimization of CVaR and its characteresitics, and introduce some examples about CVaR minimization in [6]. J. Gotoh and Y. Takano analyze about a single-period news vendor problem with CVaR, and suggest some Mean-CVaR models in [4].

The structure of this paper is as follows. In Section 2, we introduce about two valuation methods, namely, the net present value (NPV, for short) method and the ROV method. In Section 3, we suggest a risk minimization model for a production system by minimizing CVaR. In this section, we first denote the notation which using for the model, and define the expected cash flow and the NPV. Then, we introduce the calculation method of the call-option value by the binomial lattice model. Furthermore, we define β-VaR and β-CVaR, and refer to two theorems about them. Finally, we construct the CVaR minimization model for a production system applying the ROV method. In Section 4, we analyze the sensitivity of the model.

2. Valuation Methods for a Project

First of all, we introduce two valuation methods for a project, namely, the NPV method and the ROV method. The NPV method estimates a value of a project by calculating the NPV. If a value of the NPV is positive, the project is adopted. On the other hand, if negative, the project is rejected. Here, let $t,\ t = 1, 2, \ldots, T$ be a period of production, let CF_t be the expected cash flow in period t, let r be the discount factor, and let I_0 be an initial investment cost of a project. Then, the NPV is obtained by the following formula:

$$NPV = \frac{CF_1}{1+r} + \frac{CF_2}{(1+r)^2} + \cdots + \frac{CF_T}{(1+r)^T} - I_0 = \sum_{t=1}^{T} PV_t - I_0, \tag{1}$$

where PV_t is the present value of a project in period t, it is defined as follows:

$$PV_t := \frac{CF_t}{(1+r)^t}, \quad t = 1, 2, \ldots, T. \tag{2}$$

In addition, the calculation of the NPV is described as Figure 1.

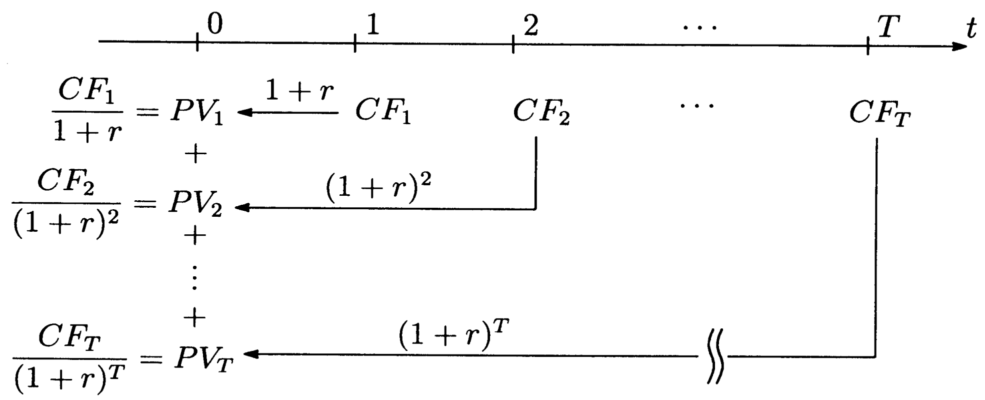

Figure 1: Calculation of the NPV

On the other hand, the ROV method is a valuation method applied the financial option theory, which estimates a value of a profect about a decision making under the uncertainty in business. The ROV method is comparatively superior to the NPV method in a point of view that the ROV method can be considered flexibility of a decision making about postponement, expansion, contraction, and so on.

3. Risk Minimization Model for a Production System

In this paper, we suggest a risk minimization model for a production system applying the ROV method, and we use the option called *option to transfer* to the model. In [12], *option to transfer* has some characterestics as follows. Underlying asset price is the present value of the expected cash flow increasing by transfer, strike price is total cost of transfer, and type of option is call-option. By using *option to transfer*, we consider about decision of optimal quantities of production dealing with the uncertainty of demand change. Furthermore, we refer to VaR and CVaR as downside risk measure, and construct a risk minimization model using them.

3.1. **Notation.** We first show below the notation which using for the model.

i : index for a product $(i = 1, \dots, m)$;
t : index for a period $(t = 1, \dots, T)$;
x_{it} : quantity of production for a product i in period t (decision variable) $(i = 1, \dots, m;\ t = 1, \dots, T)$;
w_{it} : quantity of transfer for a product i in period t (decision variable) $(i = 1, \dots, m;\ t = 1, \dots, T)$;
y_{it} : quantity of inventory for a product i in period t $(i = 1, \dots, m;\ t = 1, \dots, T)$;
ζ_{it} : demand quantity for a product i in period t (random variable) $(i = 1, \dots, m;\ t = 1, \dots, T)$;
c_i : production cost per unit for a product i $(i = 1, \dots, m)$;
p_i : selling price per unit for a product i $(i = 1, \dots, m)$;
h_i : holding cost of inventory per unit for a product i $(i = 1, \dots, m)$;
s_i : shortage penalty per unit for a product i $(i = 1, \dots, m)$;
I_0 : initial investment cost of a project;
I_{it} : transfer cost for a product i in period t $(i = 1, \dots, m;\ t = 1, \dots, T)$;
I_t : transfer cost in period t, $(t = 1, \dots, T)$;
I : total transfer cost;
r : discount factor;
u : up-rate for an underlying asset price;
d : down-rate for an underlying asset price;
β : confidence level, $\beta \in (0, 1)$.

Here, transfer cost in period t is defined as follows:

$$I_t := w_{1t}I_{1t} + \cdots + w_{mt}I_{mt} = \sum_{i=1}^{m} w_{it}I_{it}, \qquad t = 1, 2, \dots, T. \tag{3}$$

Then, total transfer cost I is obtained by

$$I = I_1 + I_2 + \cdots + I_T = \sum_{t=1}^{T} I_t. \tag{4}$$

3.2. **Expected Cash Flow and NPV.** Let a function CF_{it} from $\boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}$ into $\boldsymbol{R}$ be the expected cash flow for a product i in period t, which is defined by the following formula:

$$(5) \quad CF_{it}(x_{it}, w_{it}, \zeta_{it}) = \\ p_i \cdot \min\{x_{it} + w_{it}, \zeta_{it}\} - c_i(x_{it} + w_{it}) - h_i \cdot \max\{y_{it}, 0\} + s_i \cdot \min\{y_{it}, 0\}, \\ i = 1, 2, \ldots, m, \quad t = 1, 2, \ldots, T,$$

where $\boldsymbol{R}$ is a real space. Let a function CF_t from $\boldsymbol{R}^n \times \boldsymbol{R}^n \times \boldsymbol{R}^n$ into $\boldsymbol{R}$ be the total expected cash flow in period t, and it is defined by the following formula:

$$(6) \qquad CF_t(x_t, w_t, \zeta_t) := \sum_{i=1}^{m} CF_{it}(x_{it}, w_{it}, \zeta_{it}), \quad t = 1, 2, \ldots, T,$$

where $x_t = (x_{1t}, \ldots, x_{mt})^\top$, $w_t = (w_{1t}, \ldots, w_{mt})^\top$, $\zeta_t = (\zeta_{1t}, \ldots, \zeta_{mt})^\top$, and $\boldsymbol{R}^n$ is a real n-dimensional Euclidean space. Then, the NPV is defined as V by the following formula:

$$(7) \qquad V := \sum_{t=1}^{T} \frac{CF_t(x_t, w_t, \zeta_t)}{(1+r)^t} - I_0.$$

3.3. **Option Value by the Binomial Lattice Model.** First, let S be a underlying asset price and let X be a strike price. Then, the call-option value in the maturity P is given by $P := \max\{S - X, 0\}$, and we calculate the call-option value within all periods by the call-option pricing formula [11]. Let r be the risk-free rate. And we assume that $R := 1 + r$ and $u > R > d > 0$. If the risk-neutral probability q is given by $q := (R-d)/(u-d)$, then the call-option value C denoted by the binomial lattice model is given as follows:

$$(8) \qquad C = \frac{1}{R}\{qC_u + (1-q)C_d\},$$

where C_u and C_d are the call-option values, C_u is the value when a price of underlying asset rises, and C_d is the value when a price of underlying asset falls. For example, the binomial lattice in two-periods is shown as Figure 2.

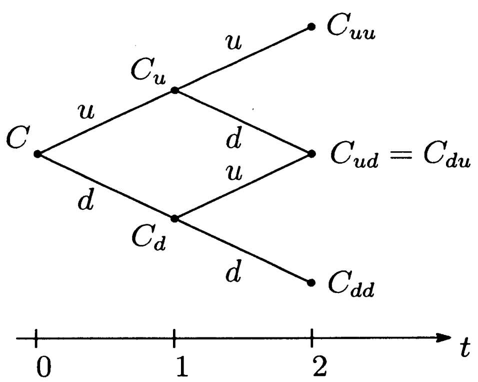

Figure 2: Binomial lattice in two-periods

In Figure 2, the call-option values in the last nodes are calculated as follows:

$$C_{uu} = \max\{u^2S - X, 0\}, \tag{9}$$

$$C_{ud} = C_{du} = \max\{udS - X, 0\}, \tag{10}$$

$$C_{dd} = \max\{d^2S - X, 0\}. \tag{11}$$

Then, by the formula (8),

$$C_u = \frac{1}{R}\{qC_{uu} + (1-q)C_{ud}\}, \tag{12}$$

$$C_d = \frac{1}{R}\{qC_{ud} + (1-q)C_{dd}\}. \tag{13}$$

Therefore, we finally obtain the call-option value C as follows:

$$\begin{aligned} C &= \frac{1}{R}\{qC_u + (1-q)C_d\} \\ &= \frac{1}{R^2}\left\{q^2C_{uu} + 2q(1-q)C_{ud} + (1-q)^2C_{dd}\right\}. \end{aligned} \tag{14}$$

Here, we assign numbers k, $k = 1, \ldots, t$ to the last nodes in the binomial lattice. Then, the call-option value in the first period is calculated by the following formula going back from $t = T$ to $t = 1$:

$$C_{Tk} = \max\left\{u^{(T-k)}d^{(k-1)}V - I, 0\right\}, \quad t = T, \quad k = 1, 2, \ldots, T, \tag{15}$$

$$\begin{aligned} C_{tk} = \frac{1}{R}\{qC_{t+1,k} + (1-q)C_{t+1,k+1}\}, \\ t = 1, 2, \ldots, T-1, \quad k = 1, 2, \ldots, t. \end{aligned} \tag{16}$$

We ragard $L := -C_{11}$ as the loss in a production system, and suggest a risk minimization model using CVaR in the next subsection.

3.4. **CVaR Minimization.** We refer to definitions and theorems about VaR and CVaR. The β-VaR and β-CVaR will be denoted by $\alpha_\beta(x)$ and $\phi_\beta(x)$.

Definition 1 (β-VaR). *Let X be a certain subset of $\boldsymbol{R}^n$ and let $\beta \in (0,1)$ be the confidence level. Then, for all $x \in X$ and $\alpha \in \boldsymbol{R}$, β-VaR is defined as follows:*

$$\alpha_\beta(x) := \min\{\alpha : \Phi(x, \alpha) \geq \beta\}, \tag{17}$$

where a function Φ from $X \times \boldsymbol{R}$ into $(0,1)$ is a continuous cumulative distribution function for $x \in X$.

Definition 2 (β-CVaR). *Let X be a certain subset of $\boldsymbol{R}^n$ and let $\beta \in (0,1)$ be a confidence level. Let a function f from $X \times \boldsymbol{R}^n$ into $\boldsymbol{R}$ be a certain function for $x \in X$ and $y \in \boldsymbol{R}^n$, and let a function p from $\boldsymbol{R}^n$ into $\boldsymbol{R}$ be a continuous probability density function. Then, β-CVaR is defined as follows:*

$$\phi_\beta(x) := \frac{1}{1-\beta}\int_{f(x,y)\geq\alpha_\beta(x)} f(x,y)p(y)dy. \tag{18}$$

Here, we give a function F_β from $X \times \boldsymbol{R}$ into $\boldsymbol{R}$ defined by

$$F_\beta(x,\alpha) := \alpha + \frac{1}{1-\beta}\int_{y\in\boldsymbol{R}^n} [f(x,y)-\alpha]^+ p(y)dy, \tag{19}$$

where $[\cdot]^+ := \max\{\cdot, 0\}$. Then, according to R. T. Rockafellar and S. Uryasev, two theorems about $F_\beta(x,\alpha)$ and $\phi_\beta(x)$ hold.

Theorem 1 (R. T. Rockafellar and S. Uryasev, 2000 [6]). *$F_\beta(x,\alpha)$ is convex and continuously differentiable with respect to α. Furthermore, the following formula holds:*

$$\phi_\beta(x) = \min_{\alpha\in\boldsymbol{R}} F_\beta(x,\alpha). \tag{20}$$

In this formula, the set consisting of the values of α, i.e.,

$$A_\beta(x) = \operatorname*{argmin}_{\alpha\in\boldsymbol{R}} F_\beta(x,\alpha) \tag{21}$$

is a nonempty, closed, and bounded interval.

Theorem 2 (R. T. Rockafellar and S. Uryasev, 2000 [6]). *Minimizing β-*CVaR *with $x\in X$ is equivalent to minimizing $F_\beta(x,\alpha)$ with $(x,\alpha)\in X\times\boldsymbol{R}$, i.e.,*

$$\min_{x\in X}\phi_\beta(x) = \min_{(x,\alpha)\in X\times\boldsymbol{R}} F_\beta(x,\alpha). \tag{22}$$

We consider the approximation function $\tilde{F}_\beta(x,\alpha)$ for $F_\beta(x,\alpha)$ obtained by sampling from the probability distribution in ζ, i.e.,

$$\tilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{(1-\beta)n}\sum_{j=1}^{n}[f(x,y_j)-\alpha]^+. \tag{23}$$

Furthermore, the following function $\hat{F}_\beta(x,\alpha)$ using auxiliary real variables $v_j,\ j=1,\dots,n$, i.e.,

$$\hat{F}_\beta(x,\alpha) = \alpha + \frac{1}{(1-\beta)n}\sum_{j=1}^{n} v_j, \tag{24}$$

subject to the following constrains

$$v_j \geq f(x,y_j) - \alpha, \quad v_j \geq 0 \tag{25}$$

is equivalent to $\tilde{F}_\beta(x,\alpha)$. We set $\hat{F}_\beta(x,\alpha)$ on the model as a objective function. Here, we set two constraints about quantities of inventory and transfer as follows:

$$y_{it} = x_{it} + w_{it} + y_{i,t-1} - \zeta_{it}, \quad i=1,2,\dots,m, \quad t=1,2,\dots,T, \tag{26}$$

and

$$w_{it} \leq \sum_{l=1,l\neq i}^{m} x_{lt}, \quad i=1,2,\dots,m, \quad t=1,2,\dots,T, \tag{27}$$

where $y_{i0} = a$ $(a \geq 0)$. The former means relation between present quantities of inventory and previous one. On the other hand, the latter means that quantities of transfer for a product i is not over total quantity of production except for a product i in period t. Thus, we show below the CVaR minimization model for a production system with real option approach.

[CVaR minimization model]

$$\begin{array}{|ll} \textbf{minimize} & \alpha + \dfrac{1}{(1-\beta)n}\displaystyle\sum_{j=1}^{n} v_j \\ & \\ \textbf{subject to} & \text{the following constraints (28)} \sim \text{(40).} \end{array}$$

(28) $y_{ijt} = x_{ijt} + w_{ijt} + y_{ij,t-1} - \zeta_{ijt}$ $(i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

(29) $w_{ijt} \leq \displaystyle\sum_{l=1, l\neq i}^{m} x_{ljt}$ $\quad (i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

(30) $$CF_{ijt}(x_{ijt}, w_{ijt}, \zeta_{ijt}) = p_i \cdot \min\{x_{ijt} + w_{ijt}, \zeta_{ijt}\} - c_i(x_{ijt} + w_{ijt}) - h_i \cdot \max\{y_{ijt}, 0\} + s_i \cdot \min\{y_{ijt}, 0\}$$
$(i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

(31) $CF_{jt}(x_{jt}, w_{jt}, \zeta_{jt}) = \displaystyle\sum_{i=1}^{m} CF_{ijt}(x_{ijt}, w_{ijt}, \zeta_{ijt})$ $\quad (j = 1, \ldots, n;\ t = 1, \ldots, T)$

(32) $V_j = \displaystyle\sum_{t=1}^{T} \frac{CF_{jt}(x_{jt}, w_{jt}, \zeta_{jt})}{(1+r)^t} - I_0$ $\quad (j = 1, \ldots, n)$

(33) $C_{jTk} = \max\left\{u^{(T-k)}d^{(k-1)}V_j - I_j, 0\right\}$ $(j = 1, \ldots, n;\ t = T;\ k = 1, \ldots, T)$

(34) $C_{jtk} = \dfrac{1}{R}\left\{qC_{j,t+1,k} + (1-q)C_{j,t+1,k+1}\right\}$
$(j = 1, \ldots, n;\ t = 1, \ldots, T-1;\ k = 1, \ldots, t)$

(35) $L_j = -C_{j11} = -\dfrac{1}{R}\left\{qC_{j21} + (1-q)C_{j22}\right\}$ $\quad (j = 1, \ldots, n)$

(36) $v_j \geq L_j - \alpha$ $\quad (j = 1, \ldots, n)$

(37) $v_j \geq 0$ $\quad (j = 1, \ldots, n)$

(38) $x_{ijt} \geq 0$ $\quad (i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

(39) $w_{ijt} \geq 0$ $\quad (i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

(40) $\zeta_{ijt} \geq 0$ $\quad (i = 1, \ldots, m;\ j = 1, \ldots, n;\ t = 1, \ldots, T)$

4. Sensitivity Analysis

We analyze the sensitivity of the model. We use the mathematical programming solver NUOPT (ver.10.1.0) for Windows, on a personal computer with Pentium 4 processor (2.26 GHz) and 512 MB memory. In sensitivity analysis, the sample data of demand ζ are generated under the normal distribution that mean is 200 and variance is 50. We set the following conditions:

- products: $i = 1, 2, 3$;
- periods: $t = 1, 2, 3$;
- initial quantity of inventory: $y_{i0} = 0$ $(i = 1, 2, 3)$;
- risk-free rate: $r = 0.2$;
- up-rate for an underlying asset price: $u = 1.3$;
- down-rate for an underlying asset price: $d = 0.9$;
- confidence level: $\beta = 95\%$.

As results of sensitivity analysis, a value of CVaR is 0.003, VaR is 0.001, and average of the NPV is 847.915. Optimal quantities of production, transfer, and inventory are shown by Tables 1 through 3 and Figures 3 through 5.

Table 1: Optimal quantities of production

$i \setminus t$	1	2	3
1	75.0	69.4	73.9
2	65.8	58.0	61.7
3	67.6	63.9	65.6

Table 2: Optimal quantities of transfer

$i \setminus t$	1	2	3
1	53.9	51.3	54.1
2	43.5	39.9	41.9
3	46.9	45.5	46.4

Table 3: Optimal quantities of inventory

$i \setminus t$	1	2	3
1	47.1	44.4	44.6
2	51.2	54.7	52.7
3	71.7	73.6	72.9

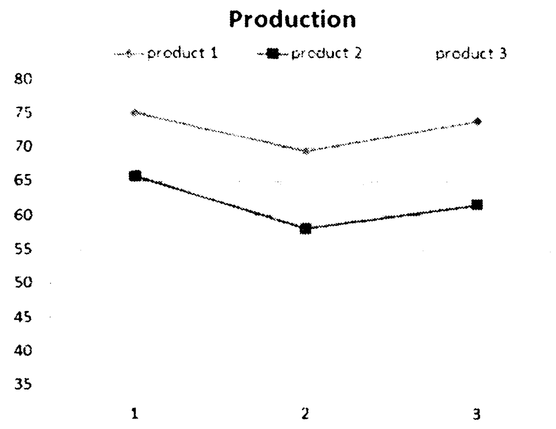

Figure 3: Optimal quantities of production

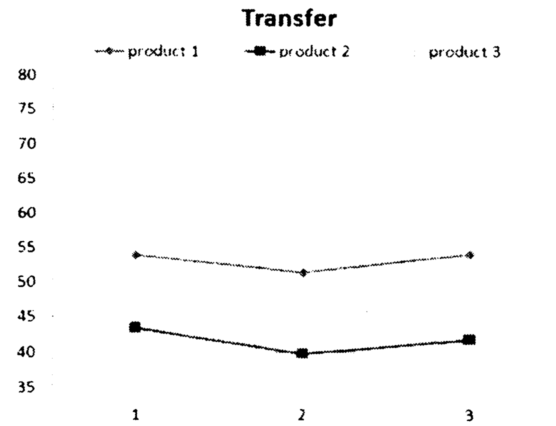

Figure 4: Optimal quantities of transfer

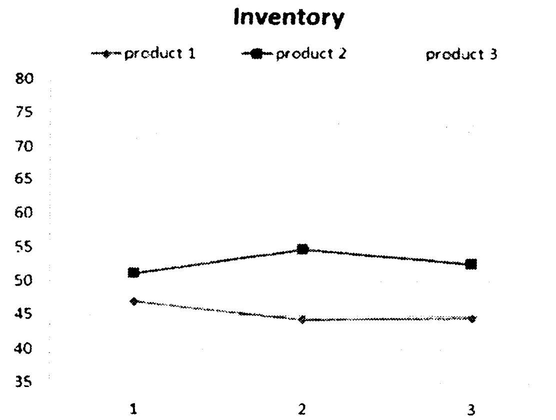

Figure 5: Optimal quantities of inventory

In Figures 3 through 5, optimal quantities of production and transfer show the same tendency. However, optimal quantities of inventory indecate the reverse tendency to them. This fact means that optimal quantities of production and transfer are decided considering demand change of sample data, and that optimal quantities of inventory are decided in conjection with them to reverse.

5. Concluding Remarks

In this paper, we suggested a risk minimization model for a production system which considered the uncertainty of demand change. So, we could decide optimal quantities of production, transfer, and inventory considering flexibility for demand change by applying the ROV method.

In a production system, however, there are many cases which compounded with multiple options (for instance, *option to expansion/contract*, *option to abandon/entry*, and *cancellation option*) in a management actually. In addition, since a production system generally is not in single period, we need to consider a multi-period optimization problem for a production system. Therefore, as a future problem, we will try to construct a risk minimization model considering the compounded cases with multiple options in a multi-period optimization problem for a production system.

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