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A remark on global weak solution in time for a system of compressible viscous fluid with large external potential force

Dedicated to Professor Kenji Nishihara on his sixtieth birthday

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1 Introduction and Main Theorem

In this short note, we make a survey of the recent work [5] concerning with global weak solution in time for a barotropic model system of compressible viscous fluid, which is described by the following conservation laws of mass and momentum:

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mu\Delta u - (\mu+\lambda)\nabla(\mathrm{div} u) + \nabla p(\rho) = \rho f, \tag{1.2}$$

where $x \in R^3, t > 0$ and $\rho,\ u,\ p = p(\rho)$ represent the fluid density, velocity, and pressure respectively, $f = \nabla\phi$ is the external force given by a potential ϕ, and the viscosity coefficients μ, λ are assumed to be constants and satisfy $\mu > 0$ and $3\lambda + 2\mu \geq 0$.

As for the global strong solutions in time, Matsumura-Nishida [7] considered the Cauchy problem without external forces and proved the global existence in a small H^3 neighborhood of a constant state $(\rho, u) = (\rho_\infty, 0)\ (\rho_\infty > 0)$. Although this result was extended by several works to the cases where the external force is small, there have been no remarkable results in the case with large external forces except for that of Matsumura-Padula [8], who proved the asymptotic stability in H^3 of the stationary state for the interior problems.

On the other hand, as for the weak global solutions in time, Hoff [2, 3] extended the result of [7] (for the Cauchy problem without external forces) to a weaker space including discontinuous functions, and proved the global existence of weak solutions for $p = a\rho^\gamma (\gamma \geq 1)$ in a small neighborhood of a constant state $(\rho_\infty, 0)\ (\rho_\infty > 0)$. In the cases of 'large data' and weak solutions in an 'energy finite class', big progress has been made (for example, see Lions [6], Feireisl [1]), but these results are not applicable to the case around a constant state $(\rho_\infty, 0)\ (\rho_\infty > 0)$.

Under these backgrounds, placing emphasis on large external forces and weak solutions around a constant state $(\rho_\infty, 0)\ (\rho_\infty > 0)$, we consider the Cauchy problem of (1.1)(1.2) with the initial data

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad \inf \rho_0 > 0, \quad \lim_{|x|\to\infty} \rho_0(x) = \rho_\infty. \tag{1.3}$$

For this Cauchy problem, under the conditions $\phi \in H^4(R^3)$ and

$$\left\| |x| |\nabla \phi| \right\|_{L^\infty} + \left\| |x|^2 \left| D^2 \phi \right| \right\|_{L^\infty} < \infty, \tag{1.4}$$

with $D^2\phi := \{(\partial/\partial x)^\alpha \phi | \, |\alpha| = 2\}$, Matsumura-Yamagata [9] considered the isentropic model, i.e., the pressure $p = a\rho^\gamma$ with $a > 0, \gamma \geq 1$. When the initial perturbation is suitably small in $L^2 \cap L^\infty$ for density and in H^1 for velocity, they obtained the global existence of weak solutions under an additional condition that $\gamma - 1$ is suitably small, which, however, excludes many significant physical models.

In our recent paper [5], we supposed that $\phi \in H^3(R^3)$ and p satisfy

$$p = p(\rho) \in C^2((0, \infty)) \text{ with } p(\rho) > 0, \; p'(\rho) > 0 \;\; (\rho > 0) \tag{1.5}$$

which includes the typical polytropic model

$$p = a\rho^\gamma, \gamma > 0, a > 0,$$

and showed the global existence of weak solutions when there exists a unique steady state away from vacuum and the initial perturbation is suitably small in $L^2 \cap L^\infty$ for density and in H^1 for velocity. Thus, we succeeded in relaxing the conditions on p and removing the essential conditions, smallness of $\gamma - 1$ and (1.4) in [9].

To state the main theorem, we first give the definition of weak solutions.

Definition 1.1 *We say that (ρ, u) is a weak solution to Cauchy problem* (1.1)-(1.3) *provided that $\rho \in L^\infty_{\text{loc}}(0, \infty; L^\infty(R^3)), u \in L^\infty_{\text{loc}}(0, \infty; H^1(R^3))$ and for all test functions $\psi \in \mathcal{D}(R^3 \times (-\infty, \infty))$,*

$$\int_{R^3} \rho_0 \psi(\cdot, 0) dx + \int_0^\infty \int_{R^3} (\rho \psi_t + \rho u \cdot \nabla \psi) \, dx dt = 0,$$

and for $j = 1, 2, 3$,

$$\int_{R^3} \rho_0 u_0^j \psi(\cdot, 0) dx + \int_0^\infty \int_{R^3} \left(\rho u^j \psi_t + \rho u^j u \cdot \nabla \psi + p(\rho) \psi_{x_j}\right) dx dt$$
$$- \int_0^\infty \int_{R^3} \left(\mu \nabla u^j \cdot \nabla \psi + (\mu + \lambda)(\text{div} u) \psi_{x_j}\right) dx dt = \int_0^\infty \int_{R^3} \rho f_{x_j} \psi dx dt.$$

We next consider a steady solution (ρ_s, u_s) for $\phi \in H^3$ satisfying the condition

$$(\rho_s, u_s)(x) \to (\rho_\infty, 0) \text{ as } |x| \to \infty.$$

Since the steady solution turns out to be unique and u_s be zero, it suffices to look for the steady solution in the form $(\rho_s, 0)$. Then by (1.2)

$$\nabla p(\rho_s) = \rho_s \nabla \phi, \tag{1.6}$$

which implies

$$\int_{\rho_\infty}^{\rho_s} \frac{p'(\rho)}{\rho} \, d\rho = \phi.$$

Therefore, in order to have uniformly positive ρ_s, we assume

$$- \int_0^{\rho_\infty} \frac{p'(\rho)}{\rho} \, d\rho < \inf \phi(x) \leq \sup \phi(x) < \int_{\rho_\infty}^\infty \frac{p'(\rho)}{\rho} \, d\rho. \tag{1.7}$$

Then we have

Lemma 1.1 *Assume that p satisfies* (1.5) *and* $f \in H^3(R^3)$ *satisfies* (1.7). *Then there exists a unique solution* ρ_s *of* (1.6) *satisfying* $\rho_s - \rho_\infty \in H^3(R^3)$. *Moreover, there exist constants* $\overline{\rho}$, $\underline{\rho} > 0$ *depending on* $\|f\|_{H^3}$ *such that*

$$\underline{\rho} < \inf \rho_s \le \sup \rho_s < \overline{\rho}. \tag{1.8}$$

Now it is ready to state our main theorem.

Theorem 1.2 *Fix a positive constant* ρ_∞. *Assume that the pressure p satisfying* (1.5) *and the potential* $\phi \in H^3(R^3)$ *satisfying* (1.7) *are given. Then, there exists a positive constant* ε_0, *depending only on* $\|\phi\|_{H^3}$ *and* ρ_∞ *such that: if*

$$\|\rho_0 - \rho_s\|_{L^\infty} + \|\rho_0 - \rho_s\|_{L^2} + \|u_0\|_{H^1} \le \varepsilon_0,$$

then the Cauchy problem (1.1)-(1.3) *has a global weak solution* (ρ, u) *satisfying*

$$\begin{cases} \rho - \rho_s \in C\left([0,\infty); H^{-1}(R^3)\right), \rho(\cdot,t) - \rho_s \in \left(L^2 \cap L^\infty\right)(R^3), \; a.e. \; t > 0, \\ u \in C\left([0,\infty); L^2(R^3)\right); \end{cases}$$

$$0 < \inf \rho \le \sup \rho < \infty.$$

Moreover, $(\rho, u) \to (\rho_s, 0)$ *as* $t \to \infty$ *in the sense that, for all* $q \in (2, \infty]$,

$$\lim_{t\to\infty} \|(\rho(\cdot,t) - \rho_s, u(\cdot,t))\|_{L^q} = 0.$$

The basic strategy of the proof is due to Hoff [2, 3], and the key step to have the a priori estimates is that to derive the uniform time-independent L^2-estimate of the gradients of the velocity. However, due to the arbitrariness of both the external force f and the pressure p, we cannot directly generalize the approaches in [3, 9]. To overcome these difficulties, we use an idea due to Huang-Li-Xin [4] (that is, we first normalize the momentum equation (1.2) by dividing it by ρ_s), and modify the "effective viscous flux" which played essential roles in [2, 3] to

$$F := \rho_s^{-1}\left((\lambda + 2\mu)\mathrm{div} u - (p(\rho) - p(\rho_s))\right). \tag{1.9}$$

Then combining the arguments on the a priori estimates in [3, 9] and a compactness argument in Feireisl [1] carefully, we can prove the theorem.

In what follows, we denote the usual norm L^p in the spatial direction by $\|\cdot\|_p$, in particular the L^2 norm by $\|\cdot\|$ for simplicity.

2 *A Priori* estimates

In this section, we show a rough sketch of how to show the desired *a priori* estimates. Let (ρ, u) be a smooth solution of (1.1)(1.2) defined up to a positive time T, where "(ρ, u) is smooth" means $(\rho - \rho_\infty, u) \in C^4 \cap H^4(R^3 \times [0,T])$. We set

$$\Psi(T) = \|\eta\|^2_{L^\infty(R^3\times[0,T])}, \quad \eta = \rho - \rho_s,$$

and

$$\Phi(t) = \Phi_1(t) + \Phi_2(t) + \Phi_3(t)$$

where

$$\begin{aligned}\Phi_1(T) &= \sup_{0\le t\le T}\left(\|u(t)\|^2+\|\eta(t)\|^2\right)+\int_0^T\|\nabla u(t)\|^2dt,\\ \Phi_2(T) &= \sup_{0\le t\le T}\|\nabla u(t)\|^2+\int_0^T\|\dot u(t)\|^2dt,\\ \Phi_3(T) &= \sup_{0\le t\le T}\sigma(t)\|\dot u(t)\|^2+\int_0^T\sigma(t)\|\nabla\dot u(t)\|^2dt,\end{aligned}$$

for $\sigma(t)=\min\{1,t\}$, and $\dot u$ is the material derivative of u given by

$$\dot f=\frac{D}{Dt}f=f_t+u^jf_j,\quad (f_j=f_{x_j}).$$

Moreover we set

$$C_0=\|\rho_0-\rho_s\|_{L^\infty}+\|\rho_0-\rho_s\|_{L^2}+\|u_0\|_{H^1}.$$

In what follows, we assume that $\rho\in[\underline{\rho},\,\overline{\rho}]$, and $\Phi+\Psi\le 1$, $C_0\le 1$, and use C as generic positive constants which may depend on $\|f\|_{H^3}$ and ρ_∞ but not on T.

Then, our goal is to obtain the following a priori estimate:

Proposition 2.1 (a priori estimate) *There exist positive constants ε_0 and C independent of T such that, if $\Phi(T)+\Psi(T)\le\varepsilon_0$, then $\Phi(T)+\Psi(T)\le CC_0^{1/4}$.*

Once we obtain Proposition 2.1, the remaining arguments to obtain the global weak solution and its asymptotic behavior, that is Theorem 1.2, are almost the same as that of the previous works [1,3,9]. The Proposition 2.1 is proved by the following series of lemmas. Let us start with the most basic energy estimate.

Lemma 2.1 *It holds that*

$$\Phi_1(T)=\sup_{0\le t\le T}\left(\|u(t)\|^2+\|\eta(t)\|^2\right)+\int_0^T\|\nabla u(t)\|^2dt\le CC_0.$$

Proof. Using the mass equation (1.1) and (1.6) we rewrite the momentum equation (1.2) in the form

$$\rho\dot u+\rho\left(\frac{\nabla p(\rho)}{\rho}-\frac{\nabla p(\rho_s)}{\rho_s}\right)-\mu\Delta u-(\lambda+\mu)\nabla(\mathrm{div}u)=0. \tag{2.1}$$

Multiplying (2.1) by u and integrating the resultant equation, we obtain after integration by parts,

$$\begin{aligned}&\int_0^t\int\rho u\cdot\dot u+\int_0^t\int\rho u\cdot\left(\frac{\nabla p(\rho)}{\rho}-\frac{\nabla p(\rho_s)}{\rho_s}\right)\\ &\quad+\int_0^t\int\left(\mu|\nabla u|^2+(\lambda+\mu)(\mathrm{div}u)^2\right)=0.\end{aligned} \tag{2.2}$$

Here and in what follows we omit the symbols of integral variables, e.g. '$dxdt$', 'dx' and so on, in integral notation unless we are confused. Now noting that

$$\int\rho\dot f dx=\frac{d}{dt}\int\rho f dx$$

holds in general, the first term on the left hand side of (2.2) is

$$\int_0^t \int \rho \frac{D}{Dt}\frac{|u|^2}{2} = \frac{1}{2}\int \rho|u|^2\Big|_0^t.$$

The second one is

$$\begin{aligned}\int_0^t \int \rho u \cdot \nabla \int_{\rho_s}^{\rho} \frac{p'(s)}{s} ds &= -\int_0^t \int \mathrm{div}(\rho u) \int_{\rho_s}^{\rho} \frac{p'(s)}{s} ds \\ &= \int_0^t \int \rho_t \int_{\rho_s}^{\rho} \frac{p'(s)}{s} ds \\ &= \int G(\rho)\Big|_0^t,\end{aligned}$$

where

$$G(\rho) := \int_{\rho_s}^{\rho} \int_{\rho_s}^{r} \frac{p'(s)}{s} ds dr.$$

Thus, we obtain the energy equality

$$\int \left(\frac{1}{2}\rho|u|^2 + G(\rho)\right)\Big|_0^t + \int_0^t \int (\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2) = 0.$$

An easy observation that

$$C^{-1}\eta^2 \le G(\rho) \le C\eta^2$$

which follows by use of $G(\rho_s) = G'(\rho_s) = 0$ and $\rho \in [\underline{\rho}, \overline{\rho}]$, completes the proof.

Next, we proceed to the most important estimates for $\Phi_2(T)$ and $\Phi_3(T)$ which essentily differ from that in [9].

Lemma 2.2 *It holds that*

$$\Phi_2(T) \le C\left(C_0 + \Psi^{1/3}(T)R_1(T) + R_2(T)\right), \tag{2.3}$$

$$\Phi_3(T) \le C\left(C_0 + \Psi^{1/3}(T)R_1(T) + R_2(T) + R_3(T)\right) \tag{2.4}$$

where

$$R_1(T) = \int_0^T \|\eta\|_{10/3}^{10/3} dt,\ R_2(T) = \int_0^T \|\nabla u(t)\|_3^3 dt,\ R_3(T) = \int_0^T \sigma(t)\|\nabla u(t)\|_4^4 dt.$$

Proof. The key step to prove (2.3) is the following observation on the density and pressure deviation due to [4]. Noticing that

$$\begin{aligned}&\rho_s^{-1}\left(\nabla\left(p(\rho) - p(\rho_s)\right) - \eta\rho_s^{-1}\nabla p(\rho_s)\right) \\ &= \nabla\left(\rho_s^{-1}(p(\rho) - p(\rho_s))\right) - (p(\rho) - p(\rho_s))\nabla\rho_s^{-1} - \rho_s^{-2}\eta\nabla p(\rho_s) \\ &= \nabla\left(\rho_s^{-1}(p(\rho) - p(\rho_s))\right) + \left(p(\rho) - p(\rho_s) - p'(\rho_s)\eta\right)\rho_s^{-2}\nabla\rho_s \\ &= \nabla\left(\rho_s^{-1}(p(\rho) - p(\rho_s))\right) + \eta^2\rho_s^{-2}\nabla\rho_s \int_0^1\int_0^1 p''(\rho_s + \sigma\lambda\eta) d\sigma d\lambda,\end{aligned} \tag{2.5}$$

we rewrite the momentum equation (2.1) as

$$\begin{aligned}
&\rho\dot{u}-\mu\Delta u-(\lambda+\mu)\nabla(\mathrm{div}u)+\rho_s\nabla\left(\rho_s^{-1}(p(\rho)-p(\rho_s))\right)\\
&\quad+\eta^2\rho_s^{-1}\nabla\rho_s\int_0^1\int_0^1 p''(\rho_s+\sigma\lambda\eta)d\sigma d\lambda=0. \qquad (2.6)
\end{aligned}$$

Multiplying (2.6) by $\dot{u}$ and integrating, we thus get

$$\begin{aligned}
&\int_0^t\int\rho|\dot{u}|^2-\mu\int_0^t\int\dot{u}\cdot\Delta u-(\lambda+\mu)\int_0^t\int\dot{u}\cdot\nabla(\mathrm{div}u)\\
&=-\int_0^t\int\rho_s^{-1}\dot{u}\cdot\nabla\rho_s\eta^2\int_0^1\int_0^1 p''(\rho_s+\sigma\lambda\eta)d\sigma d\lambda\\
&\quad-\int_0^t\int\rho_s\dot{u}\cdot\nabla\left(\rho_s^{-1}(p(\rho)-p(\rho_s))\right)\\
&=-\int_0^t\int\rho_s^{-1}\dot{u}\cdot\nabla\rho_s\eta^2\int_0^1\int_0^1 p''(\rho_s+\sigma\lambda\eta)d\sigma d\lambda\\
&\quad+\int_0^t\int\mathrm{div}u_t(p(\rho)-p(\rho_s))+\int_0^t\int\mathrm{div}(u\cdot\nabla u)(p(\rho)-p(\rho_s))\\
&\quad+\int_0^t\int u_t\cdot\nabla\rho_s\rho_s^{-1}(p(\rho)-p(\rho_s))+\int_0^t\int u\cdot\nabla u\cdot\nabla\rho_s\rho_s^{-1}(p(\rho)-p(\rho_s))\\
&=\sum_{i=1}^5 I_i. \qquad (2.7)
\end{aligned}$$

First, after integration by parts, we can estimate the second term on the left of (2.7) as follows:

$$\begin{aligned}
-\mu\int_0^t\int u_t^j u_{ll}^j-\mu\int_0^t\int u^j u_j^k u_{ll}^k &= \mu\int_0^t\int u_{tl}^j u_l^j+\mu\int_0^t\int\left(u_l^j u_j^k u_l^k+u^j u_{jl}^k u_l^k\right)\\
&= \frac{\mu}{2}\int|\nabla u|^2\Big|_0^t+\int_0^t\int\mathcal{O}\left(|\nabla u|^3\right).
\end{aligned}$$

By similar calculations, the third one on the left of (2.7) is

$$\frac{\mu+\lambda}{2}\int(\mathrm{div}u)^2\Big|_0^t+\int_0^t\int\mathcal{O}\left(|\nabla u|^3\right).$$

Next, we can estimate the terms $I_i(i=1,\cdots,5)$ as follows:

$$I_1\le\delta\int_0^t\int|\dot{u}|^2+C_\delta\Psi^{1/3}(t)R_1(t);$$

It follows from (1.1) that

$$(p(\rho))_t=-\mathrm{div}(p(\rho)u)-\left(\rho p'(\rho)-p(\rho)\right)\mathrm{div}u, \qquad (2.8)$$

which yields that

$$\begin{aligned}
I_2 &= \int\mathrm{div}u(p(\rho)-p(\rho_s))\Big|_0^t+\int_0^t\int\left(\rho p'(\rho)-p(\rho)\right)(\mathrm{div}u)^2\\
&\quad-\int_0^t\int p(\rho)u\cdot\nabla\mathrm{div}u. \qquad (2.9)
\end{aligned}$$

Integration by parts leads to

$$I_3=\int_0^t\int p(\rho)\partial_i u\cdot\nabla u^i+\int_0^t\int p(\rho)u\cdot\nabla\mathrm{div}u+\int_0^t\int u\cdot\nabla u\cdot\nabla p(\rho_s),$$

which together with (2.9) yields that

$$\begin{aligned}I_2+I_3+I_5 &\le \int \mathrm{div}u(p(\rho)-p(\rho_s))\Big|_0^t+C\int_0^t\left(\|\nabla u\|^2+\|\nabla u\|\|u\|_6\|\nabla\rho_s\|_3\right)\\ &\le \int \mathrm{div}u(p(\rho)-p(\rho_s))\Big|_0^t+C\int_0^t\|\nabla u\|^2.\end{aligned}$$

We use (2.8) again to estimate I_4 as follows:

$$\begin{aligned}I_4 &= \int u\cdot\nabla\rho_s\rho_s^{-1}(p(\rho)-p(\rho_s))\Big|_0^t-\int_0^t\int\rho_s^{-1}p(\rho)u\cdot\nabla u\cdot\nabla\rho_s\\ &\quad -\int_0^t\int p(\rho)u^iu\cdot\nabla\left(\rho_s^{-1}\partial_i\rho_s\right)+\int_0^t\int u\cdot\nabla\rho_s\rho_s^{-1}(\rho p'(\rho)-p(\rho))\mathrm{div}u\\ &\le \int u\cdot\nabla\rho_s\rho_s^{-1}(p(\rho)-p(\rho_s))\Big|_0^t+C\int_0^t\|\nabla u\|^2,\end{aligned}$$

where we have used the following simple fact:

$$\begin{aligned}&-\int_0^t\int p(\rho)u^iu\cdot\nabla\left(\rho_s^{-1}\partial_i\rho_s\right)\\ &=-\int_0^t\int(p(\rho)-p(\rho_s))u^iu\cdot\nabla\left(\rho_s^{-1}\partial_i\rho_s\right)+\int_0^t\int\rho_s^{-1}\partial_i\rho_s\mathrm{div}\left(p(\rho_s)u^iu\right)\\ &\le C\int_0^t\|\eta\|_3\|u\|_6^2\|\nabla\rho_s\|_{W^{1,3}}+C\int_0^t\|u\|_6^2\|\nabla\rho_s\|_3^2+C\int_0^t\|\nabla u\|\|u\|_6\|\nabla\rho_s\|_3\\ &\le C\int_0^t\|\nabla u\|^2.\end{aligned}$$

Substituting these estimates back into (2.7) and applying the previous bound in the Lemma 2.1, we then obtain (2.3) after choosing δ suitably small. To show (2.4), noticing that

$$\left[\frac{\partial}{\partial t}+\mathrm{div}(u\,\cdot)\right](\rho\phi)=\rho\dot{\phi},$$

and (1.2) is equivalent to

$$\rho\dot{u}-\mu\Delta u-(\lambda+\mu)\nabla(\mathrm{div}u)=-\nabla(p(\rho)-p(\rho_s))+\eta\nabla f, \tag{2.10}$$

we operate $\sigma\dot{u}^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ to $(2.10)^j$ and integrate it. Then we get

$$\begin{aligned}&\int_0^t\sigma\int\rho\frac{D}{Dt}\frac{|\dot{u}|^2}{2}-\mu\int_0^t\sigma\int\dot{u}^j\left(\Delta u_t^j+\mathrm{div}(\Delta u^ju)\right)\\ &\qquad -(\lambda+\mu)\int_0^t\sigma\int\dot{u}^j\left(\mathrm{div}u_{jt}+\mathrm{div}(u\mathrm{div}u_j)\right)\\ &= -\int_0^t\sigma\int\dot{u}^j\left((p(\rho))_{jt}+\mathrm{div}\left(u(p(\rho)-p(\rho_s))_j\right)\right)\\ &\qquad +\int_0^t\sigma\int\dot{u}^j\left(f_j\rho_t+\mathrm{div}\left(u\eta f_j\right)\right).\end{aligned}\tag{2.11}$$

Estimating each term carefully in the same way as the above proof of (2.3), we can get the desired inequality (2.4). We omit the details.

In order to complete the estimate for $\Phi(T)$, we need to estimate $R_1(T), R_2(T)$ and $R_3(T)$.

Lemma 2.3 *If ε_0 is suitably small, then it holds*

$$\begin{aligned} R_1(T) &\leq C\left(C_0^{5/3} + \Phi^{5/3}(T)\right), &(2.12)\\ R_2(T) &\leq C\left(C_0 + \Phi^{3/2}(T)\right), &(2.13)\\ R_3(T) &\leq C\left(C_0 + \Phi^2(T)\right). &(2.14)\end{aligned}$$

To prove Lemma 2.3, we shall at first state important estimates based on singular integral operator theory, with a formal proof. For more details, see Hoff [2].

Lemma 2.4 *For $p, q \in (1, \infty), t > 0$, it holds that*

$$\|\nabla u(t)\|_p \leq C_p\left(\|F(t)\|_p + \|\mathrm{curl} u(t)\|_p + \|\eta(t)\|_p\right), \quad (2.15)$$
$$\|\nabla F(t)\|_q + \|\nabla \mathrm{curl} u(t)\|_q \leq C_q\left(\|\dot{u}(t)\|_q + \|\nabla u(t)\|_q + \left\|\eta^2(t)\right\|_q\right), \quad (2.16)$$

where $F = \rho_s^{-1}\left((\lambda + 2\mu)\mathrm{div} u - (p(\rho) - p(\rho_s))\right)$ as in (1.9).

Proof. The estimate (2.15) follows easily from the following well-known inequality:

$$\|\nabla\phi\|_p \leq C_p\left(\|\mathrm{div}\phi\|_p + \|\mathrm{curl}\phi\|_p\right), \quad \phi \in W^{1,p}.$$

To prove (2.16), we rewrite the equation (2.6) using the definition of F as in the form

$$\nabla F - \mu \mathrm{curl}\left(\rho_s^{-1}\mathrm{curl} u\right) = \rho_s^{-1}\rho\dot{u} + \mathcal{O}\left(|\nabla u| + \eta^2\right). \quad (2.17)$$

Operating $\nabla\Delta^{-1}\mathrm{div}$, we get the desired estimate for ∇F which, together with (2.17) and the estimate

$$\|\nabla \mathrm{curl} u\|_q \leq C\|\mathrm{curl}(\rho_s^{-1}\mathrm{curl} u)\|_q + C\|\nabla u\|_q,$$

yields the the desired estimate for $\nabla \mathrm{curl} u$.

Proof of Lemma 2.3. We only give the proof of (2.12), and omit the details for (2.13) and (2.14). We rewrite equation (1.1) as

$$(2\mu + \lambda)\frac{D}{Dt}\eta + \rho(p(\rho) - p(\rho_s)) = -\rho\rho_s F - (2\mu + \lambda)u \cdot \nabla\rho_s. \quad (2.18)$$

Multiplying (2.18) by $\rho\eta^{7/3}$, we then obtain

$$\begin{aligned}&\frac{3(2\mu + \lambda)}{10}\rho\frac{D}{Dt}\eta^{10/3} + \rho^2\eta^{7/3}(p(\rho) - p(\rho_s))\\ &= -\rho^2\rho_s\eta^{7/3}F - (2\mu + \lambda)\rho\eta^{7/3}u \cdot \nabla\rho_s,\end{aligned}$$

which implies that there exists a positive constant ν such that

$$\rho\frac{D}{Dt}\eta^{10/3} + \nu\eta^{10/3} \leq CF^{10/3} + C|u|^{10/3}.$$

Integrating (2.19), we have by use of (2.16) that

$$\begin{aligned}
&\int \rho\eta^{10/3}\Big|_0^t + \nu\int_0^t\int \eta^{10/3}\\
&\le C\int_0^t \left(\|F\|^{4/3}\|\nabla F\|^2 + \|u\|^{4/3}\|\nabla u\|^2\right)\\
&\le C\Phi^{2/3}(t)\int_0^t \|\nabla F\|^2 + CC_0^{5/3}\\
&\le C\Phi^{2/3}(t)\int_0^t (\|\dot{u}\|^2 + \|\nabla u\|^2 + \|\eta\|_4^4) + CC_0^{5/3}\\
&\le CC_0^{5/3} + C\Phi^{5/3}(t) + C\Phi^{2/3}(t)\Psi^{1/3}(t)\int_0^t\int \eta^{10/3}.
\end{aligned}$$

Thus, if $\Phi(T) + \Psi(T) \le \varepsilon_0$ and ε_0 is suitably small, (2.12) holds. This completes the proof.

Combining the above Lemma 2.1, 2.2 and 2.3, we easily can reach at the desired estimate for $\Phi(T)$.

Lemma 2.5 *If ε_0 is suitably small, then it holds that*

$$\Phi(T) \le CC_0. \tag{2.19}$$

Finally, we need to provide the desired point-wise estimate for the density.

Lemma 2.6 *If ε_0 is suitably small, then it holds that*

$$\Psi(T) = \|\eta\|^2_{L^\infty(R^3\times[0,T])} \le CC_0^{1/4}. \tag{2.20}$$

Once the Lemma 2.6 is proved, the proof of Proposition 2.1 easily follows from the Lemma 2.5 and Lemma 2.6.

Proof of Lemma 2.6 When $T \le 1$, integrating (2.18) along each particle path $x(t)$, we obtain that, for $t \in (0, T]$,

$$|\eta|(x(t),t) \le C\left(C_0^{1/2} + \int_0^t (\|F\|_\infty + \|u\|_\infty) + \int_0^t |\eta|(x(s),s)ds\right).$$

Applying Grönwall's inequality in light of $t \le 1$, and taking appropriate supremums, we then get

$$\Psi(T)^{1/2} \le C\left(C_0^{1/2} + A(T)\right),$$

where

$$\begin{aligned}
A(T) &= \int_0^T (\|F\|_\infty + \|u\|_\infty)\\
&\le C\int_0^T (\|u\| + \|F\| + \|\nabla u\|_4 + \|\nabla F\|_4)\\
&\le C\int_0^T (\|u\| + \|F\| + \|\nabla u\|_4 + \|\eta\|_4 + \|\dot{u}\|_4)\\
&\le CC_0^{1/2} + C\int_0^T (\|\nabla u\|_4 + \|\eta\|_4 + \|\dot{u}\|_4).
\end{aligned}$$

It follows from (2.14) and (2.19) that

$$\begin{aligned}
\int_0^T \|\nabla u\|_4 &\le \left(\int_0^T \sigma^{-1/3}\right)^{3/4}\left(\int_0^T \sigma\|\nabla u\|_4^4\right)^{1/4}\\
&\le C\left(C_0^{1/4}+\Phi^{1/2}(T)\right)\\
&\le CC_0^{1/4}.
\end{aligned}$$

Similarly, we can get the same bound for η. For the term for $\dot{u}$, we have

$$\begin{aligned}
\int_0^T \|\dot{u}\|_4 &\le C\int_0^T \|\dot{u}\|^{1/4}\|\nabla\dot{u}\|^{3/4}\\
&\le \left(\int_0^T \sigma^{-3/4}\right)^{1/2}\left(\int_0^T \|\dot{u}\|^2\right)^{1/8}\left(\int_0^T \sigma\|\nabla\dot{u}\|^2\right)^{3/8}\\
&\le CC_0^{1/4}.
\end{aligned}$$

When $T \ge 1$, we multiply (2.18) by $\sigma\rho\eta^3$ to get

$$\sigma\rho\frac{D}{Dt}\eta^4 + C^{-1}\sigma\eta^4 \le C\sigma\left(F^4+|u|^4\right). \tag{2.21}$$

Then, we integrate (2.21) (divided by ρ on both sides) along particle trajectories to obtain that, for $t \in [1, T]$,

$$\begin{aligned}
\|\eta(t)\|_\infty^4 &\le \|\eta(1)\|_\infty^4 + C\int_1^t \left(\|u\|_\infty^4 + \|F\|_\infty^4\right)\\
&\le \Psi^2(1) + C\int_1^t \left(\|u\|^2\|\nabla u\|^2 + \|\nabla u\|_4^4\right) + C\int_1^t \|F\|_6^{2/3}\|\nabla F\|_{10/3}^{10/3}\\
&\le CC_0^{1/2} + C\int_1^t \|\nabla F\|^{2/3}\|\nabla F\|_{10/3}^{10/3}\\
&\le CC_0^{1/2} + CC_0^{1/3}\int_1^t \|\nabla F\|_{10/3}^{10/3},
\end{aligned} \tag{2.22}$$

where we used (2.14) and (2.19). It follows from (2.16) that

$$\begin{aligned}
\int_1^t \|\nabla F\|_{10/3}^{10/3} &\le C\int_1^t \left(\|\dot{u}\|_{10/3}^{10/3} + \|\nabla u\|_{10/3}^{10/3} + \|\eta\|_{20/3}^{20/3}\right)\\
&\le C\int_1^t \left(\|\dot{u}\|_{10/3}^{10/3} + \|\nabla u\|^2 + \|\nabla u\|_4^4 + \|\eta\|_4^4\right),
\end{aligned}$$

which together with the Lemma 2.3, (2.22) and

$$\begin{aligned}
\int_1^t \|\dot{u}\|_{10/3}^{10/3} &\le C\int_1^t \|\dot{u}\|^{4/3}\|\nabla\dot{u}\|^2\\
&\le C\sup_{1\le t\le T}\|\dot{u}\|^{4/3}\int_1^T \|\nabla\dot{u}\|^2\\
&\le C\Phi^{5/3}(T),
\end{aligned}$$

yields the desired estimate (2.20). This completes the proof of the Lemma 2.6.

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