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Minimization of the Principal Eigenvalue for an Elliptic Boundary Value Problem with Indefinite Weight

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This article is based on a joint work with Chiu-Yen Kao and Yuan Lou of the Ohio State University. We consider the eigenvalue problem

$$\text{(EVP)} \qquad \begin{cases} \Delta\phi + \lambda m(x)\phi = 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $m(x)$ is an indefinite weight. If the eigenvalue problem (EVP) has a positive eigenfunction $\phi \in H^1(\Omega)$, then λ is called a principal eigenvalue. Clearly, $\lambda = 0$ is a principal eigenvalue with an associated eigenfunction $\phi \equiv 1$.

Here, we explain biological background of the above problem. Let us consider the logistic equation

$$\begin{cases} \dfrac{d}{dt}u(t) = \lambda u(t)\{m - u(t)\}, \\ u(0) = a > 0, \end{cases}$$

where t is the time, $u(t)$ is the population of some biological species, $\lambda > 0$ is a selection pressure, m is an intrinsic growth rate. As is easily seen, if $m > 0$ then $u(t) \to m$ as $t \to \infty$ so that the biological species can survive. Conversely, if $m \leq 0$, then $u(t) \to 0$ as $t \to \infty$ so that the species becomes extinct.

Now, let us introduce a spatial variable x, and consider the diffusive logistic model (or Fisher-KPP equation)

$$\text{(F)} \quad \begin{cases} \dfrac{\partial}{\partial t} u = \Delta u + \lambda u\{m(x) - u\} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ u(x,0) > 0, & \text{in } \overline{\Omega}. \end{cases}$$

In this model, Ω denotes the habitat and $u(x,t)$ represents the population density.

When $m(x)$ changes its sign depending on x, then a region with $m(x) > 0$ is favorable to the species whereas a region with $m(x) < 0$ is unfavorable. In the case where $m(x)$ changes its sign, can the species survive? In order to answer this question, it suffices to consider the stability of the trivial steady state $u \equiv 0$. Indeed, it holds that

$$u = 0 \text{ is unstable} \iff \text{solutions leave away from } 0 \iff \text{survival},$$

$$u = 0 \text{ is stable} \iff \text{solutions tend to } 0 \iff \text{extinction}.$$

On the other hand, the stability of the trivial solution can be determined by analyzing the linearized eigenvalue problem

$$\text{(LEP)} \quad \begin{cases} \mu\Phi = \Delta\Phi + \lambda m(x)\Phi & \text{in } \Omega, \\ \dfrac{\partial \Phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let μ_0 be the maximal eigenvalue. Then

$$\mu_0 > 0 \Longleftrightarrow \text{unstable} \Longleftrightarrow \text{survival},$$

$$\mu_0 < 0 \Longleftrightarrow \text{stable} \Longleftrightarrow \text{extinction}.$$

By the eigenvalue analysis, we can show the following about the maximal eigenvalue of (LEP):

Case I: If $m(x) < 0$ on Ω, then $\mu_0 < 0$ for any $\lambda > 0$. ... extinction,

Case II: If $\int_\Omega m(x)\,dx > 0$, then $\mu > 0$ for any $\lambda > 0$. ... survival.

Case III: If $m(x)$ changes its sign and $\int_\Omega m(x)\,dx < 0$, then the positive principal eigenvalue $\lambda_p > 0$ of (EVP) has the following properties:

(i) If $0 < \lambda < \lambda_p$, then $\mu_0 < 0$. ... extinction

(ii) If $\lambda > \lambda_p$, then $\mu_0 > 0$. ... survival.

For an endangered species, we control $m(x)$ by some means in order to reserve the species. In this case, if the positive principal eigenvalue λ_p is small, then the species has a better chance to survive. Under a limited resource, how can we minimize λ_p? This question is formulated mathematically as follows. For the eigenvalue problem, we impose the following conditions:

(A1) $\Omega_+ := \{x \in \Omega : m(x) > 0\}$ has positive measure.

(A2) $\int_\Omega m(x)dx < 0.$

(A3) For a fixed constant $\kappa > 0$, $m(x)$ satisfies $-1 \le m(x) \le \kappa$ a.e. on Ω.

(A4) For a fixed constant $0 < \mu < 1$, $m(x)$ satisfies

$$\int_\Omega m(x)dx \le (-1+\mu)|\Omega|.$$

Let $\mathcal{M}$ denote the set of $m(x)$ satisfying the above conditions. In (A3), $m(x) = -1$ corresponds to a growth rate without any protection, and $m(x) = \kappa$ is a growth rate in the optimal environment. In (A4), the constant μ corresponds to the maximal supply of the resource and the inequality $\mu < 1$ implies that the resource is not sufficient.

Under the constraints (A1) and (A2), Brown-Lin [1] and Senn-Hess [7] showed that (EVP) has a unique positive principal eigenvalue $\lambda_p(m)$. Saut-Scheurer [6] proved that

$$\lambda_{inf} := \inf_{m \in \mathcal{M}} \lambda_p(m) > 0.$$

Cantrell-Cosner [2] addressed the following question: Among all $m(x) \in \mathcal{M}$, which $m(x)$ minimizes $\lambda_p(m)$? They studied some simple cases, but their answer was not satisfactory. In the following, we derive some general properties for this problem, and determined the minimizer in some specific cases. First, concerning the minimizer of $\lambda_p(m)$, the following bang-bang property holds:

Theorem 1. ([5]) *The infimum of λ_{inf} of $\lambda_p(m)$ is attained by some $m \in \mathcal{M}$, and such m is expressed as follows by using a measurable set $E \subset \Omega$:*

$$m(x) = \begin{cases} \kappa & \text{for } x \in E, \\ -1 & \text{for } x \notin E, \end{cases} \qquad a.e.$$

Proof First, by the variational principle, the positive principal eigenvalue of (EVP) is characterized by using the Rayleigh quotient

$$\lambda_p(m) = \inf_{U \in \mathcal{S}(m)} \frac{\displaystyle\int_\Omega |\nabla U|^2}{\displaystyle\int_\Omega m(x) U^2},$$

where

$$\mathcal{S}(m) := \left\{ U \in H^1(\Omega) : \int_\Omega m(x) U^2 > 0 \right\}.$$

Moreover, $\lambda_p(m)$ is simple, and the infimum of the Rayleigh quotient is attained only by associated eigenfunctions that do not change sign. Suppose now that $m_1(x)$ is not of bang-bang type, and let phi_1 be a positive eigenfunction associated with the principal eigenvalue $\lambda_p(m_1)$. Let $m_2(x)$ be a weight function obtained by moving resource from a region with smaller ϕ_1 to that with larger ϕ_1, and let ϕ_2 be an positive eigenfunction associated with the principal eigenvalue $\lambda_p(m_2)$. Then we have

$$\lambda_p(m_1) = \frac{\int_\Omega |\nabla \phi_1|^2}{\int_\Omega m_1(x)\phi_1^2} > \frac{\int_\Omega |\nabla \phi_1|^2}{\int_\Omega m_2(x)\phi_1^2} \geq \frac{\int_\Omega |\nabla \phi_2|^2}{\int_\Omega m_2(x)\phi_2^2} = \lambda_p(m_2).$$

Therefore, if $m(x)$ is not of bang-bang type, the it cannot be a minimizer. (We omit details for the existence of a minimizer.) □

By Theorem 1, it suffices to determine the set $E \subset \Omega$ to fine a global minimizer. Let us consider the one-dimensional problem as the simplest case

$$\text{(EVP1)} \quad \begin{cases} \phi_{xx} + \lambda m(x)\phi = 0, & x \in (0,1), \\ \phi_x(0) = \phi_x(1) = 0. \end{cases}$$

In this case, the constraints are described as

$$-1 \leq m(x) \leq \kappa, \qquad -1 < \int_0^1 m(x)dx \leq -1 + \mu < 0.$$

Theorem 2. ([5]) *In the eigenvalue problem (EVP1), $\lambda_p(m) = \lambda_{inf}$ holds if and only if*

$$m(x) = \begin{cases} \kappa & \text{for } x \in (0,\alpha), \\ -1 & \text{for } x \in (\alpha,1), \end{cases} \qquad a.e.$$

or

$$m(x) = \begin{cases} -1 & \text{for } x \in (0, 1-\alpha), \\ \kappa & \text{for } x \in (1-\alpha, 1), \end{cases} \qquad a.e.,$$

where $0 < \alpha < 1$ *is chosen to be*

$$\int_0^1 m(x)dx = -1 + \mu.$$

Proof. Given a weight function $m_1(x)$ and its associated eigenfunction $\phi_1(x)$, we define its spatial rearrangement by

$$m_2(\xi(s)) = s, \qquad \xi(s) = \text{measure}\{x : m_1(x) > s\},$$

$$U(\eta(s)) = s, \qquad \eta(s) = \text{measure}\{x : \phi_1(x) > s\}.$$

Then we have

$$|\nabla U| \leq |\nabla \phi_1|, \qquad \int_\Omega m_1(x)\phi_1^2 \leq \int_\Omega m_2(x)U^2.$$

Hence

$$\lambda_p(m_1) = \frac{\int_\Omega |\nabla \phi_1|^2}{\int_\Omega m_1(x)\phi_1^2} > \frac{\int_\Omega |\nabla U|^2}{\int_\Omega m_2(x)U^2} > \lambda_p(m_2).$$

□

By Theorem 2, in the Fisher-KPP model, the biological species has the maximal chance to survive if the weight is at one end of the interval.

Corolllary 1. *Let* $\Omega = (0,1)$. *If* $\lambda \leq \lambda_{inf}$, *then for any* $m \in \mathcal{M}$, *the trivial solution of Fisher-KPP model is globally stable.*

Next, we consider a limiting problem of thin cylindrical domains:

$$\text{(EVP2)} \quad \begin{cases} \dfrac{1}{a(x)}\{a(x)\phi_x\}_x + \lambda m(x)\phi = 0, & x \in (0,1), \\ \phi_x(0) = \phi_x(1) = 0, \end{cases}$$

where $a(x)$ is a positive smooth function representing the width of the thin domain. Without loss of generality, we may assume

$$\int_0^1 a(x)dx = 1$$

and assume also that

$$-1 \leq m(x) \leq \kappa, \quad -1 < \int_0^1 m(x)a(x)dx \leq -1 + \mu < 0.$$

Theorem 3. ([4]) *For the eigenvalue problem (EVP2), both*

$$m(x) = \begin{cases} \kappa & \text{for } x \in (0, \alpha), \\ -1 & \text{for } x \in (\alpha, 1), \end{cases} \quad a.e.$$

and

$$m(x) = \begin{cases} -1 & \text{for } x \in (0, 1-\beta), \\ \kappa & \text{for } x \in (1-\beta, 1), \end{cases} \quad a.e.$$

are local minimizers, where $0 < \alpha,\ \beta < 1$ *are constants such that*

$$\int_0^1 m(x)a(x)dx = -1 + \mu.$$

The proof is obtained by showing that the principal eigenvalue becomes smaller if we perturb a positive region through the Rayleigh quotient. We note that the variational principle for (EVP2) is formulated as

$$\lambda_p(m) = \inf_{U \in \mathcal{S}(m)} \frac{\displaystyle\int_0^1 a(x)\,|\nabla U|^2}{\displaystyle\int_0^1 a(x)m(x)U^2},$$

where

$$\mathcal{S}(m) := \left\{ U \in H^1(\Omega) : \int_0^1 a(x)m(x)U^2 > 0 \right\}.$$

We note that this result does not necessarily imply that there are other local minimizers. In fact, there is an example of $a(x)$ for which another local minimizer exists.

If the thin domain is not so constricted, then one of the local minimizers obtained in Theorem 3 becomes a global minimizer.

Theorem 4. ([4]) For the eigenvalue problem (EVP2), there exists an $R = R(\kappa, \mu) > 1$ such that if

$$\frac{\max\ a(x)}{\min\ a(x)} < R,$$

then

$$m(x) = \begin{cases} \kappa & \text{for } x \in (0, \alpha), \\ -1 & \text{for } x \in (\alpha, 1), \end{cases} \qquad a.e.$$

or

$$m(x) = \begin{cases} -1 & \text{for } x \in (0, 1-\beta), \\ \kappa & \text{for } x \in (1-\beta, 1), \end{cases} \qquad a.e.$$

is a global minimizer.

The value of the constant R is important, but its best constant is not clear.

Next, let us consider the case where Ω be a rectangular domain

$$\Omega = \{(x, y) \in (0,1) \times (0, a)\} \subset \mathbb{R}^2$$

and that $m(x, y)$ is positive on a strip-like domain given by

$$E = (0, c) \times (0, a).$$

Theorem 5. ([3]) *Fixing other parameters, there exists a critical value* $c^* > 0$ *such that if* $c < c^*$, *then the strip-like pattern is not local minimizer.*

Proof. We perturb the strip-like region E as follows by using a small parameter $\varepsilon > 0$:

$$E_\varepsilon = (0, c + \varepsilon \cos(j\pi/a)) \times (0, a), \quad j = 1, 2, \ldots$$

Expanding λ_ε and the associated eigenfunction ϕ_ε as

$$\lambda_\varepsilon = \lambda_0 + \varepsilon^2 \lambda_2 + o(\varepsilon^2),$$

$$\phi_\varepsilon = \phi_0 + \varepsilon^2 \phi_2 + o(\varepsilon^2),$$

substituting these expansions in the equation, and comparing termwise, then we can determine the sign of λ_2 as

$$c < {}^{\exists}c^* \iff \lambda_2 < 0 \implies \text{not a local minimizer} .$$

Based on this formal analysis, we can obtain a rigorous proof by using the Rayleigh quotient. □

Formal argument suggests that if $c > c^*$, then the strip-like pattern is a local minimizer, but it is difficult to verify it rigorously.

Theorem 5 implies that if the resource is not enough, then the strip-like pattern is not a local minimizer. By numerical computation with a projection gradient method, a strip-like pattern in another side or a disc pattern seems to be a global minimizer. Here the idea of the projection gradient method is to start from an initial guess for $m(x)$, evolve it along the gradient direction until it reaches the optimal configuration. Since the gradient direction may result in the violation of the constraint, a projection approach is used to project $m(x)$ back to the feasible set. Furthermore, we propose a new binary update for $m(x)$ to preserve the bang-bang structure. See [3] for more details.

Theorem 6. ([4]) *In a rectangular domain, any global minimizer and its associated eigenfunction are both monotone in x-direction and y-direction.*

The proof is based on spatial rearrangement. In fact, rearrangement in both x-direction and y-direction make the principal eigenvalue smaller.

The minimization problem in more general domains is an extremely difficult question. If we consider a singular limit of the problem, it may be possible to characterize the minimizer. Even if we cannot characterize a global minimizer, it is desired to make clear its properties. It is conjectured that if the domain is convex, then the set E is simply connected, but it is still open.

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