A Study on Optimal Maintenance Policies for Deteriorating Queueing Systems

Junji Koyanagi

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Preface

As service systems are used by many people, once the failure occurs, the damage may be enormous. To prevent the failure, therefore, preventive maintenance should be considered. However, the maintenance operations should not be applied frequently, because the customers in the systems are not served or even lost during the maintenance period. The problem of performing the maintenance suitably has been dealt with in queueing theory and reliability theory; the decision is made on the basis of only queue length in queueing theory and on the basis of only deterioration level in reliability theory. However, in service systems, it is important to make a decision with consideration on both the number of users who are troubled by the maintenance and the deterioration level of the system. For example, in a computer network system, more than one user is concurrently served, and the number of users changes as the time goes on. Since there may be bugs in programs, we sometimes update the program to remove such bugs. If the program is needed to operate the computer, especially if it is an operating system, we must stop the computer to update the program. To avoid many users being troubled by stopping the computer, we usually update the program while there are only few users, or we do not update the program if the defect is minor. If the defect is major, we must update the program immediately even if we lose several users. The decision when to update the program has, therefore, crucial influence on the system performance.

In this thesis, we propose some models to analyze the optimal maintenance policy in such situations. The aim of this thesis is to characterize the structure of the optimal policy. We mainly deal with the *switch curve struc-*

ture, which indicates that the preventive maintenance should be performed if the queue length is shorter than a threshold which is increasing with the deterioration level. Though the existence of such switch curve structure seems to be obvious, we need some mathematical conditions to prove it rigorously. The switch curve structure and other properties proved in this thesis are helpful to compute the optimal policy in such systems.

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Junji Koyanagi
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Contents

Chapter 1

Introduction

1.1 Introduction

Service systems such as telecommunication systems, e-mail systems and ATM systems at banks support our lives. In service systems, if service time and inter-arrival time are deterministic and service time is less than inter-arrival time, customers does not wait. However, in real service systems, the inter-arrival time and/or service time are stochastic and the number of servers is finite. In this case, the customers may have to wait because it is possible that more customers than servers may arrive in a service time. For the analysis of these service systems, queueing theory has been studied to evaluate various performance measures such as waiting time distribution, expected queue length and loss probability of customers when the number of waiting customers is restricted.

Though it is desirable that service system operates all the time, services in a real system may be interrupted by breakdowns such as hard disk crashes in a computer system. As a result of breakdowns, the customers in the system may be lost or have to wait until the system is recovered. Since a goal of the service system is to serve as many customers as possible or to serve customers as fast as possible, the loss of customers or the delay of the service should be avoided.

A preventive maintenance such as hard disk replacement in a computer system may be useful to decrease the loss or delay. Some of the systems can be maintained while no customers use the system. For example, an ATM system can be maintained after the bank is closed. However, in a system with many users such as a web server, it is hard to find the time with no users in the system. Even if the system is empty at the start of maintenance, there are some customers who arrive during the maintenance. Therefore some customers are lost or kept waiting because most of the systems cannot serve the customer during the maintenance.

For an appropriate maintenance in such systems, we need to know two stochastic processes, deterioration process and queue length process. The deterioration process tells us how old the server is and when it is likely to break down. The queue length process tells us how many customers exist in the system and will exist in the system in the future. The queue length includes the customers in service throughout this thesis. Thus two processes tell us when the server will fail and how many customers will be troubled by the failure. However, the traditional reliability theory has considered only the preventive maintenance problems that contain deterioration process. There are few papers dealing with a maintenance problem of a deteriorating system with another stochastic process such as queue length process as we review in Section 1.2. Furthermore, in the maintenance problems studied in queueing theory, the time to failure is assumed to be exponentially distributed. With this assumption, the preventive maintenance is not necessary because the time to the next failure is independent of the operation time, which means the used system is as good as new. Thus, no preventive maintenance has been considered and only the queue length upon failure is concerned with the maintenance problems of queueing systems. The maintenance problems of queueing systems are reviewed in Section 1.3.

In this thesis we analyze some maintenance problems, in which the maintenance decision is made on the basis of the information of two stochastic processes, queueing process and deteriorating process. In a real system we tend to maintain the machine while it has less customers because the trouble caused by the maintenance are smaller than the trouble caused by the failure. Therefore the maintenance decision is often made by considering both

the deterioration level of the machine and the number of users. We discuss such problems in Section 1.4.

In maintenance problems, the decision is made on the basis of the *state*, which is the relevant information needed to describe the current condition of the system. The objective is to determine the optimal policy, which tells us the optimal action for each state. However, it is rare to obtain the optimal policy without numerical calculation. In the infinite state problems, which often appear in the queueing system when the queue length is not restricted, even the numerical calculation is hard to carry out because the memory size of the computer is restricted. Thus, without numerical calculation, several papers have discussed the properties of optimal policies. Various structures of optimal policies are introduced in Section 1.5. In this thesis we mainly discuss the switch curve structure, which often appears in problems with two state variables, queue length and deterioration level in our case.

Maintenance problems in reliability and queuing theory are briefly reviewed in Section 1.2 and Section 1.3, respectively. The maintenance models dealt with in this thesis are introduced in Section 1.4. The typical structures of optimal policies in queueing systems are introduced in Section 1.5. The outline of this thesis is given in the last section.

1.2 Maintenance problems in reliability theory

There are useful machines such as cars and computers. Unfortunately, these machines sometimes fail and we suffer from those failures. Reliability theory aims to avoid the failures as much as possible or to make the damage as small as possible. For this purpose, various theoretical models have been studied, among which several typical models are discussed in Barlow [2]. In these models, the damage by the failure is considered as a cost. In addition, the operating cost is incurred while the system works, and preventive maintenance cost is charged when it is performed.

One objective of reliability theory is to minimize the total cost, which includes preventive maintenance cost, operating cost, corrective maintenance cost, and so forth. One way to minimize the cost is to perform the preventive maintenance on the basis of either the age of the system or the system state, where the age means the elapsed time since the system became as good as new, and the system state is assumed to indicate the deterioration level of the system. As a preventive maintenance policy based on age, the age replacement is introduced in Barlow [2]. Under the age replacement, the system is replaced when the age of the system reaches the planned time T which is determined in advance. If the system fails before T, it is also replaced at the replacement cost which is more expensive than that for the planned replacement. Given the failure distribution function $F(t)$ (the probability that the system fails at age less than t), the optimal value of the planned time T that minimizes the total cost is obtained by renewal theory (Çinlar [5]).

The idea of the age maintenance is also applied to backup computer hard disks. To restore the system at a hard disk failure, it is required to keep the files on hard disks in a computer system by copying them on other media such as magnetic tapes. However, it is not desirable to backup the files very frequently because a backup takes much time and cost. Thus the optimal backup policy has been discussed in several papers. Sandoh[27] studies the optimal backup policy to maximize availability, which is the expected fraction of the time in the long run that the system operates satisfactory. The minimization of the expected cost per unit time over an infinite time span is also studied in Sandoh[28].

There is another type of reliability model which is based on semi-Markov process and we call it semi-Markovian degradation model. In this model, the system state which indicates deterioration level of the system is usually denoted by a discrete number. The system begins to operate from the initial state, which is usually the state as good as new. After a random time period elapses, the system state changes to another state. If this random time period is exponentially distributed, the model is called a Markovian degradation model. There is one absorbing state called failure state in the semi-Markov process. If the system state becomes the failure state after some transitions, the system fails and needs to be repaired.

This reliability model can be adopted if we can observe whether the system is about to fail or in a good state by inspecting the system. For example, in the maintenance of a building, the number of cracks on the wall may be a suitable measure of the system state because many cracks in the wall indicate that the building may collapse soon but no crack indicates that it is as good as new. If there are many cracks, the building will be rebuilt, which is considered as a preventive maintenance. As we see in the above example, the maintenance in a semi-Markovian degradation model is usually performed by observing the system state.

Kawai [11] studies a Markovian degradation model in which a spare of the system is needed to replace the system upon failure or to apply a preventive maintenance. To keep one spare for the above purpose, the holding cost is incurred, indicating that it is costly to always keep the spare. However, if the spare is not at hand, it takes a random time before it is delivered after ordering. Therefore the damage caused by a failure without a spare is larger than the case of a failure with a spare. Thus it is important to determine whether to order the spare or not (ordering policy) in the states with no spare and whether to replace the system or not (replacement policy) in the states with a spare. It is shown that the optimal policy is (n, N)-policy type under a condition which implies that the system is more degraded as the state number becomes large. The (n, N)-policy means that a spare should be ordered if the state number is higher than n and there is no spare, and the replacement should be performed if the state number is higher than N and there is a spare.

In the above model, the system state is assumed to be completely observable. However, in some real systems, the system state is not always observable. In such models, inspections that find out (or estimate) the system state are conducted. To determine when to inspect, the optimal inspection policy is also important to study as well as the optimal replacement policy. Kawai [12], Mine [21], and Ohnishi [22] deal with the optimal inspection policy in incomplete information models.

1.3 Queueing systems with breakdowns

In queueing theory, main focus has been given to the analysis of such measures as the expected waiting time and the expected queue length. Wolff [38] discusses several queueing models with detailed explanations and their applications. The analysis of queueing systems is useful in designing various queueing systems; for example, to determine the number of ATMs in a bank, and to determine the buffer size in a computer system. These design problems are considered as static control problems.

In queueing systems, dynamic control problems such as service control, arrival control and customer assignment in a multiserver system are also studied in order to operate the system efficiently. The control of the number of ATMs in a bank is a typical example of the service control problem. In a bank, the number of ATMs is usually designed to serve the customers in busy hours. Thus all ATMs will work during busy hours, but in other hours, not all ATMs are necessary to serve customers. In such case, some of ATMs can be turned off, which makes the operating cost cheaper compared with uncontrolled systems.

An assignment problem in a parallel queueing system is also familiar to us. A parallel queueing system is a multiserver system in which each server has its own queue and all servers are identical in their service ability. Each customer joins one of the queues upon arrival and cannot change the queue until the end of service. Although it seems natural to join the shortest queue (shortest queue discipline) to minimize the waiting time or total number of customers in the system, it may not be true if the service time distribution is general (Whitt [36]). Winston [37] shows shortest queue discipline is optimal if the service time in each server is exponentially distributed. Weber [34] proposes an assignment policy which sends the arriving customer to the queue with the least expected waiting time. This policy is optimal if the service time has an increasing hazard rate. The increasing hazard rate is defined to be that the probability of the service end is increasing as the elapsed service time increases. Johri [10] considers the system in which the service time depends on the queue length. It is shown the shortest

queue discipline is also optimal for this system under some mathematical conditions. Other various control problems are discussed in Walrand [33].

Although usual control problems assume that servers do not fail or are available anytime, Awi [1] deals with a server subject to breakdown. The system is subject to breakdowns according to a Poisson process when the system has customers, and according to another Poisson process when it is empty. Upon breakdowns, a decision between fast (and expensive) repair and slow (and cheap) repair must be made to recover the system. The customers are kept waiting during the breakdown. The operating cost is incurred while the system is working, and the holding cost per customer per unit time is also incurred. Under some conditions of holding cost and operating cost, it is proved that a threshold policy is optimal; i.e., if the system has more customers than the threshold upon breakdown, the fast repair should be chosen, otherwise, the slow repair should be chosen.

As we reviewed the maintenance problem in reliability theory in Section 1.2, preventive maintenance is usually considered in reliability models. However, in Awi [1], no preventive maintenance is considered because the failure time distribution is assumed to be an exponential distribution. Since exponential distribution indicates that the distribution of the time to failure for a used system is the same as the one for a new system, it is no use to replace the system before failure. Instead of considering preventive maintenance, in queueing system, the damage by a failure is considered to depend on the number of customers upon the failure, which is not considered in reliability theory.

There are similar topics in stochastic scheduling problems. In scheduling problems there are several jobs in the queue. For each unfinished job in the queue, the holding cost is incurred. To decrease the holding cost, one of the jobs in the queue is selected and processed at the server for a stochastic service time. After the service is finished, the job is removed from the queue. Since the jobs are different in terms of holding cost and service time distribution, the total expected cost depends on the order of processed jobs. Thus, the objective is to determine the optimal sequence of jobs to minimize the total expected cost.

If the server fails in the above scheduling system, the holding cost is also incurred during the failure, which indicates that a breakdown while many jobs are in the queue is very costly. Glazebrook [8] deals with these scheduling problems for discrete and continuous time models in which a failure occurs after the time with a Bernoulli distribution in discrete time model and with an exponential distribution in continuous time model, respectively. It is proved that the optimal policy is an index policy. The index policy means that the index for each job can be calculated from the holding costs, the service distribution functions of all jobs and other parameters such as discount factor, and the job with the biggest index in the queue should be served at the server. Pinedo [24] discusses similar models under various cost criteria. These models are further discussed in Birge [3], [4]. In stochastic scheduling problems, the failure distributions are Bernoulli or exponential. Therefore preventive maintenance is not considered for the same reason as in a queueing system.

1.4 Preventive maintenance in queueing systems

As we reviewed in Section 1.2 and Section 1.3, no preventive maintenance policy which considers both the queueing process and the deteriorating process has been proposed. This thesis discusses the optimal preventive maintenance problems in deteriorating queueing systems.

In service systems, the server deterioration may yield inconveniences such as the increase of failure probability and the slow service. Therefore the maintenance should be performed to renew the system. It is usually undertaken in such a manner that the loss caused by the maintenance becomes as small as possible. In this thesis, the loss refers to the users who cannot use the system due to the maintenance and the failure. For examples, in a road maintenance, each car must choose other inconvenient roads during the maintenance and it brings in a loss. Thus the maintenance is usually planned at night because the traffic is less than that during daylight. In a

computer system, the maintenance should be planned while the numbers of users and jobs in the system are small, because the users and the jobs will be lost or halted by the maintenance.

Though the maintenance without hazarding customers is ideal, if the system is about to fail, we cannot wait for the system to become empty to avoid the large loss caused by the failure. From this point of view, this thesis proposes the optimal maintenance policy which considers both stochastic processes, queueing process and deterioration process. Our main interest is to analyze how the optimal action of maintenance changes as the observed data of the two processes changes. Typical properties of an optimal policy have been studied in various systems. We review the study of such properties in the next section.

1.5 Structures of optimal controls

In controlling queueing systems, simple properties usually exist between the state and the optimal action. These relations may be helpful to find out or calculate the optimal policy. For examples, the shortest queue discipline in parallel queueing systems and the index policy in scheduling problems determine the optimal policy. Although it is desirable that the optimal policy is obtained explicitly, it is impossible in most cases to obtain the optimal policy from the stated properties. Thus the properties which show us rough structure of optimal policies are important from the theoretical point of view.

A common property that holds in many systems is the threshold property. Lin [20] and Walrand [32] study the threshold policy in a slow server system, in which there are two servers, a fast one and a slow one. In each server, the service time distribution is exponential, and the fast server has larger service rate than the slow server. Customers whose arrival process is assumed to be Poisson form a single queue, and the customer at the top of the queue can be assigned to either the fast or slow server, or kept waiting. Once the customer is assigned to either of the servers, he/she must stay there until the

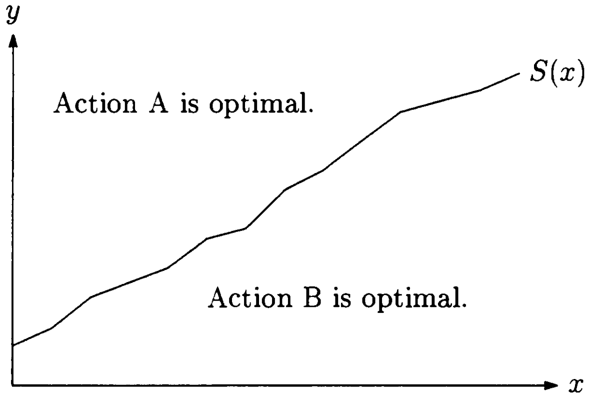

Figure 1.1: The switch curve structure

end of service. For this model, Lin [20] and Walrand [32] study the optimal policy to minimize the total expected discounted holding cost. They prove that the customer should be assigned to the fast server whenever it is idle, while the customer should be assigned to the slow server only when the fast server is occupied and queue length exceeds the threshold. In this problem, the utilization of the fast server and slow server depends on the queue length only.

In the threshold policy, the decision depends on the one variable (i.e., the queue length in the above case). If the decision between two actions (say A and B) is concerned with two variables (say x and y), the switch curve structure of optimal policy is usually argued. The switch curve structure indicates that for each fixed value $x = x'$ there is a threshold $S(x')$; i.e., action A is optimal for all (x', y) with $y \geq S(x')$ and action B for all (x', y) with $y < S(x')$. This $S(x)$ is an increasing function of x or could be parallel to y axis (Fig. 1.1). The function $S(x)$ is called the switching curve. From the switch curve structure, we can know that the changes of optimal action only happen at most once in the direction of each axis.

Hajek [9] deals with the switch curve structure in a control problem of a queueing network with two servers (Fig. 1.2), referred to as server 1 and server 2.

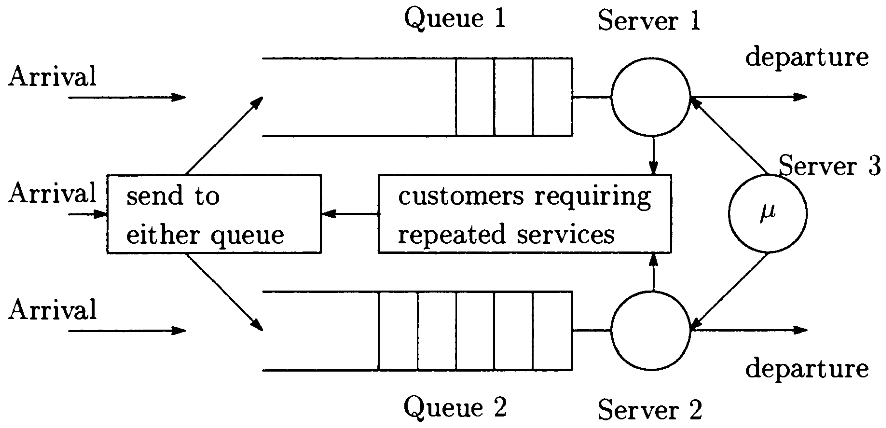

Figure 1.2: The queueing network in Hajek [9]

Each server has its own queue, queue 1 for server 1 and queue 2 for server 2. The service time distribution is exponential and the holding cost depends on the queue lengths of queue 1 and queue 2. For this system, three controls are considered, the assignment of arrivals, the service rate control and the assignment of the customer requiring a repeated service. These controls aim at minimizing the total expected discounted holding cost over the infinite time horizon.

Firstly, let us consider the assignment control of the arrivals. The arrival process consists of three Poisson processes; each process may have a different rate from other processes. The arrivals in the first process must be assigned to queue 1, and the ones in the second one to queue 2. However, the arrivals in the third one can be assigned to either queue 1 or queue 2. Hajek [9] shows that the optimal assignment policy of this arrivals has a switch curve structure if the holding cost is linear with respect to two queue lengths. In detail, the x and y axes in the state space corresponding to queue 1 length and queue 2 length, respectively, is divided by a switching curve. The customer should be sent to queue 1 when the state is in the left upper side of the curve (i.e., queue 2 is too long compared with queue 1), and should be sent to queue 2 when the state is in the right lower side of the curve.

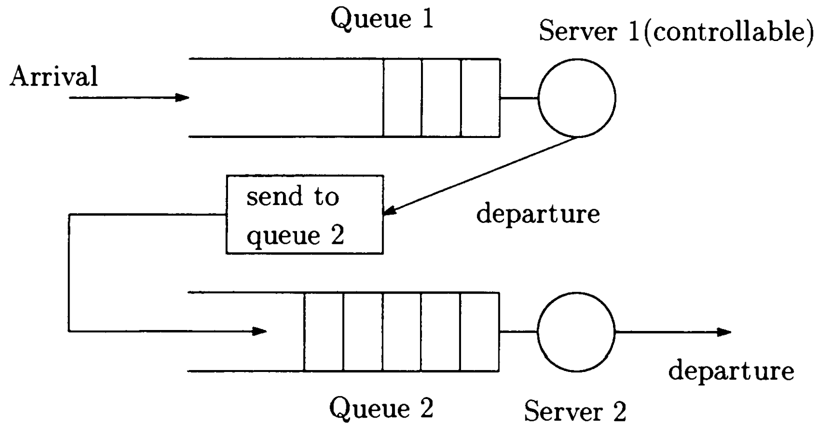

Figure 1.3: The tandem queue in Rosberg [25]

We next explain the service rate control. If no control is applied to the system, the servers process the customer with the given service rate (the service rate may be different from the other). The service rate control can increase the service rates of two servers, but the sum of the increases of two service rates is restricted to be less than constant μ. The optimal service rate control also has a switch curve structure, that is, if the state is in the right lower side of the curve, we should give the maximum increase μ to server 1 (no increase to server 2), and vice versa. The switching curve for this control is generally different from the one for the arrival control.

The last control deals with the customers who require repeated services. In this system, some customers reenter the system after his service. These customers must be assigned to queue 1 or queue 2. It is shown that the optimal assignment policy is similar to the arrival control.

Rosberg [25] discusses the optimal control in a tandem queue system (Fig. 1.3). In the tandem queue system, two queueing systems are connected in series. Customers arrive at the first queue (queue 1), and after the service in the first server, they are sent to the second queue (queue 2). They depart from the system after the service in the second server is over. Rosberg [25] discusses the case in which the arrival to queue 1 is governed by a Poisson process, and both service distributions are exponential. The service rate of

the first server is controllable in the interval $[0, \mu]$. The holding cost depends on two queue lengths, and the objective is to minimize the total expected discounted holding cost. It is shown that the optimal policy has the switch curve structure, which means that the service rate should be 0 when the state is in the left upper side of the switching curve and μ in the other side, where x axis and y axis represents the lengths of queue 1 and queue 2 respectively. See Veatch [31] and Glasserman [7] for more general control problems.

1.6 Outline of the thesis

In this thesis we deal with five maintenance models. We first introduce the basic approach for our models in Chapter 2. Semi-Markov decision process, value iteration method and uniformization are explained there.

A simple combination of the queue length process and the deterioration process is an $M/M/1$ queueing system and a server with a failure distribution function. The notation of $M/M/1$ means that the arrival process is a Poisson process, the service time distribution is exponential and one server exists in the system. Chapter 3 deals with this model in which the server fails after a random time and the maintenance can be performed periodically. This chapter is adapted from Koyanagi [15].

From Chapter 4 to Chapter 7, the deterioration process is expressed by a Markov process. The combination of an $M/M/1$ queueing system and Markovian degradation process is easily extended to the model in Chapter 5. Before Chapter 5, we consider another extension in Chapter 4, which is adapted from Koyanagi [16]. Namely we discuss a maintenance policy for $M/G/1$ queueing system, which means that the service distribution is allowed to be general. Since general distribution includes exponential distribution, it is an extension of $M/M/1$ queueing system. Though the decision epochs in this model are more restricted than those in $M/M/1$ queue, the problem in Chapter 4 is considered as an extension of a maintenance problem of $M/M/1$ queue.

In Chapter 3 and Chapter 4, the deterioration is assumed to affect only the failure probability. From Chapter 5 to Chapter 7, we discuss the models, where the deterioration affects the queueing process as well.

In Chapter 5, the maintenance problem of $M/M/N$ queueing system is discussed. We assume that the service rate is affected by both the server state and the queue length. It is natural to consider that the service rate becomes slower as the server deteriorates. We consider such a model in Chapter 5, which is adapted from Koyanagi [18].

In Chapter 6 and Chapter 7, we assume that the server deterioration affects the arrival rate, but that the system never fails. This model is considered as a replacement problem of an amusement machine. As the time goes, the customers who enjoy the machine will decrease because the machine becomes out-of-date. The system charges fees from customers, and the maximization of the collected fees is discussed. To regain the customers, the old machine is replaced to the new one. In Chapter 6, the customers in the queue can be cancelled by paying a cancel fee if there are customers upon maintenance. In Chapter 7, we deal with the case in which the customers in the queue can not be cancelled; i.e., in performing the maintenance, the system is first closed, serve all customers in the queue and then the maintenance is performed. These chapters are adapted from Koyanagi [19] and [17], respectively.

In these chapters, it is shown that the optimal maintenance policy in each case has a switch curve structure. Other properties are also discussed in Chapters 5, 6 and 7.

Chapter 2

Elements of Semi-Markov Decision Processes

In our problems, semi-Markov decision processes are often used to determine the optimal policy. This chapter provides a summary of the properties of semi-Markov decision processes, and explains the value iteration method and uniformization (Serfozo [29]) which are used to obtain an optimal policy.

2.1 Semi-Markov decision process

A semi-Markov decision process is specified by 'state set', 'action set', 'transition probability' and 'cost function'.

State set S: The state set S is the set of the system state, where state is the information relevant to making decision.

Action set A_s: The action set A_s is the set of actions which can be taken in state $s \in S$.

Transition probability $Q^a_{ss'}(t)$: The transition probability $Q^a_{ss'}(t)$ specifies the transition probability to state s' within t time when the action a is taken in state s.

Cost function $C^a_s(t)$: The cost function $C^a_s(t)$ is the expected cumulative cost when the action a is taken in state s and transition time is t. Typically, $C^a_s(t)$ is expressed by

$$C^a_s(t) = f^a_s + g^a_s t, \tag{2.1}$$

where f_s^a is the cost incurred upon the decision epoch and g_s^a is the cost per unit time incurred until the next transition happens.

In this thesis, we assume that the state set S is a discrete and m-dimensional vector space. For $s = (s_1, s_2, \ldots, s_m) \in S$, we assume s_k is a nonnegative integer. The action set A_s is also assumed to be discrete and finite. The transition probability $Q_{ss'}^a(t)$ is often expressed in the form

$$Q_{ss'}^a(t) = \int_0^t P_{ss'}^a(x) dF_s^a(x), \tag{2.2}$$

where $F_s^a(x)$ is the distribution function of the transition time for action a in state s, and $P_{ss'}^a(x)$ is the conditional transition probability from state s to s', given the transition time x.

By these items, we can define a semi-Markov decision process as follows.

(1) The process is observed at time 0, and the initial state $s \in S$ is identified.

(2) For the observed state s, action a is selected from the action set A_s.

(3) The next state becomes s' within t time with probability $Q_{ss'}^a(t)$, after taking action a in state s.

(4) The cost incurred is given by the cost function $C_s^a(x)$.

(5) The above steps are repeated indefinitely after the transition to state s'.

In an infinite time horizon problem, the discounted cost is usually considered. In the discounted cost problem, the one unit cost after time x is evaluated as $e^{-\alpha x}$ at time 0, where α is called discount rate. Then, the cost function in the discounted cost problem is evaluated by

$$\int_0^t e^{-\alpha x} dC_s^a(x). \tag{2.3}$$

To minimize the total expected discounted cost, the optimal policy is determined by solving the following optimality equation:

$$\begin{aligned} V(s) &= \min_{a \in A_s} \Big\{ \sum_{s' \in S} \int_0^\infty \int_0^t e^{-\alpha x} dC_s^a(x) dQ_{ss'}^a(t) \\ &\quad + \sum_{s' \in S} \int_0^\infty e^{-\alpha t} V(s') dQ_{ss'}^a(t) \Big\} \\ &= \min_{a \in A_s} \Big\{ r_s^a + \sum_{s' \in S} q_{ss'}^a V(s') \Big\}, \end{aligned} \tag{2.4}$$

where

$$r_s^a \equiv \sum_{s' \in S} \int_0^\infty \int_0^t e^{-\alpha y} dC_s^a(y) dQ_{ss'}^a(t), \tag{2.5}$$

$$q_{ss'}^a \equiv \int_0^\infty e^{-\alpha t} dQ_{ss'}^a(t). \tag{2.6}$$

This equation can be solved by various methods. Here, we introduce the value iteration method.

2.2 Value iteration method

The value iteration method updates the value $V^n(s)$ as follows.

Step 0. Give the initial value $V^0(s)$ for each $s \in S$.

Step 1. Calculate $V^{n+1}(s)$ by

$$V^{n+1}(s) := \min_{a \in A_s} \Big\{ r_s^a + \sum_{s' \in S} q_{ss'}^a V^n(s') \Big\}. \tag{2.7}$$

Step 2. $n := n + 1$ and return to Step 1.

There are various sets of conditions to secure the convergence of this iteration. To secure the convergence $\lim_{n \to \infty} V^n(s) = V(s)$, we assume that for all $s, s' \in S$ ($s = (s_1, s_2, \ldots, s_m)$, $s' = (s'_1, s'_2, \ldots, s'_m)$), the process satisfies the next conditions.

Condition 2.1

There are constants $C_k \geq 0$ $(k = 0, \ldots, m)$ *and* $0 < \beta < 1$ *that satisfy*

(1) $|r^a_s| \leq C_0 + \sum_{k=1}^{m} C_k s_k$ *for all* s, a.

(2) $\sum_{s' \in S} q^a_{ss'} \Big(C_0 + \sum_{k=1}^{m} C_k s'_k\Big) \leq \beta \Big(C_0 + \sum_{k=1}^{m} C_k s_k\Big)$ *for all* s, a.

(3) $|V^0(s)| \leq C_0 + \sum_{k=1}^{m} C_k s_k$ *for all* s.

Condition 2.1(1) states that the cost between transition increases at most linearly in s_k, and Condition 2.1(3) states that the initial value $V^0(s)$ should have the same property. Condition 2.1(2) assures that the mapping from $V^n(\cdot)$ to $V^{n+1}(\cdot)$ is a contraction mapping with respect to weighted supremum norm (Wessels[35]).

In this thesis, we deal with the decision process with two-dimensional state $s = (s_1, s_2)$. The variable s_1 indicates the queue length and s_2 indicates the deterioration level. To satisfy Condition 2.1, we assume the following conditions.

Condition 2.2

(1) *Arrival process is assumed to be a Poisson process and the cost between decision epochs increases by* 1 *per lost arrival.*

(2) *For all* (s_1, s_2), *the cost* f^a_s *incurred upon the decision epoch satisfies* $f^a_s \leq C + s_1$. *This means that the cost increases at most 1 as the queue length* s_1 *increases by 1, and has an upper bound with respect to* s_2.

(3) *If the transition time has a general distribution* $H(x)$, *we assume* $H(0) = 0$ *and*

$$0 < \varepsilon \leq \int_0^{\infty} t dH(t) \leq \tau < \infty. \tag{2.8}$$

This means the expected transition time is positive and finite, which is usually assumed for semi-Markov decision process.

Let us denote the arrival rate by λ, which means the expected number of arrivals is λt for time t. Then the expected number of lost customers and the expected increase of queue length are less than λt. Therefore, from Conditions 2.2(1), the cost function $C^a_s(t)$ satisfies the inequality

$$|C^a_s(t)| \leq C + s_1 + \lambda t. \tag{2.9}$$

Furthermore, for $s' = (s'_1, s'_2)$,

$$\sum_{s' \in S} P_{ss'}(t) s'_1 \leq s_1 + \lambda t. \tag{2.10}$$

Using inequalities (2.8), (2.9) and (2.10), we can check Condition 2.1(1) and Condition 2.1(2) as follows.

(1) Condition 2.1(1) is satisfied because

$$\begin{aligned} |r^a_s| &\leq \sum_{s' \in S} \int_0^{\infty} \int_0^{t} e^{-\alpha y} dC^a_s(y) dQ^a_{ss'}(t) \\ &\leq \sum_{s' \in S} \int_0^{\infty} (C + s_1 + \lambda t) dQ^a_{ss'}(t) \\ &\leq C + s_1 + \lambda \tau. \end{aligned}$$

By defining $C_0 = C + \lambda\tau$, $C_1 = 1$ and $C_2 = 0$, Condition 2.1(1) is satisfied.

(2) Condition 2.1(2) is checked as follows.
First, there exists a ρ such that

$$\int_0^{\infty} e^{-\alpha t} dH(t) \leq \rho < 1$$

because

$$0 < \varepsilon \leq \int_0^{\infty} t dH(t) \leq \tau < \infty.$$

Next, for $s' = (s'_1, s'_2)$,

$$\sum_{s' \in S} q^a_{ss'} \big(s'_1 + C_0\big)$$

$$
\begin{aligned}
&= \sum_{s'\in S}\int_0^\infty e^{-\alpha t}P_{ss'}(t)dH(t)\Big(s_1' + C_0\Big)\\
&\le \int_0^\infty e^{-\alpha t}\Big(s_1 + C_0 + \lambda t\Big)dH(t)\\
&\le \rho\Big(s_1 + C_0\Big) + \lambda\tau.
\end{aligned}
$$

By taking β as $\rho < \beta < 1$ and redefining $C_0 = (C + \lambda\tau)/(\beta - \rho)$, it is shown that

$$\sum_{s'\in S} q^a_{ss'}(s_1' + C_0) \le \beta(s_1 + C_0). \tag{2.11}$$

(3) Condition 2.1(3) is easily satisfied, for example, by taking $V^0(s) := 0$.

2.3 Uniformization

In a queueing system, it is common to assume exponential distribution for the arrival and service time. In a semi-Markov decision problems, if the transition time is exponential, the transition rate from state s to s' can be denoted by $\lambda^a_{ss'}$, for given states s, s' and the action a. This means that $Q^a_{ss'}(t)$ is expressed by

$$Q^a_{ss'}(t) = (1 - e^{-\Lambda^a_s t})\frac{\lambda^a_{ss'}}{\Lambda^a_s}, \tag{2.12}$$

where $\Lambda^a_s \equiv \sum_{s'\in S}\lambda^a_{ss'}$. We consider that the actions for each state s are numbered from 1 to ζ_s. We also assume that the transition time is exponential for actions $a = 1, 2, \ldots, \xi_s$, but is not exponential for actions $a = \xi_s + 1, \ldots, \zeta_s$. Eq. (2.4) then becomes

$$
\begin{aligned}
V(s) &= \min\Bigg\{\min_{1\le a\le\xi_s}\Big\{r^a_s + \sum_{s'\in S}\int_0^\infty e^{-\alpha t}V(s')dQ^a_{ss'}(t)\Big\},\\
&\qquad \min_{\xi_s+1\le a\le\zeta_s}\Big\{r^a_s + \sum_{s'\in S}\int_0^\infty e^{-\alpha t}V(s')dQ^a_{ss'}(t)\Big\}\Bigg\}\\
&= \min\Bigg\{\min_{1\le a\le\xi_s}\Big\{r^a_s + \sum_{s'\in S}\frac{\lambda^a_{ss'}}{\Lambda^a_s + \alpha}V(s')\Big\},
\end{aligned}
$$

$$\min_{\xi_s+1\le a\le\zeta_s}\Big\{r^a_s + \sum_{s'\in S}\int_0^\infty e^{-\alpha t}V(s')dQ^a_{ss'}(t)\Big\}\Bigg\}. \tag{2.13}$$

Since Λ^a_s depends on s and a, it is inconvenient to compare

$$r^a_s + \sum_{s'\in S}\frac{\lambda^a_{ss'}}{\Lambda^a_s + \alpha}V(s')$$

in terms of s and a. In this case, it is useful to apply uniformization to the original equation. Consider a modified optimality equation:

$$
\begin{aligned}
U(s) &= \min\Bigg\{\min_{1\le a\le\xi_s}\Big\{r^a_s\frac{\Lambda^a_s + \alpha}{\Lambda + \alpha} + \sum_{s'\in S}\frac{\lambda^a_{ss'}}{\Lambda + \alpha}U(s') + \frac{\Lambda - \Lambda^a_s}{\Lambda + \alpha}U(s)\Big\},\\
&\qquad \min_{\xi_s+1\le a\le\zeta_s}\Big\{r^a_s + \sum_{s'\in S}\int_0^\infty e^{-\alpha t}U(s')dQ^a_{ss'}(t)\Big\}\Bigg\},
\end{aligned} \tag{2.14}
$$

where Λ is a constant that satisfies $\Lambda > \sup_{s,a}\Lambda^a_s$. This equation is satisfied if we substitute $U(s) := V(s)$. We can check this by considering the optimal action a^* in Eq. (2.13).

Case 1. If $1 \le a^* \le \xi_s$ in Eq. (2.13), equation

$$V(s) = r^{a^*}_s + \sum_{s'\in S}\frac{\lambda^{a^*}_{ss'}}{\Lambda^{a^*}_s + \alpha}V(s') \tag{2.15}$$

and inequalities

$$V(s) \le r^a_s + \sum_{s'\in S}\frac{\lambda^a_{ss'}}{\Lambda^a_s + \alpha}V(s'), \tag{2.16}$$

$$V(s) \le r^a_s + \sum_{s'\in S}\int_0^\infty e^{-\alpha t}V(s')dQ^a_{ss'}(t) \tag{2.17}$$

for $a \ne a^*$ holds, because a^* is the optimal action.

From these relations, it is obvious

$$V(s) = r^{a^*}_s\frac{\Lambda^{a^*}_s + \alpha}{\Lambda + \alpha} + \sum_{s'\in S}\frac{\lambda^{a^*}_{ss'}}{\Lambda + \alpha}V(s') + \frac{\Lambda - \Lambda^{a^*}_s}{\Lambda + \alpha}V(s) \tag{2.18}$$

and

$$V(s) \leq r_s^a \frac{\Lambda_s^a + \alpha}{\Lambda + \alpha} + \sum_{s' \in S} \frac{\lambda_{ss'}^a}{\Lambda + \alpha} V(s') + \frac{\Lambda - \Lambda_s^a}{\Lambda + \alpha} V(s), \quad (2.19)$$

$$V(s) \leq r_s^a + \sum_{s' \in S} \int_0^\infty e^{-\alpha t} V(s') dQ_{ss'}^a(t) \quad (2.20)$$

for $a \neq a^*$. Thus, $V(s)$ satisfies Eq. (2.14).

Case 2. If $\xi_s + 1 \leq a^* \leq \zeta_s$, the proof is similar to Case 1.

Thus, we obtain the solution of Eq. (2.13) by solving Eq. (2.14) if the solution is unique. Since the unique convergence is guaranteed by the contraction mapping, we can obtain the optimal value $V(s)$ and optimal policy by applying the value iteration method to Eq. (2.14) instead of Eq. (2.13).

These results are used in the subsequent chapters of this thesis.

Chapter 3

An $M/M/1$ Queue under Periodic Observation

3.1 Introduction

This chapter studies an optimal maintenance policy for an $M/M/1$ queueing system under a periodic observation. Customers arrive at the system in a Poisson stream and form a single queue. They are served at the server and their service times are exponentially distributed. For this queueing system, we assume that the server fails after a random time. The system is checked by regularly timed observations, and the failure can be detected upon the observation epoch. Upon the detection of the failure, the corrective maintenance which takes a random time starts to recover the system. The customer is lost whenever the server does not work. Thus the failure could produce a larger loss of the customer because the custome upon the failure, during the failure (until it is detected) and during the maintenance are lost.

However, it is possible to avoid the failure and the subsequent corrective maintenance by performing a preventive maintenance upon the observation time. Although the customers upon the start of the maintenance and during the maintenance are lost, the loss may become smaller by the preventive maintenance than that caused by the future failure. Since the queueing system should serve as many customers as possible, we consider to minimize the number of the lost customers. For this purpose, in this thesis, we formulate this problem as a semi-Markov decision process and determine the optimal maintenance policy. The decision is based on the number of customers in the system and the age of the server. We then derive a switch curve structure of the optimal policy under some conditions.

The organization of this chapter is as follows. In the next section we provide a detailed explanation of our model. In Section 3.3 we formulate our problem as a semi-Markov decision process. In Section 3.4 we derive properties of the value functions. In Section 3.5, the switch curve structure of the optimal policy is derived from the properties obtained in Section 3.4. The last section concludes our results.

3.2 Model

We consider an $M/M/1$ queueing system with arrival rate $\lambda(>0)$ and service rate $\mu(>0)$. The server fails after a random time and the failure time distribution function is denoted by $F(x)$. We assume it has density function $f(x)$. The queue length process and the failure process is assumed to be independent. To prevent a failure and to recover from a failure, we have a preventive maintenance action and a corrective maintenance action, respectively. The server is observed at periodic interval T measured from the instant when the server begins operating or restarts after either the preventive or the corrective maintenance. The opportunity of maintenance comes upon an observation epoch. If the server fails, we detect the failure upon the next observation and start the corrective maintenance. If a failure occurs, we lose all the customers who are in the system upon the failure or arrive at the system while the system is in failure or the corrective maintenance is taken place (Fig. 3.1). If the server is working upon the observation time, preventive maintenance can be performed, though the customers are lost upon the start of the maintenance and during the maintenance (Fig. 3.2).

The distribution functions of the preventive and the corrective maintenance time are denoted by $H_1(x)$ and $H_2(x)$, respectively. The system becomes new by maintenance, and the next observation time is measured from the end of maintenance. We consider that the unit cost is incurred for each lost customer, and we minimize the total expected discounted cost with

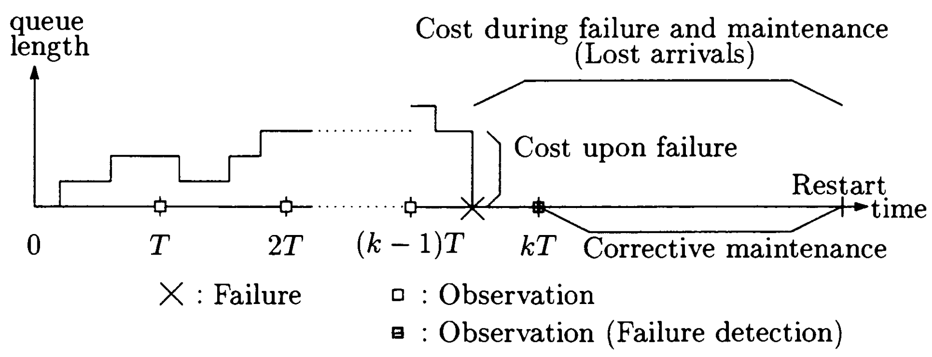

Figure 3.1: Corrective maintenance case

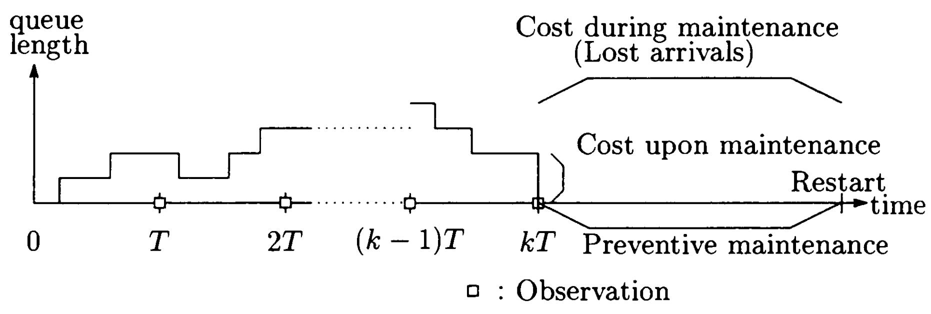

Figure 3.2: Preventive maintenance case

discount factor α.

3.3 Formulation

We formulate our problem as a semi-Markov decision process. While the system is working, the decision epoch comes every T time, which is the observation epoch. The end of the maintenance, i.e, the restart of the system is also a decision epoch. Upon an observation epoch, if state (i, k) is observed, it shows that the queue length is i (including the customer in service) and observation time kT has been measured from the epoch when the server started to work. For convenience, we denote the failure state by $k = \infty$. The decision is made on the basis of the state upon the observation.

Upon an observation epoch, we may find that the server is failed. In this case, we only take the corrective maintenance action. If the server is working upon the observation, we choose whether performing the preventive maintenance or continuing service to minimize the lost customers.

The lost customers are

(1) the customers who were in the system upon the failure or arrived during the period of failure and corrective maintenance,
(2) the customers who were in the system upon the preventive maintenance or who arrived during the period of preventive maintenance.

We consider to determine the optimal action for state (i,k) to minimize the total expected discounted cost. The semi-Markov decision problem consists of the following elements.

State and Action

The state set is given by $S=\{(i,k)|i=0,1,\ldots,\text{ and } k=0,1,\ldots,\infty\}$. For a state $s\in S$, the action set A_s is $\{1,2\}$ for (i,k) $(k<\infty)$ and $\{3\}$ for (i,∞) where Action 1, Action 2 and Action 3 indicate the preventive maintenance, the service and the corrective maintenance, respectively.

Transition Probability

We consider two stochastic processes, queue length process and failure process. For the queue length process, $P_{ij}(x)$ denotes the probability that the j customers are in the system at time x, when i customers were in the system at time 0. For the failure process, the probability f_k that the system is working at time kT and fails before the next observation is expressed by

$$f_k \equiv \frac{\overline{F}(kT)-\overline{F}(kT+T)}{\overline{F}(kT)}, \tag{3.1}$$

where $\overline{G}(x)$ means $1-G(x)$.

Since two processes are independent, we have the following transition probabilities. If Action 1 is taken at (i,k) $(k<\infty)$, the next transition time has a distribution function $H_1(t)$ and the next state becomes $(0,0)$. If Action 2 is taken at (i,k) $(k<\infty)$, the next transition will occur after T time. The transition probability from (i,k) to $(j,k+1)$ is $P_{ij}(T)(1-f_k)$ and the transition probability from (i,k) to $(0,\infty)$ is f_k. Since the customers are lost upon and during the failure, the system is empty when we detect the system failure. At $(0,\infty)$, only Action 3 can be taken. The next transition time has a distribution function $H_2(t)$ and the next state becomes $(0,0)$.

Cost Function

The expected cumulative cost (lost arrivals) is λt when the system stops t time, because of the Poisson arrivals with rate λ. Thus, the expected discounted arrivals $N(t)$ for the time interval t is

$$N(t) \equiv \int_0^t e^{-\alpha x}\lambda dx = \frac{\lambda}{\alpha}(1-e^{-\alpha t}). \tag{3.2}$$

(1) For Action 1 in state (i,k), the expected cost until the next transition is

$$i+\int_0^\infty N(t)dH_1(t) = i+\lambda h_1, \tag{3.3}$$

where

$$h_m \equiv \int_0^\infty e^{-\alpha t}\overline{H}_m(t)dt \quad (m=1,2). \tag{3.4}$$

By Condition 2.2(3), $0<h_m<1/\alpha$ holds.

(2) For Action 2 in state (i,k), the expected cost until the next transition is

$$\int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_i(x)+N(T-x)]dx, \tag{3.5}$$

where

$$L_i(t) \equiv \sum_{j=0}^{\infty} jP_{ij}(t). \tag{3.6}$$

The function $L_i(t)$ indicates the expected queue length at time t, starting from queue length i at time 0,

(3) For Action 3, the expected cost until the next transition is

$$\int_0^\infty N(t)dH_2(t) = \lambda h_2. \tag{3.7}$$

For the queueing process, the following lemma should be noted.

Lemma 3.1
The following properties hold for $P_{ij}(t)$ and $L_i(t)$ of Eq. (3.6).

(1) $\frac{d}{dt}L_i(t) \le \lambda$.

(2) $\sum_{j=l}^{\infty}[P_{i+1j}(t) - P_{ij}(t)] \ge 0$ *for all* l.

(3) $L_{i+1}(t) - L_i(t) \le 1$.

Proof.
The first property is obvious from the definition of Poisson arrival, which means that the arrival probability for a small time interval Δt is $\lambda\Delta t + o(\Delta t)$.

For the second and the third properties, define $m_{ij}^{(n)}$ as follows:

$$m_{ij}^{(0)} \equiv \begin{cases} 1 & (i=j) \\ 0 & \text{(otherwise)}, \end{cases} \tag{3.8}$$

$$m_{ij}^{(1)} \equiv \begin{cases} \mu/(\lambda+\mu) & (i=j=0) \\ \lambda/(\lambda+\mu) & (j=i+1) \\ \mu/(\lambda+\mu) & (j=i-1) \\ 0 & \text{(otherwise)}, \end{cases} \tag{3.9}$$

$$m_{ij}^{(n)} \equiv \sum_{k=0}^{\infty} m_{ik}^{(n-1)} m_{kj}^{(1)}. \tag{3.10}$$

Then we have the following expression of $P_{ij}(t)$.

$$P_{ij}(t) = \sum_{n=0}^{\infty} m_{ij}^{(n)} e^{-(\lambda+\mu)t}\frac{((\lambda+\mu)t)^n}{n!}. \tag{3.11}$$

It is easy to show that $\sum_{j=k}^{\infty} m_{ij}^{(n)}$ is increasing in i for all k, and $\sum_{j=1}^{\infty}[jm_{i+1j}^{(n)} - jm_{ij}^{(n)}] \le 1$. Properties (2) and (3) are easily proved from these inequalities. □

Optimality Equation

To derive the optimality equation, we define the following functions with respect to (i,k).

$V(i,k)$: The optimal cost function for state (i,k).

$M(i,k)$: The cost function when we perform preventive maintenance upon a decision epoch, and operate the system optimally thereafter.

$W(i,k)$: The cost function when we continue service upon the decision epoch, and operate the system optimally thereafter.

$D(i,k)$: The optimal action for state (i,k),

$$D(i,k) = \begin{cases} 1 & \text{if preventive maintenance is optimal,} \\ 2 & \text{if continuing service is optimal.} \end{cases}$$

Through the standard use of semi-Markov decision process, we derive the following equations for this problem, where $\beta \equiv e^{-\alpha T}$:

$$\begin{aligned} M(i,k) &= i + \lambda h_1 + \int_0^\infty V(0,0)e^{-\alpha t}dH_1(t) \\ &= i + \lambda h_1 + (1-\alpha h_1)V(0,0), \end{aligned} \tag{3.12}$$

$$\begin{aligned} W(i,k) &= \beta(1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V(j,k+1) \\ &\quad + \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_i(x) + N(T-x)]dx \\ &\quad + \beta f_k[\lambda h_2 + (1-\alpha h_2)V(0,0)]. \end{aligned} \tag{3.13}$$

Since the preventive maintenance time does not depend on k, $M(i,k)$ is independent of k. Therefore, $M(i,k)$ is denoted by $M(i)$ in the rest of this chapter. The first term of Eq. (3.13) denotes the discounted cost in the case that the server does not fail until the next observation, and the second and third terms are the costs incurred before the failure detection and after the failure when the server fails.

By these functions, the values of $V(i,k)$ and $D(i,k)$ are obtained as follows:

$$V(i,k) = \min\{M(i), W(i,k)\}, \tag{3.14}$$

$$D(i,k) = \begin{cases} 1 & \text{when } M(i) < W(i,k) \\ 2 & \text{when } M(i) \geq W(i,k). \end{cases} \tag{3.15}$$

We apply the value iteration method of Section 2.2 to obtain $V(i,k)$.

Value iteration method

Step 0. $n := 0$ and $V^0(i,k) := 0$ for all i,k.

Step 1.

$$M^{n+1}(i) := i + \lambda h_1 + (1-\alpha h_1)V^n(0,0), \tag{3.16}$$

$$\begin{aligned} W^{n+1}(i,k) := {} & \beta(1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+1) \\ & + \beta f_k[\lambda h_2 + (1-\alpha h_2)V^n(0,0)] \\ & + \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_i(x) + N(T-x)]dx, \end{aligned} \tag{3.17}$$

$$V^{n+1}(i,k) := \min\{M^{n+1}(i), W^{n+1}(i,k)\}. \tag{3.18}$$

Step 2. $n := n+1$ and return to Step 1.

We derive some properties of the cost functions in the next section.

3.4 Properties of the cost function

The following conditions are assumed in this section to examine cost functions. The properties of the cost functions proved in this section are needed to prove the switch curve structure of the optimal policy in Section 3.5.

Condition 3.1
The distribution functions $F(x)$, $H_1(x)$ and $H_2(x)$ satisfy the following conditions.

(1) *The failure distribution function $F(x)$ has the IFR (increasing failure rate) property, i.e., $(\overline{F}(x) - \overline{F}(x+y))/\overline{F}(x)$ is increasing in x for all $y > 0$.*

(2) *The inequality $\overline{H}_1(x) \leq \overline{H}_2(x)$ holds for all $x \geq 0$.*

(3) *$F(x)$, $H_1(x)$ and $H_2(x)$ satisfy Condition 2.2(3).*

Condition 3.1(1) tells that the failure probability is increasing as the time goes. Condition 3.1(2) tells that the time for corrective maintenance is stochastically longer than that for preventive maintenance. Note that f_k is increasing by Condition 3.1(1), and $h_1 \leq h_2$ holds by Condition 3.1(2).

Lemma 3.2
The function $V^n(i,k)$ in the value iteration method satisfies

(1) $V^n(0,0) \leq \lambda/\alpha$,

(2) $V^n(i,k) \leq V^{n+1}(i,k) \leq i + \lambda/\alpha$ *for all* i,k,

(3) $\lambda h_1 + (1-\alpha h_1)V^n(0,0) \leq \lambda h_2 + (1-\alpha h_2)V^n(0,0)$.

Proof of (1).
Lemma 3.2(1) is obvious for $n=0$. If $V^n(0,0) \leq \lambda/\alpha$, then

$$\begin{aligned} V^{n+1}(0,0) \leq M^{n+1}(0) &= \lambda h_1 + (1-\alpha h_1)V^n(0,0) \\ &\leq \lambda h_1 + (1-\alpha h_1)\lambda/\alpha = \lambda/\alpha. \end{aligned}$$

Thus, by induction, $V^n(0,0) \leq \lambda/\alpha$ holds for all n.
Proof of (2).
It is obvious that $V^0(i,k) \leq V^1(i,k)$ holds. It is also easy to show that if $V^{n-1}(i,k) \leq V^n(i,k)$, then $V^n(i,k) \leq V^{n+1}(i,k)$. Hence, the inequality $V^{n+1}(i,k) \leq i + \lambda/\alpha$ is obtained since

$$V^{n+1}(i,k) \leq M^{n+1}(i) = i + \lambda h_1 + (1-\alpha h_1)V^n(0,0) \leq i + \lambda/\alpha.$$

Proof of (3).
The third property is proved as follows.

$$\lambda h_1+(1-\alpha h_1)V^n(0,0)-\lambda h_2-(1-\alpha h_2)V^n(0,0) = (h_1-h_2)(\lambda-\alpha V^n(0,0)) \leq 0.$$

We apply Condition 3.1(2) and Lemma 3.2(1) to obtain the last inequality. This completes the proof. □

Lemma 3.3
$W^n(i,k)$ is increasing in k.

Proof.
First we prove that if $V^n(i,k)$ is increasing in k, then $W^{n+1}(i,k)$ is also increasing in k. From Eq. (3.17), we have

$$\begin{aligned} W^{n+1}(i,k) &= \beta(1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+1) \\ &\quad + \beta f_k[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)] \\ &\quad - \beta f_k L_i(T) + \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_i(x) + N(T-x)]dx \\ &= \beta R^n(i,k) + S^n(i,k). \end{aligned} \tag{3.19}$$

Here, we define

$$\begin{aligned} R^n(i,k) \equiv (1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+1) \\ + f_k[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)], \end{aligned} \tag{3.20}$$

$$S^n(i,k) \equiv \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_i(x) + N(T-x)]dx - \beta f_k L_i(T). \tag{3.21}$$

Let us now prove that both $R^n(i,k)$ and $S^n(i,k)$ are increasing in k. It is seen that

$$\begin{aligned} R^n(i,k+1) &- R^n(i,k) \\ &= (1-f_{k+1})\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+2) \\ &\quad + f_{k+1}[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)] \\ &\quad - (1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+1) \\ &\quad - f_k[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)] \\ &\geq (1-f_{k+1})\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+2) \\ &\quad + f_{k+1}[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)] \\ &\quad - (1-f_k)\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+2) \\ &\quad - f_k[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T)] \\ &= (f_{k+1} - f_k)[\lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T) \\ &\quad - \sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+2)]. \end{aligned} \tag{3.22}$$

From Condition 3.1(1), the relation $f_{k+1} - f_k \geq 0$ holds. For $n=0$, we find that $\lambda h_2+(1-\alpha h_2)V^0(0,0)+L_i(T)-\sum_{j=0}^{\infty} P_{ij}(T)V^0(j,k+2)] = \lambda h_2+L_i(T) \geq 0$ holds. For $n > 0$, we have

$$\begin{aligned} &\sum_{j=0}^{\infty} P_{ij}(T)V^n(j,k+2) \\ &\leq \sum_{j=0}^{\infty} P_{ij}(T)[j + \lambda h_1 + (1-\alpha h_1)V^{n-1}(0,0)] \\ &\leq \sum_{j=0}^{\infty} P_{ij}(T)[j + \lambda h_1 + (1-\alpha h_1)V^{n}(0,0)] \\ &\leq \lambda h_2 + (1-\alpha h_2)V^n(0,0) + L_i(T). \end{aligned}$$

Therefore, $R^n(i,k+1) \geq R^n(i,k)$ holds from (3.22). Thus, the function $R^n(i,k)$ is increasing in k.

For $S^n(i,k)$, we apply the integration by parts, using the following relations:

$$f(kT+x) = \frac{d}{dx}(\overline{F}(kT) - \overline{F}(kT+x)), \tag{3.23}$$

$$\frac{d}{dx}(e^{-\alpha x}[L_i(x) + N(T-x)]) = -e^{-\alpha x}\left(\alpha L_i(x) - \frac{d}{dx}L_i(x) + \lambda\right). \tag{3.24}$$

Then, we have

$$\begin{aligned} S^n(i,k) &= \left[\frac{\overline{F}(kT) - \overline{F}(kT+x)}{\overline{F}(kT)}\left(e^{-\alpha x}L_i(x) + \frac{\lambda}{\alpha}(e^{-\alpha x} - e^{-\alpha T})\right)\right]_0^T \\ &\quad + \int_0^T e^{-\alpha x}\frac{\overline{F}(kT) - \overline{F}(kT+x)}{\overline{F}(kT)}(\alpha L_i(x) - \frac{d}{dx}L_i(x) + \lambda)dx \\ &\quad - \beta f_k L_i(T) \\ &= \int_0^T e^{-\alpha x}\frac{\overline{F}(kT) - \overline{F}(kT+x)}{\overline{F}(kT)}(\alpha L_i(x) - \frac{d}{dx}L_i(x) + \lambda)dx. \end{aligned} \tag{3.25}$$

Since $(\alpha L_i(x) - \frac{d}{dx}L_i(x) + \lambda) \geq 0$ holds and $(\overline{F}(kT) - \overline{F}(kT+x))/\overline{F}(kT)$ is increasing in k, $S^n(i,k)$ is increasing in k. We can conclude that $W^{n+1}(i,k)$ is increasing in k, because both $R^n(i,k)$ and $S^n(i,k)$ are increasing in k.

It is obvious that if $W^{n+1}(i,k)$ is increasing in k, then $V^{n+1}(i,k)$ is also increasing in k. Therefore $W^n(i,k)$ is increasing in k for all n. This completes the proof. □

Lemma 3.4
$V^n(i+1,k) - V^n(i,k) \leq 1$ and $W^n(i+1,k) - W^n(i,k) \leq 1$ hold for all i,k.

Proof.
First, it is obvious that $V^0(i+1,k) - V^0(i,k) \leq 1$ holds. We prove that $W^{n+1}(i+1,k) - W^{n+1}(i,k) \leq 1$ with the inductive hypothesis $V^n(i+1,k) - V^n(i,k) \leq 1$. Let us define

$$v^n(i,k) = \begin{cases} V^n(0,k) & \text{for } i = 0, \\ V^n(i,k) - V^n(i-1,k) & \text{for } i \geq 1. \end{cases} \tag{3.26}$$

By inductive hypothesis, $v^n(i,k) \leq 1$ holds for $i \geq 1$. Using Lemma 3.1(2) and (3), we can find that $W^{n+1}(i,k)$ satisfies the following inequality.

$$\begin{aligned} &W^{n+1}(i+1,k) - W^{n+1}(i,k) \\ &= \beta(1-f_k)\Big(\sum_{j=0}^{\infty} P_{i+1\,j}(T)\sum_{m=0}^{j} v^n(m,k+1) \\ &\quad - \sum_{j=0}^{\infty} P_{ij}(T)\sum_{m=0}^{j} v^n(m,k+1)\Big) \\ &\quad + \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}[L_{i+1}(x) - L_i(x)]dx \\ &\leq \beta(1-f_k)\left[\sum_{m=0}^{\infty} v^n(m,k+1)\sum_{j=m}^{\infty}(P_{i+1j}(T) - P_{ij}(T))\right] \\ &\quad + \int_0^T e^{-\alpha x}\frac{f(kT+x)}{\overline{F}(kT)}dx \\ &= \beta(1-f_k)\left[\sum_{m=1}^{\infty} v^n(m,k+1)\sum_{j=m}^{\infty}(P_{i+1j}(T) - P_{ij}(T))\right] \\ &\quad - e^{-\alpha T}\frac{\overline{F}(kT+T)}{\overline{F}(kT)} + 1 - \int_0^T \alpha e^{-\alpha x}\frac{\overline{F}(kT+x)}{\overline{F}(kT)}dx \\ &\leq \beta(1-f_k)\left[\sum_{m=1}^{\infty}\sum_{j=m}^{\infty}(P_{i+1j}(T) - P_{ij}(T))\right] \\ &\quad + \beta(f_k - 1) + 1 - \int_0^T \alpha e^{-\alpha x}\frac{\overline{F}(kT+x)}{\overline{F}(kT)}dx \\ &\leq \beta(1-f_k)[L_{i+1}(T) - L_i(T)] \\ &\quad + \beta(f_k - 1) + 1 - \int_0^T \alpha e^{-\alpha x}\frac{\overline{F}(kT+x)}{\overline{F}(kT)}dx \\ &\leq 1 - \int_0^T \alpha e^{-\alpha x}\frac{\overline{F}(kT+x)}{\overline{F}(kT)}dx \leq 1. \end{aligned}$$

To complete the induction, we show that $V^{n+1}(i+1,k+1) - V^{n+1}(i,k+1) \leq 1$ holds by $W^{n+1}(i+1,k+1) - W^{n+1}(i,k+1) \leq 1$. This is easy by $\min\{a,b\} - \min\{c,d\} \leq \max\{a-c, b-d\}$. Thus by induction, the inequalities $V^n(i+1,k) - V^n(i,k) \leq 1$ and $W^n(i+1,k) - W^n(i,k) \leq 1$ hold. This completes the proof. □

Since this model satisfies Condition 2.2, $\lim_{n\to\infty} W^n(i,k) = W(i,k)$ and $\lim_{n\to\infty} V^n(i,k) = V(i,k)$ hold. Thus, we have the following lemma.

Lemma 3.5

(1) *$W(i,k)$ is increasing in k.*

(2) *$V(i+1,k) - V(i,k) \leq 1$ and $W(i+1,k) - W(i,k) \leq 1$ hold.*

3.5 Structure of the optimal policy

Based on Lemma 3.5, we obtain the following theorem that describes a structure of the optimal policy.

Theorem 3.1
If $D(i,l) = 2$, then $D(j,k) = 2$ for all $k \leq l$ and $j \geq i$.

Proof.
First, we note that $W(i,l) \leq M(i)$ holds by $D(i,l) = 2$. Then we can prove the inequality $W(j,k) \leq M(j)$ as follows:

$$\begin{aligned} W(j,k) &\leq W(j,l) \leq W(i,l) + j - i \\ &\leq M(i) + j - i = M(j). \end{aligned}$$

Thus $D(j,k) = 2$. Lemma 3.5(1) and Lemma 3.5(2) are used for the first and the second inequality, respectively. This completes the proof. □

This theorem states that the optimal policy has a switch curve structure (Fig. 3.3). The switch curve structure indicates that the two-dimensional state space is divided by an increasing function, with the optimal action changes across the function. In other words, the optimal action changes at most once as the queue length i (the observation time k) increases for the fixed observation time k (the queue length i).

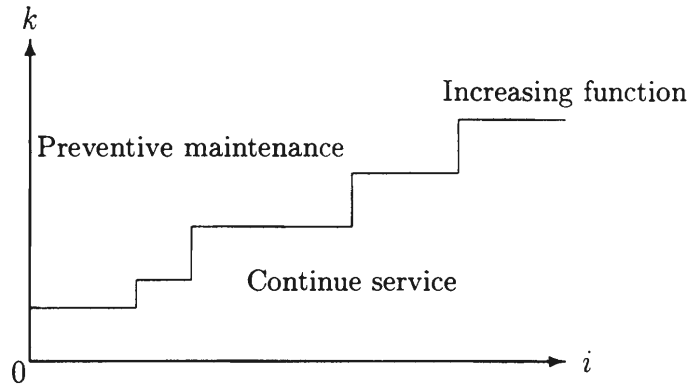

Figure 3.3: The switch curve structure of the optimal policy

3.6 Conclusion

In this chapter, we investigated a maintenance problem for an $M/M/1$ queue with a failure distribution function of the system. The system is observed periodically and the maintenance can be started upon the observation time. We considered the discounted cost of the lost customers which is incurred when the system stops by the failure and the maintenance. The semi-Markov decision process was used to analyze the optimal policy. Our policy was defined on the two-dimensional state space, where the state indicates the queue length and the number of the observations. We showed that the optimal maintenance policy has a switch curve structure under some mathematical conditions.

Chapter 4

An $M/G/1$ Queue with Deteriorating Server

4.1 Introduction

This chapter deals with an optimal maintenance policy for a deteriorating server. The customer arrives at the system in a Poisson process and he/she is served by a server with a general service time distribution. A Markov process is considered to represent the deteriorating process of the server. The process has multiple states that indicate the deterioration level of the server. The server state changes after an exponential time whose transition rate may depends on the state. There is one failure state and the failure occurs when the server state becomes the failure state. When the server fails, the failure is detected immediately and the corrective maintenance starts. The deteriorating process and the queue length process is always monitored. With this observation, the preventive maintenance can be taken to avoid the failure. After the maintenance, the server state becomes as good as new, though we lose the customers in the system upon the start of the maintenance and the arrivals during the maintenance. A unit cost per lost customer is incurred. Our objective is to find the optimal maintenance policy that minimizes the total expected discounted cost over an infinite time horizon. The problem is formulated as a semi-Markov decision process whose state space is a pair of the queue length and the server state. It is shown that the optimal control has a switch curve structure under some conditions.

The organization of this chapter is as follows. The next section provides a detailed explanation of our model. In Section 4.3, we formulate our problem as a semi-Markov decision process. In Section 4.4, we derive the properties of the value functions. In Section 4.5, the switch curve structure of the optimal policy is derived from the properties proved in Section 4.4. The last section concludes our results.

4.2 Model

We consider a single server queue with Poisson arrivals at rate λ. The customers form a single queue and their service times are i.i.d. with a distribution function $G(x)$, which is assumed to have a density function $g(x)$. The server has $s+2$ states that are numbered from 0 to $s+1$. The number indicates the deterioration level of the server. It is assumed that the server deteriorates in this order, i.e., state 0 indicates the server is as good as new and state $s+1$ is the failure state. The transitions are governed by a Markov process whose transition rate from state i to j is denoted by γ_{ij}. By uniformization, we can assume $\sum_{j=0}^{s+1} \gamma_{ij}$ is equal to a constant Γ for all i.

We consider a maintenance problem for this system. The corrective maintenance is immediately performed when the server fails. The decision between the preventive maintenance and the operation of the system is made upon the decision epochs. The decision epochs are the departure time, the transition time of the server state when the system is empty, and the arrival time to the empty system (Fig. 4.1). Note that no decisions during service are made even if the server state changes. The functions $H_1(x)$ and $H_2(x)$ denote the distribution functions of the time for preventive maintenance and corrective maintenance, respectively. Upon the beginning of the maintenance, all the customers in the system are lost and the arriving customers during maintenance are also lost. One unit cost per lost customer is incurred and we consider the minimization of the total expected discounted cost over an infinite time horizon with discount factor α.

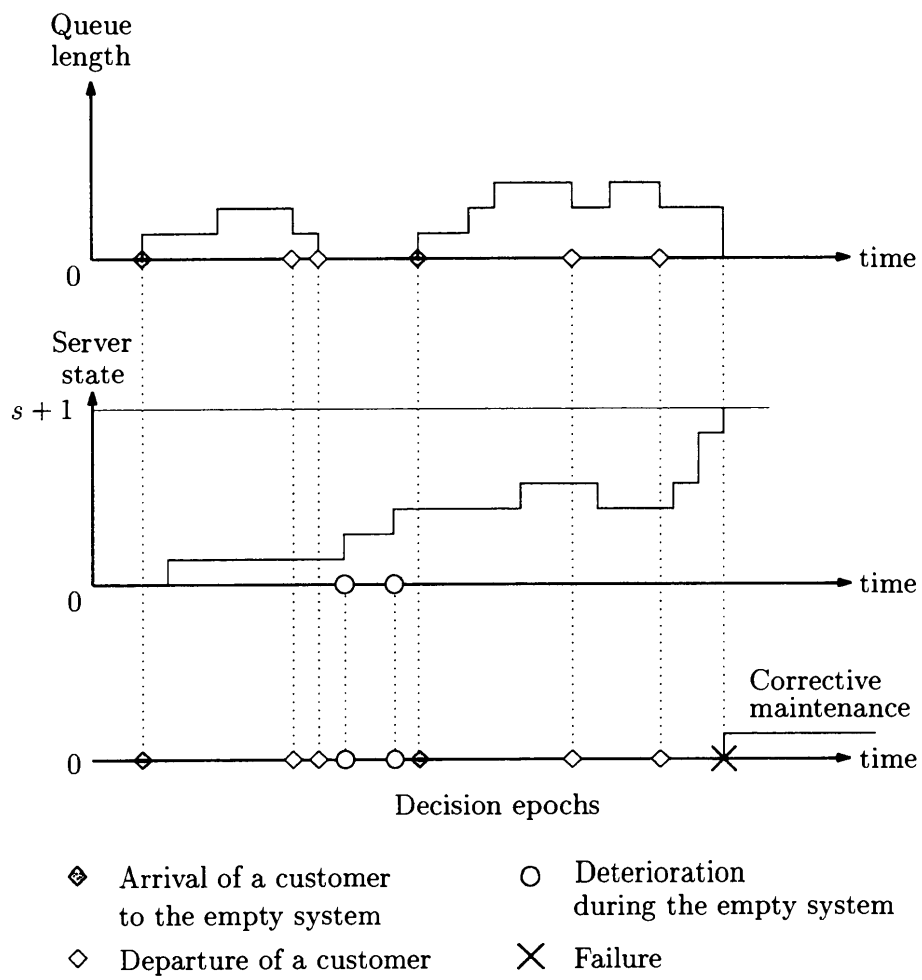

Figure 4.1: The decision epochs

4.3 Formulation

In this section, we define the semi-Markov decision process for our problem.

State and Action set

The system state is expressed by a pair of the queue length i ($i \geq 0$) and the server state k ($0 \leq k \leq s+1$). The preventive maintenance is referred to as Action 1 and continuing service is referred to as Action 2, which can be taken in state (i,k) ($k \leq s$). Action 3 is the corrective maintenance, which must be taken in state $(i, s+1)$.

Transition Probability

The transition probabilities are expressed by using following probabilities.

$P_i(x)$: The probability that i customers arrive at the queue during time x. Since the Poisson arrival is assumed, it holds that

$$P_i(x) = \frac{(\lambda x)^i}{i!} e^{-\lambda x}.$$

Note that

$$\sum_{i=0}^{\infty} i P_i(x) = \lambda x.$$

$T_{kl}(x)$: The probability that the server state is l after time x, given the initial state k.

By these notations, we can express as follows the transition probability $Q^a_{((i,k),(j,l))}(t)$ from state (i,k) to (j,l) within time t for each action a.

(1) For Action 1, the next state becomes $(0,0)$ and the transition time distribution is $H_1(t)$. Thus,

$$Q^1_{((i,k),(0,0))}(t) = H_1(t).$$

(2) For Action 2, we have the following cases:

$$Q^2_{((i,k),(j,l))}(t) = \begin{cases} \int_0^t P_{j-i+1}(x) T_{kl}(x) g(x) dx & \text{if } i \geq 1 \text{ and } l \leq s, \\ \int_0^t P_{j-i}(x) \overline{G}(x) dT_{k\,s+1}(x) & \text{if } i \geq 1 \text{ and } l = s+1, \\ \int_0^t \gamma_{kl} e^{-(\Gamma+\lambda)x} dx & \text{if } i = 0 \text{ and } j = 0, \\ \int_0^t \lambda e^{-(\Gamma+\lambda)x} dx & \text{if } i = 0,\ j = 1 \text{ and } k = l. \end{cases}$$

(3) For Action 3, the next state becomes $(0,0)$ and the transition time distribution is $H_2(t)$. Thus,

$$Q^3_{((i,s+1),(0,0))}(t) = H_2(t).$$

Cost Function

The cost functions are calculated in a manner similar to Chapter 3.

(1) For Action 1 in state (i,k) ($k \leq s$) and Action 3 in state $(i, s+1)$, the expected discounted costs from the start to the end of maintenance are $i + \lambda h_1$ and $i + \lambda h_2$, respectively (h_m is defined by Eq. (3.4)).

(2) For Action 2, no cost is incurred until the next transition.

Optimality Equation

For the optimality equation, we define the following cost functions.

$V(i,k)$: The optimal cost function for state (i,k).

$M(i,k)$: The cost function when Action 1 is taken upon the transition to (i,k) and taking the optimal policy thereafter. For the same reason as in Chapter 3, $M(i,k)$ is denoted by $M(i)$.

$W(i,k)$: The cost function when Action 2 is taken upon the transition to (i,k) and taking the optimal policy thereafter.

$D(i,k)$: The optimal action for state (i,k).

$$D(i,k)=\begin{cases}1 & \text{when } M(i)\le W(i,k),\\ 2 & \text{when } M(i)>W(i,k).\end{cases}$$

By a similar argument as in Chapter 3, the optimality equation becomes as follows.

$$W(0,k)=\frac{1}{\alpha+\lambda+\Gamma}\Big[\sum_{l=0}^{s+1}\gamma_{kl}V(0,l)+\lambda V(1,k)\Big], \tag{4.1}$$
$$M(i)=i+\lambda h_1+(1-\alpha h_1)V(0,0), \tag{4.2}$$
$$W(i,k)=\sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty}e^{-\alpha x}P_j(x)T_{kl}(x)V(i+j-1,l)g(x)dx$$
$$+\sum_{j=0}^{\infty}\int_0^{\infty}e^{-\alpha x}P_j(x)\overline{G}(x)V(i+j,s+1)dT_{k\,s+1}(x)\quad (i\ge 1), \tag{4.3}$$
$$V(i,k)=\min[M(i),\ W(i,k)]\quad (k\le s), \tag{4.4}$$
$$V(i,s+1)=i+\lambda h_2+(1-\alpha h_2)V(0,0). \tag{4.5}$$

The value iteration method for this problem is defined as follows.

Value iteration method

Step 0. $n:=0,\ V^0(i,k):=W^0(i,k):=M^0(i):=i+\lambda/\alpha$.

Step 1.
$$W^{n+1}(0,k):=\frac{1}{\alpha+\lambda+\Gamma}\Big[\sum_{l=0}^{s+1}\gamma_{kl}V^n(0,l)+\lambda V^n(1,k)\Big],$$
$$W^{n+1}(i,k):=\sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty}e^{-\alpha x}P_j(x)T_{kl}(x)g(x)V^n(i+j-1,l)dx$$
$$+\sum_{j=0}^{\infty}\int_0^{\infty}e^{-\alpha x}P_j(x)\overline{G}(x)V^n(i+j,s+1)dT_{k\,s+1}(x)\quad (i\ge 1),$$
$$M^{n+1}(i):=i+\lambda h_1+(1-\alpha h_1)V^n(0,0),$$
$$V^{n+1}(i,k):=\min[M^{n+1}(i),\ W^{n+1}(i,k)]\quad (k\le s),$$
$$V^{n+1}(i,s+1):=i+\lambda h_2+(1-\alpha h_2)V^n(0,0).$$

Step 2. $n:=n+1$ and return to Step 1.

We derive some properties of $V(i,k)$ and $W(i,k)$ in the next section.

4.4 Properties of the cost function

To derive the property of the cost functions, we assume that the following conditions hold.

Condition 4.1
The transition rate γ_{ij} and the distribution functions $G(x)$, $H_1(x)$ and $H_2(x)$ satisfy the following conditions.

(1) $\sum_{l=m}^{s+1}\gamma_{kl}$ *is increasing in k for all m.*

(2) $\overline{H}_2(x)\ge\overline{H}_1(x)$ *for all x.*

(3) $\lambda\le g(x)/\overline{G}(x)$ *for all x.*

(4) $H_1(x)$, $H_2(x)$ *and* $G(x)$ *satisfy Condition* 2.2(3).

Condition 4.1(1) tells that the server is more likely to move to deteriorated states as its deterioration level becomes higher. Condition 4.1(2) tells that the time for corrective maintenance is stochastically larger than that for preventive maintenance. This condition is a sufficient condition for $h_1\le h_2$. Condition 4.1(3) tells that the service rate is always larger than the arrival rate, which secures that the queue length does not grow infinitely as the time goes.

Lemma 4.1 *(Stoyan [30])*
Under Condition 4.1(1), *the following relations holds.*

(1) $\sum_{l=m}^{s+1}T_{kl}(x)$ *is increasing in k for all m.*

(2) *For any increasing sequence* f_l, *the sums* $\sum_{l=0}^{s+1} T_{kl}(x) f_l$ *and* $\sum_{l=0}^{s+1} \gamma_{kl} f_l$ *are both increasing in* k.

Lemma 4.2
$V^n(i,k)$ *and* $W^n(i,k)$ *are decreasing in* n.

Proof.
First, we show $V^1(i,k) \le V^0(i,k)$ and $W^1(i,k) \le W^0(i,k)$.

$$\begin{aligned} W^1(0,k) &= \frac{1}{\alpha+\lambda+\Gamma}\left[\sum_{l=0}^{s+1} \gamma_{kl}\lambda/\alpha + \lambda(1+\lambda/\alpha)\right] \\ &= \frac{1}{\alpha+\lambda+\Gamma}\left[\Gamma\lambda/\alpha + \alpha(\lambda/\alpha) + \lambda(\lambda/\alpha)\right] \\ &= \lambda/\alpha = W^0(0,k) \end{aligned}$$

For $i \ge 1$,

$$\begin{aligned} W^1(i,k) &= \sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^\infty e^{-\alpha x} P_j(x) T_{kl}(x) g(x)(i+j-1+\lambda/\alpha)dx \\ &\quad + \sum_{j=0}^{\infty}\int_0^\infty e^{-\alpha x} P_j(x)\overline{G}(x)(i+j+\lambda/\alpha)dT_{k\,s+1}(x) \\ &= \int_0^\infty e^{-\alpha x}\overline{T}_{k\,s+1}(x) g(x)(i+\lambda x - 1 + \lambda/\alpha)dx \\ &\quad + \int_0^\infty e^{-\alpha x}\overline{G}(x)(i+\lambda x+\lambda/\alpha)dT_{k\,s+1}(x) \\ &= \int_0^\infty e^{-\alpha x}(i+\lambda x+\lambda/\alpha)\frac{d}{dx}(-\overline{T}_{k\,s+1}(x)\overline{G}(x))dx \\ &\quad - \int_0^\infty e^{-\alpha x}\overline{T}_{k\,s+1}(x) g(x)dx \\ &= \left[e^{-\alpha x}(-\overline{T}_{k\,s+1}(x)\overline{G}(x))(i+\lambda x+\lambda/\alpha)\right]_0^\infty \\ &\quad + \int_0^\infty \overline{T}_{k\,s+1}(x)\overline{G}(x)(\lambda e^{-\alpha x} - (i+\lambda x+\lambda/\alpha)\alpha e^{-\alpha x})dx \\ &\quad - \int_0^\infty e^{-\alpha x}\overline{T}_{k\,s+1}(x) g(x)dx \end{aligned}$$

$$\begin{aligned} &\le i+\lambda/\alpha + \int_0^\infty \overline{T}_{k\,s+1}(x)(\lambda\overline{G}(x) - g(x))e^{-\alpha x}dx \\ &\le i+\lambda/\alpha = W^0(i,k). \end{aligned}$$

Condition 4.1(3) was applied to obtain the above last inequality.

The following calculation shows that $M^1(i) \le i+\lambda/\alpha$ and $V^1(i,s+1) \le i+\lambda/\alpha$ hold.

$$\begin{aligned} M^1(i) &\le i+\lambda h_1 + (1-\alpha h_1)\lambda/\alpha = i+\lambda/\alpha, \\ V^1(i,s+1) &\le i+\lambda h_2 + (1-\alpha h_2)\lambda/\alpha = i+\lambda/\alpha. \end{aligned}$$

Since $M^1(i)$, $W^1(i,k)$ and $V^1(i,s+1)$ are less than or equal to $i+\lambda/\alpha$, it is obvious that $V^1(i,k) \le i+\lambda/\alpha$. Thus $W^1(i,k) \le W^0(i,k)$ and $V^1(i,k) \le V^0(i,k)$ hold.

Next, we show $W^{n+2}(i,k) \le W^{n+1}(i,k)$ and $V^{n+2}(i,k) \le V^{n+1}(i,k)$, using the inductive hypothesis $W^{n+1}(i,k) \le W^n(i,k)$ and $V^{n+1}(i,k) \le V^n(i,k)$.

$$\begin{aligned} &W^{n+2}(0,k) - W^{n+1}(0,k) \\ &= \frac{1}{\alpha+\lambda+\Gamma}\left[\sum_{l=0}^{s+1}\gamma_{kl}(V^{n+1}(0,l) - V^n(0,l)) + \lambda(V^{n+1}(1,k) - V^n(1,k))\right] \\ &\le 0. \end{aligned}$$

The inequality $W^{n+2}(i,k) \le W^{n+1}(i,k)$ $(i \ge 1)$ and $M^{n+2}(i) \le M^{n+1}(i)$ are shown in a similar manner. It is also obvious that $V^{n+2}(i,k) \le V^{n+1}(i,k)$ holds because $W^{n+2}(i,k) \le W^{n+1}(i,k)$ and $M^{n+2}(i) \le M^{n+1}(i)$. This completes the proof. □

We now prove the following lemma.

Lemma 4.3

(1) $W(i,k)$ *and* $V(i,k)$ *are increasing in* k.

(2) $V(i+1,k) - V(i,k) \le 1$.

Since $W^n(i,k)$ and $V^n(i,k)$ converge to $W(i,k)$ and $V(i,k)$ as $n \to \infty$, respectively, by Condition 2.2, it is sufficient to prove above properties for $W^n(i,k)$ and $V^n(i,k)$.

Proof of (1).

It is obvious that $W^0(i,k)$ and $V^0(i,k)$ are increasing in k. Assuming $W^n(i,k)$ and $V^n(i,k)$ are increasing in k, we show below that $W^{n+1}(i,k)$ and $V^{n+1}(i,k)$ are also increasing in k.

It is easily proved that $W^{n+1}(0,k)$ is increasing in k by induction and Lemma 4.1(2). We show that $W^{n+1}(i,k)$ for $i \geq 1$ is increasing in k as follows.

$$\begin{aligned}W^{n+1}(i,k) = &\sum_{j=0}^{\infty}\sum_{l=0}^{s+1}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x)\\ &\qquad [V^n(i+j-1,l)-V^n(i+j-1,s+1)]dx\\ &+\sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x)V^n(i+j-1,s+1)dx\\ &+\sum_{j=0}^{\infty}\int_0^{\infty} e^{-\alpha x}P_j(x)\overline{G}(x)V^n(i+j,s+1)dT_{k\,s+1}(x).\end{aligned}$$

By the inductive hypothesis and Lemma 4.1(2), the first term

$$\sum_{j=0}^{\infty}\sum_{l=0}^{s+1}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x)[V^n(i+j-1,l)-V^n(i+j-1,s+1)]dx$$

is increasing in k. The second and third terms are calculated as follows.

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x)V^n(i+j-1,s+1)dx\\ &+\sum_{j=0}^{\infty}\int_0^{\infty} e^{-\alpha x}P_j(x)\overline{G}(x)V^n(i+j,s+1)dT_{k\,s+1}(x)\\ =&\int_0^{\infty} e^{-\alpha x}[\lambda x-1+V^n(i,s+1)]\overline{T}_{k\,s+1}(x)g(x)dx\\ &+\int_0^{\infty} e^{-\alpha x}[\lambda x+V^n(i,s+1)]\overline{G}(x)dT_{k\,s+1}(x)\\ =&-\int_0^{\infty} e^{-\alpha x}\overline{T}_{k\,s+1}(x)g(x)dx\end{aligned}$$

$$\begin{aligned}&-\int_0^{\infty} e^{-\alpha x}[\lambda x+V^n(i,s+1)]\frac{d}{dx}(\overline{G}(x)\overline{T}_{k\,s+1}(x))dx\\ =&-\int_0^{\infty} e^{-\alpha x}\overline{T}_{k\,s+1}(x)g(x)dx\\ &-\left[e^{-\alpha x}(\lambda x+V^n(i,s+1))\overline{T}_{k\,s+1}(x)\overline{G}(x)\right]_0^{\infty}\\ &-\alpha\int_0^{\infty} e^{-\alpha x}[\lambda x+V^n(i,s+1)]\overline{T}_{k\,s+1}(x)\overline{G}(x)dx\\ &+\lambda\int_0^{\infty} e^{-\alpha x}\overline{T}_{k\,s+1}(x)\overline{G}(x)dx\\ =&V^n(i,s+1)-\alpha\int_0^{\infty} e^{-\alpha x}[\lambda x+V^n(i,s+1)]\overline{T}_{k\,s+1}(x)\overline{G}(x)dx\\ &+\int_0^{\infty} e^{-\alpha x}(\lambda\overline{G}(x)-g(x))\overline{T}_{k\,s+1}(x)dx.\end{aligned}$$

By Condition 4.1(3), $\lambda\overline{G}(x)-g(x)\leq 0$ holds for all x, and $\overline{T}_{k\,s+1}(x)$ is decreasing in k by Lemma 4.1(1). Thus $W^{n+1}(i,k)$ is increasing in k. It is also obvious that if $W^{n+1}(i,k)$ is increasing in k, then $V^{n+1}(i,k)$ is also increasing in k.

Proof of (2).

It is obvious that $V^0(i+1,k)-V^0(i,k)\leq 1$ holds. Using the inductive hypothesis $V^n(i+1,k)-V^n(i,k)\leq 1$, we show that $V^{n+1}(i+1,k)-V^{n+1}(i,k)\leq 1$. To show $V^{n+1}(1,k)-V^{n+1}(0,k)\leq 1$, the next two cases are considered.

Case 1. $D^{n+1}(0,k)=1$.

$$V^{n+1}(1,k)-V^{n+1}(0,k)\leq M^{n+1}(1)-M^{n+1}(0)=1.$$

Case 2. $D^{n+1}(0,k)=2$.

$$\begin{aligned}&V^{n+1}(1,k)-V^{n+1}(0,k)\\ \leq\ &V^n(1,k)-W^{n+1}(0,k)\\ =\ &\frac{1}{\alpha+\lambda+\Gamma}\left[(\alpha+\lambda+\Gamma)V^n(1,k)-\sum_{l=0}^{s+1}\gamma_{kl}V^n(0,l)-\lambda V^n(1,k)\right]\\ \leq\ &\frac{1}{\alpha+\lambda+\Gamma}\left[(\alpha+\lambda+\Gamma)V^n(1,k)-\sum_{l=0}^{s+1}\gamma_{kl}V^n(0,k)-\lambda V^n(1,k)\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\alpha+\lambda+\Gamma}[(\alpha+\Gamma)V^n(1,k) - \Gamma V^n(0,k)] \\
&= \frac{1}{\alpha+\lambda+\Gamma}[\alpha V^n(1,k) + \Gamma\{V^n(1,k) - V^n(0,k)\}] \\
&\leq \frac{1}{\alpha+\lambda+\Gamma}[\alpha(1+\lambda/\alpha)+\Gamma] = 1.
\end{aligned}
$$

Next we show $V^{n+1}(i+1,k) - V^{n+1}(i,k) \leq 1$ holds for $i \geq 1$.

Case 1. $D^{n+1}(i,k) = 1$.

$$V^{n+1}(i+1,k) - V^{n+1}(i,k) \leq M^{n+1}(i+1) - M^{n+1}(i) = 1.$$

Case 2. $D^{n+1}(i,k) = 2$.

$$
\begin{aligned}
&V^{n+1}(i+1,k) - V^{n+1}(i,k) \leq W^{n+1}(i+1,k) - W^{n+1}(i,k) \\
&= \sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x) \\
&\qquad [V^n(i+j,l) - V^n(i+j-1,l)]dx \\
&\quad + \sum_{j=0}^{\infty}\int_0^{\infty} e^{-\alpha x}P_j(x)\overline{G}(x)dT_{k\,s+1}(x) \\
&\qquad [V^n(i+1+j,s+1) - V^n(i+j,s+1)]dx \\
&\leq \sum_{j=0}^{\infty}\sum_{l=0}^{s}\int_0^{\infty} e^{-\alpha x}P_j(x)T_{kl}(x)g(x)dx \\
&\quad + \sum_{j=0}^{\infty}\int_0^{\infty} e^{-\alpha x}P_j(x)\overline{G}(x)dT_{k\,s+1}(x) \\
&= \int_0^{\infty} e^{-\alpha x}\overline{T}_{k\,s+1}(x)g(x)dx + \int_0^{\infty} e^{-\alpha x}\overline{G}(x)dT_{k\,s+1}(x) \\
&\leq \int_0^{\infty} \frac{d}{dx}(-\overline{T}_{k\,s+1}(x)\overline{G}(x))dx = 1.
\end{aligned}
$$

This completes the proof. □

4.5 Structure of the optimal policy

By Lemma 4.3, we have the following theorem.

Theorem 4.1
If $D(i,l) = 2$, then $D(j,k) = 2$ for all $k \leq l$ and $j \geq i$.

Proof.
For (i,l) and (j,k), we have

$$V(j,k) \leq V(j,l) \leq j - i + V(i,l) < j - i + M(i) = M(j).$$

The first inequality holds by Lemma 4.3(1), and the second inequality is obtained by applying Lemma 4.3(2) repeatedly. The last inequality holds because $D(i,l) = 2$ implies $V(i,l) = W(i,l) < M(i)$. The strict inequality $V(j,k) < M(j)$ indicates that $V(j,k) = W(j,k)$, i.e., $D(j,k) = 2$. This completes the proof. □

This theorem states that the optimal policy has a switch curve structure as illustrated in Fig. 4.2. This figure suggests that the optimal action changes at most once as the queue length i increases for a fixed server state k and the change is from preventive maintenance to continuation of the service. It also suggests that the optimal action changes at most once as the server state $k(\leq s)$ increases for a fixed queue length i and the change is from continuation of the service to preventive maintenance.

4.6 Conclusion

This chapter studied a maintenance problem for an $M/G/1$ queue whose deteriorating process is expressed by a Markov process. The system is always observed and the preventive maintenance can be started upon the departure of a customer, upon the arrival to the empty system and upon the deterioration during the empty system. If the system fails, the corrective maintenance starts immediately. We considered the discounted cost of the lost customers which are produced when the system stops by the failure or the maintenance. For this system, a semi-Markov decision process was formulated to analyze its optimal policy, in which the state is defined by a pair

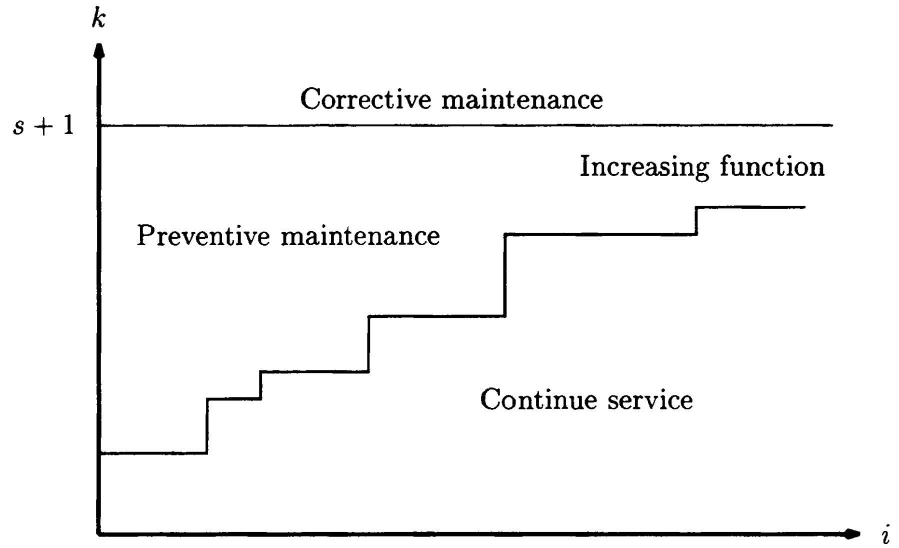

Figure 4.2: The optimal maintenance policy

of the queue length and the server state. For this problem, we showed that the optimal maintenance policy has the switch curve structure under some mathematical conditions.

Chapter 5

A Queue with Decreasing Service Rate

5.1 Introduction

In Chapter 3 and Chapter 4, we discussed the system whose service ability is not affected by server deterioration. In this chapter, we study a model in which the server state affects the service speed. The model is also extended to the situation where the server faces multiple users through terminals. By the server deterioration, the service speed at terminals becomes slower and the failure of the server means that all terminals become unavailable. The server state is recovered by two kinds of maintenance, preventive maintenance and corrective maintenance. The decision epochs are the arrival time of a customer, the departure time of a customer and the transition time of the server state. A preventive maintenance can be performed upon any decision epoch except the transition to the failure state, and the corrective maintenance must be performed upon the failure. After the maintenance, the server state returns to the new state. However, the system loses all the customers who were in the system upon the start of maintenance or who have arrived during maintenance.

Our model could be applied to the maintenance of a workstation with multiple terminals. To perform maintenance in such systems, it needs to stop service for users or transfer the customers to other systems. To avoid such inconvenience as possible, we deal with the minimization of the customers who are lost by the maintenance and the failure.

It may be intuitively obvious that the preventive maintenance should be performed when there are few users in the system and the server is in a deteriorated state. We show such a property under certain conditions, by proving a switch curve structure of the optimal policy.

5.2 Model

We consider a queueing system with Poisson arrivals at rate λ. The server has $s+2$ states numbered from 0 to $s+1$. The number indicates the deterioration level of the server. State 0 indicates that the server is as good as new and state $s+1$ indicates the failure state. States $1, \ldots, s$ indicate the increasing levels of deterioration in this order. The server state is assumed to change according to a Markov process. The transition rate from state k to l is denoted by γ_{kl}. By uniformization, we can assume that $\sum_{l=0}^{s+1} \gamma_{kl}$ is equal to a constant Γ for all k. The server has N terminals at which customers are served. The arriving customers while the system is full are lost. The system state is expressed as (i,k), indicating the queue length i $(0 \le i \le N)$, including the customers in service, and the server state k $(0 \le k \le s+1)$. The service rate $\mu(i,k)$ indicates the transition rate from (i,k) to $(i-1,k)$, i.e., one of the customers departs from the system. If the server fails, we must start the corrective maintenance immediately. Preventive maintenance can be performed when the system state changes (i.e., upon arrival, upon departure or upon server state transition). After the maintenance, the server returns to state 0. The distribution functions of preventive and corrective maintenance time are $H_1(x)$ and $H_2(x)$, respectively. It is assumed that the preventive maintenance time has the same distribution function for all k. In our model, customers in the system are lost upon the start of maintenance and the arrivals are also lost during maintenance. Since the system should serve as many customers as possible, we consider the number of the lost customers as a cost.

The problem is to determine the action (preventive maintenance or service) to minimize the total expected discounted number of lost customers, in an infinite time horizon with discount factor α.

5.3 Formulation

The problem is formulated as a semi-Markov decision process in this section.

State and Action set

The system state is expressed by a pair of the queue length i and the server state k. The preventive maintenance is referred to as Action 1 and continuing service is referred to as Action 2 which can be taken in state (i,k) $(k \le s)$. Action 3 is the corrective maintenance which is taken in state $(i, s+1)$.

Transition Probability

(1) For Action 1 and Action 3, the next state becomes $(0,0)$ and the transition time distribution are $H_1(t)$ and $H_2(t)$, respectively.

(2) We apply the uniformization in Section 2.3 for Action 2. Let us define

$$\mu \equiv \max_{i,k} \mu(i,k), \tag{5.1}$$

$$\theta \equiv \Gamma + \lambda + \mu. \tag{5.2}$$

By uniformization, the transition probability $Q^2_{ss'}(t)$ for Action 2 becomes

$$Q^2_{ss'}(t) = \begin{cases} (1-e^{-\theta t})\lambda/\theta \\ \quad (s=(i,k), s'=(i+1,k), i \le N-1) \\ (1-e^{-\theta t})\mu(i,k)/\theta \\ \quad (s=(i,k), s'=(i-1,k), i \ge 1) \\ (1-e^{-\theta t})\gamma_{kl}/\theta \\ \quad (s=(i,k), s'=(i,l), k \ne l) \\ (1-e^{-\theta t})(\mu - \mu(i,k) + \gamma_{kk})/\theta \\ \quad (s=(i,k), s'=(i,k), i \le N-1) \\ (1-e^{-\theta t})(\mu - \mu(i,k) + \lambda + \gamma_{kk})/\theta \\ \quad (s=(N,k), s'=(N,k)) \end{cases} \tag{5.3}$$

Cost Function

For Action 1 and Action 3 in state (i, k), the costs until the next transition are the same as in Chapter 4, i.e., $i + \lambda h_1$ and $i + \lambda h_2$, respectively, where h_m is defined by Eq. (3.4).

For Action 2 in state (N, k), the arrivals are lost until the next transition happens. If it takes time t before the next transition, the expected cumulative cost is λt. The cost until the next transition is then obtained by calculating Eq. (2.5),

$$\int_0^\infty \left\{ \int_0^t e^{-\alpha y} \lambda dy \right\} \theta e^{-\theta t} dt = \lambda / (\theta + \alpha). \tag{5.4}$$

Optimality Equation

We define the following cost functions for the optimality equation.

$V(i, k)$: The optimal cost function for state (i, k).

$M(i, k)$: The cost function for the policy of performing preventive maintenance upon transition to (i, k) and taking the optimal behavior thereafter. $M(i, k)$ will be denoted by $M(i)$ in the following, because it is independent of k.

$W(i, k)$: The cost function for the policy of continuing service upon transition to (i, k) and taking the optimal behavior thereafter.

$D(i, k)$: The optimal action for state (i, k),

$$D(i,\, k) = \begin{cases} 1 & \text{if preventive maintenance is optimal } (M(i) < W(i,k)), \\ 2 & \text{if continuing service is optimal } (M(i) \geq W(i,k)). \end{cases}$$

For these functions, we obtain the following optimality equations:

$$M(i) = i + \lambda h_1 + (1 - \alpha h_1) V(0, 0), \tag{5.5}$$

$$W(i,k) = \frac{1}{\Lambda} \Bigg[\sum_{l=0}^{s+1} \gamma_{kl} V(i,l) + \lambda V(i+1,k)$$

$$+ \mu(i,k) V(i-1,k) + (\mu - \mu(i,k)) V(i,k) \Bigg] \quad (0 \leq k \leq s), \tag{5.6}$$

$$V(i,k) = \min\{M(i), W(i,k)\} \quad (0 \leq k \leq s), \tag{5.7}$$

$$V(i, s+1) = i + \lambda h_2 + (1 - \alpha h_2) V(0,0), \tag{5.8}$$

where $V(-1,k) \equiv V(0,k)$, $V(N+1,k) \equiv V(N,k) + 1$ and $\Lambda \equiv \theta + \alpha$.

Value iteration method

The value iteration method is defined as follows.

Step 0. $n := 0$ and $V^0(i,k) := 0$ for all i, k.

Step 1.

$$M^{n+1}(i) := i + \lambda h_1 + (1 - \alpha h_1) V^n(0,0),$$

$$W^{n+1}(i,k) := \frac{1}{\Lambda} \Bigg[\sum_{l=0}^{s+1} \gamma_{kl} V^n(i,l) + \lambda V^n(i+1,k)$$

$$+ \mu(i,k) V^n(i-1,k) + (\mu - \mu(i,k)) V^n(i,k) \Bigg],$$

$$V^{n+1}(i,k) := \min[M^{n+1}(i), W^{n+1}(i,k)] \quad (0 \leq k \leq s),$$

$$V^{n+1}(i, s+1) := i + \lambda h_2 + (1 - \alpha h_2) V^n(0,0).$$

Step 2. $n := n+1$ and return to Step 1.

In the next section, we derive some properties of $V(i,k)$ and $W(i,k)$ to analyze the optimal policy.

5.4 Properties of the cost functions

In this chapter, we assume the following conditions:

Condition 5.1

(1) $\sum_{m=l}^{s+1} \gamma_{km}$ *is increasing in* k *for all* l.

(2) $\overline{H}_2(x) \geq \overline{H}_1(x)$ *for all* x*, and* $H_1(x)$ *and* $H_2(x)$ *satisfy Condition* 2.2(3).

(3) $\mu(i,k)$ *is decreasing in* k *and increasing in* i.

Condition 5.1(1) tells that the server is likely to enter a higher state as the level of deterioration increases. Condition 5.1(2) tells that the time for corrective maintenance is stochastically longer than that for preventive maintenance. Under Condition 5.1(2), the inequality $h_1 \le h_2$ holds. Condition 5.1(3) tells that the service rate decreases as the server deteriorates, and the service rate increases as the queue length increases.

We prove the following properties.

Lemma 5.1

(1) $V(0,0) \le \lambda/\alpha$ *for all* i, k.

(2) $V(i,s) \le V(i,s+1)$.

Proof.
By Condition 2.2, $V^n(i,k)$ converge to $V(i,k)$ as $n \to \infty$. Thus, it is sufficient to prove above properties for $V^n(i,k)$. This is used in the proofs of Lemma 5.2 and Lemma 5.3.

Both properties hold for $n = 0$.
The first property holds because if $V^n(0,0) \le \lambda/\alpha$, then

$$V^{n+1}(0,0) \le \lambda h_1 + (1-\alpha h_1)\lambda/\alpha = \lambda/\alpha.$$

The second property is shown as follows.

$$V^{n+1}(i,s+1) - V^{n+1}(i,s) = (h_2 - h_1)\{\lambda - \alpha V^n(0,0)\} \ge 0.$$

The last inequality holds by Condition 5.1(2) and $V^n(0,0) \le \lambda/\alpha$. This completes the proof. □

Lemma 5.2
$V(i,k)$ *and* $W(i,k)$ *are increasing in* i *and* k.

Proof.
It is trivial for $n = 0$.

It is obvious that if $V^n(i,k)$ and $W^n(i,k)$ are increasing in i, then $V^{n+1}(i,k)$ and $W^{n+1}(i,k)$ are also increasing in i. With the inductive hypothesis that $V^n(i,k)$ and $W^n(i,k)$ are increasing in k, we prove that $V^{n+1}(i,k)$ and $W^{n+1}(i,k)$ are increasing in k as follows:

$$W^{n+1}(i,k) = \frac{1}{\Lambda}\Big[\sum_{l=0}^{s+1}\gamma_{kl}V^n(i,l) + \lambda V^n(i+1,k) \\ + \mu(i,k)V^n(i-1,k) + (\mu - \mu(i,k))V^n(i,k)\Big].$$

The first term is increasing in k by Lemma 4.1(2). The second term is also increasing in k by the inductive hypothesis. The third and fourth terms satisfy the following.

$$\begin{aligned}&\mu(i,k)V^n(i-1,k) + (\mu - \mu(i,k))V^n(i,k)\\ &\le \mu(i,k+1)V^n(i-1,k) + (\mu - \mu(i,k+1))V^n(i,k)\\ &\le \mu(i,k+1)V^n(i-1,k+1) + (\mu - \mu(i,k+1))V^n(i,k+1).\end{aligned}$$

The first inequality holds because $V^n(i-1,k) \le V^n(i,k)$ and $\mu(i,k) \ge \mu(i,k+1)$. Thus $W^{n+1}(i,k)$ is increasing in k.

Since $W^{n+1}(i,k)$ is increasing in k and $V^{n+1}(i,s) \le V^{n+1}(i,s+1)$ holds, $V^{n+1}(i,k)$ is also increasing in k. This completes the proof. □

Lemma 5.3

(1) $W(i+1,k) - W(i,k) \le 1 - \alpha/\Lambda$.

(2) $V(i+1,k) - V(i,k) \le 1$.

Proof.
Properties (1) and (2) trivially hold for $n = 0$. For $n > 0$, it holds that

$$\begin{aligned}&W^{n+1}(i+1,k) - W^{n+1}(i,k)\\ &\le \frac{1}{\Lambda}\Big[\Gamma + \lambda + \mu + \mu(i,k) - \mu(i+1,k)\Big] \le 1 - \alpha/\Lambda.\end{aligned}$$

The first inequality holds by $V^n(i+1,k) - V^n(i,k) \le 1$, and the last inequality holds by Condition 5.1(3).

Next, $V^{n+1}(i+1,k) - V^{n+1}(i,k) \leq 1$ is obvious from

$$\min\{x,y\} - \min\{a,b\} \leq \max\{x-a, y-b\}$$

and $V^{n+1}(i+1,s+1) - V^{n+1}(i,s+1) = 1$. Then, by induction, Lemma 5.3 holds. This completes the proof. □

These lemmas are used to obtain the switch curve structure of the optimal policy and restrict the region of the switching curve.

5.5 Structure of the optimal policy

In this section we discuss properties of the optimal policy.

Theorem 5.1
If $D(i,l) = 2$, then $D(j,k) = 2$ for all $k \leq l$ and $j \geq i$.

Proof.
$M(i) \geq W(i,l)$ holds because $D(i,l) = 2$. Then,

$$M(j) = M(i)+j-i \geq W(i,l)+j-i \geq W(i,l)+W(j,l)-W(i,l) \geq W(j,k).$$

The second inequality holds by Lemma 5.3(1), and the last inequality holds by Lemma 5.2. This completes the proof. □

Furthermore, the following theorem tells that the preventive maintenance area is restricted.

Theorem 5.2
For a fixed $i \geq 1$, define $K(i)$ as follows,

$$K(i) = \max\{k \mid \gamma_{k\,s+1}\lambda(h_2-h_1) - \mu(i,k) \leq \alpha i - \lambda + \alpha h_1\lambda\}. \tag{5.9}$$

Then $D(i,k) = 2$ for $k \leq K(i)$.

Proof.
Since $V(i,k) \leq M(i)$ for $0 \leq k \leq s$ and $i \geq 1$, it holds that

$$\begin{aligned}
W(i,k) &\leq \frac{1}{\Lambda}\Big[\sum_{l=0}^{s}\gamma_{kl}M(i) + \gamma_{k\,s+1}V(i,s+1) + \lambda(M(i)+1) \\
&\quad + \mu(i,k)(M(i)-1) + (\mu - \mu(i,k))M(i)\Big] \\
&= \frac{1}{\Lambda}\Big[(\Gamma - \gamma_{k\,s+1})M(i) + \gamma_{k\,s+1}\{M(i) + (h_2-h_1)(\lambda - \alpha V(0,0))\} \\
&\quad + \lambda M(i) + \lambda + \mu M(i) - \mu(i,k)\Big] \\
&= M(i) - \frac{1}{\Lambda}\Big[\alpha M(i) + \mu(i,k) - \lambda - \gamma_{k\,s+1}(h_2-h_1)(\lambda - \alpha V(0,0))\Big] \\
&= M(i) - \frac{1}{\Lambda}\Big[\alpha i + \alpha\lambda h_1 + \alpha(1-\alpha h_1)V(0,0) + \mu(i,k) \\
&\quad - \lambda - \gamma_{k\,s+1}(h_2-h_1)(\lambda - \alpha V(0,0))\Big] \\
&= M(i) - \frac{1}{\Lambda}\Big[\alpha i + \alpha\lambda h_1 + \mu(i,k) - \lambda - \gamma_{k\,s+1}(h_2-h_1)\lambda \\
&\quad + \{\gamma_{k\,s+1}(h_2-h_1) + (1-\alpha h_1)\}\alpha V(0,0)\Big] \\
&\leq M(i) - \frac{1}{\Lambda}\Big[\alpha i + \alpha\lambda h_1 + \mu(i,k) - \lambda - \gamma_{k\,s+1}(h_2-h_1)\lambda\Big].
\end{aligned}$$

From the last inequality, for a fixed i, $W(i,k) \leq M(i)$ holds if

$$\gamma_{k\,s+1}\lambda(h_2-h_1) - \mu(i,k) \leq \alpha i - \lambda + \alpha\lambda h_1.$$

Since $\gamma_{k\,s+1}\lambda(h_2-h_1)$ is increasing in k by Condition 5.1(1) and (2), and $\mu(i,k)$ is decreasing in k by Condition 5.1(3), the above inequality holds for $k \leq K(i)$. This completes the proof. □

By Theorem 5.1 and 5.2, the optimal policy has the switch curve structure as shown in Fig. 5.1. The state space is divided by an increasing function (switching curve), with the optimal action changes across the function, and the switching curve exists above $K(i)$.

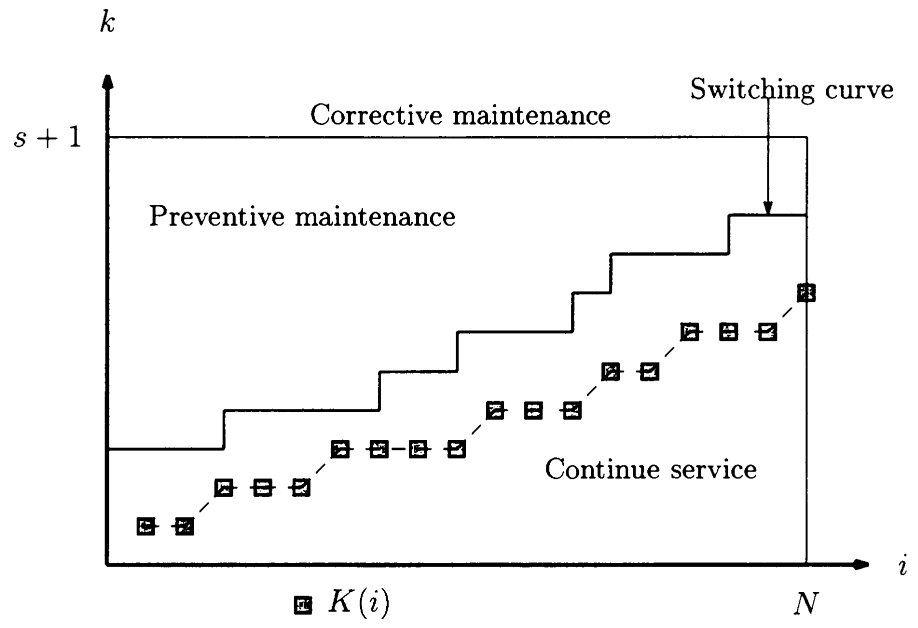

Figure 5.1: The structure of the optimal policy

5.6 Numerical example

In this section we show a numerical example of our model.

We calculate the optimal policy for a problem with $N = 10$, $s = 6$, arrival rate $\lambda = 7$, discount factor $\alpha = 0.25$, $\Gamma = 2$ and $\mu = 20$. The following table shows the values of $\mu(i, k)$.

6	3	6	9	12	12	12	12	12	12	12
5	3	6	9	12	12	12	12	12	12	12
4	4	8	12	16	16	16	16	16	16	16
3	4	8	12	16	16	16	16	16	16	16
2	5	10	15	20	20	20	20	20	20	20
1	5	10	15	20	20	20	20	20	20	20
$k = 0$	5	10	15	20	20	20	20	20	20	20
	$i = 1$	2	3	4	5	6	7	8	9	10

The service rate $\mu(i, k)$ increases linearly for i $(1 \leq i \leq 4)$. Thus, these

service rates indicate that the customers are served at four terminals and six terminals are used as waiting rooms. The transition of server deterioration is either to proceed to the next state or to enter the failure state directly. The following table shows the values of γ_{kl}, and other γ_{kl}'s are equal to zero.

	γ_{kk}	$\gamma_{k\,k+1}$	$\gamma_{k\,s+1}$
$k = 0$	1.5	0.1	0.4
1	1.4	0.1	0.5
2	0.7	0.1	1.2
3	0.6	0.2	1.2
4	0.4	0.3	1.3
5	0.3	0.4	1.3
6	0.2		1.8

In this problem, $H_1(x)$ and $H_2(x)$ are deterministic and defined as follows,

$$H_1(x) = \begin{cases} 0 & (0 \leq x < 2) \\ 1 & (2 \leq x), \end{cases} \quad H_2(x) = \begin{cases} 0 & (0 \leq x < 8) \\ 1 & (8 \leq x). \end{cases}$$

With these parameters, which satisfy Condition 5.1, $K(i)$ becomes

i	1	2	3	4	5	6	7	8	9	10
$K(i)$	0	0	1	2	2	2	2	2	2	2

From this table, we can see that, for example, it is optimal to continue service if the server state $k \leq 2$ for $i = 4, \ldots, 10$.

The optimal policy $D(i, k)$ computed by the value iteration method. The iteration is terminated when $\max_{i,k} |V^{n+1}(i,k) - V^n(i,k)|/V^n(i,k) < 10^{-6}$. After the iteration is terminated, the optimal policy is derived from Eq. (5.5)–(5.8). The following table shows the optimal action $D(i, k)$, where the number '1' in the table means that preventive maintenance is optimal, and '2' means that continuing service is optimal. $K(i)$ is expressed by '$\bar{2}$'.

Server state k	Optimal policy $D(i,k)$										
6	1	1	1	2	2	2	2	2	2	2	2
5	1	1	1	2	2	2	2	2	2	2	2
4	1	1	2	2	2	2	2	2	2	2	2
3	1	2	2	2	2	2	2	2	2	2	2
2	1	2	2	2	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$
1	2	2	2	$\bar{2}$	2	2	2	2	2	2	2
0	2	$\bar{2}$	$\bar{2}$	2	2	2	2	2	2	2	2
	0	1	2	3	4	5	6	7	8	9	10
	Queue length i										

This table illustrates the switch curve structure of optimal policy whose switching curve exists above $K(i)$.

5.7 Conclusion

We dealt with the optimal maintenance policy of a queueing system with multiple terminals handled by a deteriorating server. In this chapter, it was assumed that the deterioration affects both the time to failure and service rate.

We analyzed the structure of the optimal policy to minimize the number of lost customers, and proved a switch curve structure of the optimal policy, under some reasonable conditions, by using semi-Markov decision process. Furthermore, we derived that the region of the switching curve is restricted.

Chapter 6

A Queue with Decreasing Arrival Rate

6.1 Introduction

In Chapter 5 we studied a system whose service speed becomes slower as it deteriorates. In this chapter we study an optimal maintenance policy for a queueing system whose arrival rate decreases as it deteriorates. Customers arrive at the server in a Poisson stream and are served at a server, which is subject to multiple states. Each state indicates a level of popularity. The arrival rate depends on the server state and it decreases as the server loses popularity. The transitions of the server state are governed by a Markov process. The service time distribution is exponential and the service rate does not change by deterioration. By performing maintenance we can recover the server state completely, though all the customers in the system are lost upon the beginning of maintenance. The customers who arrive during maintenance are also lost. The system collects a fee from each customer, and, therefore, it is desirable to keep high popularity. In this chapter two systems are considered, which are different in the time when the fees are charged.

Model A: The system collects a unit fee from each customer upon arrival. If there are any customers upon the beginning of maintenance, the system pays back the fees to them upon the start of maintenance.

Model B: The system collects a unit fee from each customer upon departure and there is no repayment to customers who are lost due to maintenance.

For these two systems, our objective is to maximize the total expected discounted profit over an infinite time horizon. We use a semi-Markov decision process to formulate the problem and establish the properties of the optimal maintenance policy under certain conditions.

6.2 Model

We consider a single server queue with Poisson arrivals and a server with exponential service time. The customers form a single queue, whose capacity is denoted by N and they are served by the server with service rate μ. The server has $s+1$ states which are numbered from 0 to s. The number indicates the popularity of the server. In state 0 the server has the highest popularity, and the server becomes less popular as the state number ascends. The arrival rate depends on the server state k, and the arrival rate at state k is denoted by λ_k. The transitions of the server state are governed by a Markov process and the rate from state k to l is denoted by γ_{kl}. As stated in Chapter 5, we can assume $\sum_{l=0}^{s} \gamma_{kl} = \Gamma$ for all k. We consider no failure in this chapter, because popularity is irrelevant to failure.

The system receives a unit fee from each customer upon arrival in Model A or upon departure in Model B. Thus, it is desirable to keep the high arrival rate to earn fees. To recover the server state, we can perform maintenance of the server. After maintenance, the server returns to state 0. However, if there are customers in the system upon the beginning of maintenance, we lose the customers and pay back the fee to each lost customer in Model A but pay nothing in Model B. We also lose the customers during maintenance, i.e., we cannot earn money during maintenance. The maintenance time is generally distributed with the distribution function $H(x)$. The system state is expressed by the queue length i and the server state k. The system is continuously monitored. When the system state changes, we determine whether to perform maintenance or to continue service. Our objective is to maximize the total expected discounted reward with discount factor α.

6.3 Model with fees upon arrivals

In this section we deal with the system which receives a unit fee from each customer upon arrival. Upon the start of maintenance, if there are any customers in the system, we lose them and must pay back the fees to them. We discuss the optimal policy for this model in the following subsections.

6.3.1 Formulation

The elements of the decision process are similar to those in Chapter 5. State (i, k) is a pair of the queue length i and the server state k. The state transition is similar to those in Chapter 5 because the deterioration process is Markovian and interarrival time and service time is exponential, though the arrival rate depends on the server state. We then omit the detail of the decision process and start from the optimality equation. For the optimality equation, we define the following functions.

$V_A(i,k)$: The optimal reward function for state (i,k).

$M_A(i,k)$: The reward function for the policy of performing maintenance upon transition to (i,k) and taking optimal behavior thereafter. This will be denoted by $M_A(i)$ in the following, since it is a function of i only.

$W_A(i,k)$: The reward function for the policy of continuing service upon transition to (i,k) and taking optimal behavior thereafter.

$D_A(i,k)$: The optimal action for state (i,k), where

$$D_A(i,k) = \begin{cases} 1 & \text{if it is optimal to perform maintenance } (M_A(i) > W_A(i,k)), \\ 2 & \text{if it is optimal to continue service } (M_A(i) \le W_A(i,k)). \end{cases}$$

Note that there is no corrective maintenance because the failure never occurs.

Through the standard manner of semi-Markov decision process theory, we obtain the following optimality equation, where

$$\lambda \equiv \max_k \lambda_k, \tag{6.1}$$
$$\Lambda \equiv \lambda + \mu + \Gamma + \alpha, \tag{6.2}$$

are used.

$$V_A(i,k) = \max\{M_A(i), W_A(i,k)\}, \tag{6.3}$$
$$M_A(i) = -i + hV(0,0), \tag{6.4}$$
$$W_A(0,k) = \frac{1}{\Lambda}\Big[\lambda_k(V_A(1,k)+1) + \mu V_A(0,k) + \sum_{l=0}^{s}\gamma_{kl}V_A(0,l) + (\lambda-\lambda_k)V_A(0,k)\Big], \tag{6.5}$$
$$W_A(i,k) = \frac{1}{\Lambda}\Big[\lambda_k(V_A(i+1,k)+1) + \mu V_A(i-1,k) + \sum_{l=0}^{s}\gamma_{kl}V_A(i,l) + (\lambda-\lambda_k)V_A(i,k)\Big] \quad (1 \le i \le N-1), \tag{6.6}$$
$$W_A(N,k) = \frac{1}{\Lambda}\Big[\lambda_k V_A(N,k) + \mu V_A(N-1,k) + \sum_{l=0}^{s}\gamma_{kl}V_A(N,l) + (\lambda-\lambda_k)V_A(N,k)\Big]. \tag{6.7}$$

The constant h is defined by

$$h \equiv \int_0^\infty e^{-\alpha t}dH(t). \tag{6.8}$$

We assume Condition 2.2(3) for $H(x)$ to secure $0 < h < 1$.

By the following value iteration method, the values of $V_A(i,k)$ and $W_A(i,k)$ are obtained.

Value iteration method

Step 0. $n := 0$, and $V_A^0(i,k) := 0$ for all (i,k).

Step 1.

$$M_A^{n+1}(i) := -i + hV_A^n(0,0),$$
$$W_A^{n+1}(0,k) := \frac{1}{\Lambda}\Big[\lambda_k(V_A^n(1,k)+1) + \mu V_A^n(0,k) + \sum_{l=0}^{s}\gamma_{kl}V_A^n(0,l) + (\lambda-\lambda_k)V_A^n(0,k)\Big],$$
$$W_A^{n+1}(i,k) := \frac{1}{\Lambda}\Big[\lambda_k(V_A^n(i+1,k)+1) + \mu V_A^n(i-1,k) + \sum_{l=0}^{s}\gamma_{kl}V_A^n(i,l) + (\lambda-\lambda_k)V_A^n(i,k)\Big] \quad (1 \le i \le N-1),$$
$$W_A^{n+1}(N,k) := \frac{1}{\Lambda}\Big[\lambda_k V_A^n(N,k) + \mu V_A^n(N-1,k) + \sum_{l=0}^{s}\gamma_{kl}V_A^n(N,l) + (\lambda-\lambda_k)V_A^n(N,k)\Big],$$
$$V_A^{n+1}(i,k) := \max\{M_A^{n+1}(i), W_A^{n+1}(i,k)\}.$$

Step 2. $n := n+1$ and return to Step 1.

6.3.2 Analysis

To show the structure of the optimal policy, some properties of $V_A^n(i,k)$ and $W_A^n(i,k)$ are shown in this subsection. We assume the following conditions in this section.

Condition 6.1

(1) *The arrival rate λ_k is decreasing in k.*

(2) *For all u, $\sum_{l=u}^{s}\gamma_{kl}$ is increasing in k.*

Condition 6.1(1) tells that the arrival rate becomes lower as the server loses popularity. Condition 6.1(2) tells that the server is likely to enter a higher state as the server state becomes higher.

The following lemma gives an upper bound of $V_A^n(i,k)$.

Lemma 6.1

$$V_A^n(i,k) \le \lambda_0/\alpha.$$

Proof.

The result is trivial for $n=0$. For general n, assuming $V_A^n(i,k) \le \lambda_0/\alpha$, we show $V_A^{n+1}(i,k) \le \lambda_0/\alpha$.

$$\begin{aligned}
M_A^{n+1}(i) &= -i + hV_A^n(0,0) \le h\lambda_0/\alpha \le \lambda_0/\alpha, \\
W_A^{n+1}(i,k) &\le \frac{1}{\Lambda}[\lambda_k(\lambda_0/\alpha+1) + \mu\lambda_0/\alpha + \sum_{l=0}^{s}\gamma_{kl}\lambda_0/\alpha + (\lambda-\lambda_k)\lambda_0/\alpha] \\
&\le \frac{1}{\Lambda}[\lambda_0 + \mu\lambda_0/\alpha + \Gamma\lambda_0/\alpha + \lambda\lambda_0/\alpha] \\
&= \lambda_0/\alpha.
\end{aligned}$$

Since both $M_A^{n+1}(i) \le \lambda_0/\alpha$ and $W_A^{n+1}(i,k) \le \lambda_0/\alpha$ hold, the inequality $V_A^{n+1}(i,k) \le \lambda_0/\alpha$ follows. This completes the proof. □

Lemma 6.2

$V_A^n(i,k)$ and $W_A^n(i,k)$ have the following properties, respectively.

$$V_A^n(i+1,k) - V_A^n(i,k) \ge -1 \textit{ and } W_A^n(i+1,k) - W_A^n(i,k) \ge -1.$$

Proof.

The lemma obviously holds for $V_A^0(i,k)$. Next we prove

$$V_A^{n+1}(i+1,k) - V_A^{n+1}(i,k) \ge -1 \text{ and } W_A^{n+1}(i+1,k) - W_A^{n+1}(i,k) \ge -1$$

from the inductive hypothesis that $V_A^n(i+1,k) - V_A^n(i,k) \ge -1$.

(1) When $0 \le i \le N-2$,

$$\begin{aligned}
&W_A^{n+1}(i+1,k) - W_A^{n+1}(i,k) \\
&= \frac{1}{\Lambda}\Big[\lambda_k(V_A^n(i+2,k) - V_A^n(i+1,k)) + \mu(V_A^n(i,k) - V_A^n(i-1,k)) \\
&\qquad + \sum_{l=0}^{s}\gamma_{kl}(V_A^n(i+1,l) - V_A^n(i,l)) \\
&\qquad + (\lambda-\lambda_k)(V_A^n(i+1,k) - V_A^n(i,k))\Big] \\
&\ge \frac{1}{\Lambda}\Big[-\lambda_k - \mu - \sum_{l=0}^{s}\gamma_{kl} - (\lambda-\lambda_k)\Big] \\
&= \frac{1}{\Lambda}[-\Lambda+\alpha] \ge -1.
\end{aligned}$$

(2) When $i = N-1$,

$$\begin{aligned}
&W_A^{n+1}(N,k) - W_A^{n+1}(N-1,k) \\
&= \frac{1}{\Lambda}\Big[-\lambda_k + \mu(V_A^n(N-1,k) - V_A^n(N-2,k)) \\
&\qquad + \sum_{l=0}^{s}\gamma_{kl}(V_A^n(N,l) - V_A^n(N-1,l)) \\
&\qquad + (\lambda-\lambda_k)(V_A^n(N,k) - V_A^n(N-1,k))\Big] \\
&\ge \frac{1}{\Lambda}[-\Lambda+\alpha] \ge -1.
\end{aligned}$$

This shows $W_A^{n+1}(i+1,k) - W_A^{n+1}(i,k) \ge -1$. It is then obvious that

$$V_A^{n+1}(i+1,k) - V_A^{n+1}(i,k) \ge -1,$$

because $\max\{a,b\} - \max\{c,d\} \ge \min\{a-c, b-d\}$. This completes the proof. □

Lemma 6.3

$V_A^n(i,k)$ and $W_A^n(i,k)$ are decreasing in k.

Proof.

Lemma 6.3 is trivial for $V_A^0(i,k)$. Then we prove

$$V_A^{n+1}(i,k+1) \le V_A^{n+1}(i,k) \text{ and } W_A^{n+1}(i,k+1) \le W_A^{n+1}(i,k),$$

using the inductive hypothesis that

$$V_A^n(i,k+1) \le V_A^n(i,k).$$

We show $W_A^{n+1}(i,k+1) \le W_A^{n+1}(i,k)$ in the following.

(1) The proof for $0 \leq i \leq N-1$ is done as follows.

$$\begin{aligned}
&W_A^{n+1}(i,k+1) - W_A^{n+1}(i,k) \\
&= \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k)+1) \\
&\quad + \mu(V_A^n(i-1,k+1) - V_A^n(i-1,k)) \\
&\quad + \sum_{l=0}^{s}\gamma_{k+1l}V_A^n(i,l) - \sum_{l=0}^{s}\gamma_{kl}V_A^n(i,l) \\
&\quad + (\lambda-\lambda_{k+1})V_A^n(i,k+1) - (\lambda-\lambda_k)V_A^n(i,k)\Big] \\
&\leq \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k)+1) \\
&\quad + (\lambda-\lambda_{k+1})V_A^n(i,k+1) - (\lambda-\lambda_k)V_A^n(i,k)\Big] \\
&\leq \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k+1)+1) \\
&\quad + (\lambda-\lambda_{k+1})V_A^n(i,k+1) - (\lambda-\lambda_k)V_A^n(i,k+1)\Big] \\
&= \frac{1}{\Lambda}(\lambda_{k+1}-\lambda_k)(V_A^n(i+1,k+1) - V_A^n(i,k+1)+1) \\
&\leq 0.
\end{aligned}$$

By Lemma 4.1(2) the first inequality holds, and the last inequality holds by Condition 6.1(1) and Lemma 6.2.

(2) The proof for $i = N$ is done as follows.

$$\begin{aligned}
&W_A^{n+1}(N,k+1) - W_A^{n+1}(N,k) \\
&= \frac{1}{\Lambda}\Big[\lambda_{k+1}V_A^n(N,k+1) - \lambda_kV_A^n(N,k) \\
&\quad + \mu(V_A^n(N-1,k+1) - V_A^n(N-1,k)) \\
&\quad + \sum_{l=0}^{s}\gamma_{k+1l}V_A^n(N,l) - \sum_{l=0}^{s}\gamma_{kl}V_A^n(N,l) \\
&\quad + (\lambda-\lambda_{k+1})V_A^n(N,k+1) - (\lambda-\lambda_k)V_A^n(N,k)\Big] \\
&\leq \frac{1}{\Lambda}\Big[\lambda V_A^n(N,k+1) - \lambda V_A^n(N,k)\Big] \\
&\leq 0.
\end{aligned}$$

The inequality $V_A^{n+1}(i,k+1) \leq V_A^{n+1}(i,k)$ is obvious from $W_A^{n+1}(i,k+1) \leq W_A^{n+1}(i,k)$. This completes the proof. □

Since $V_A^n(i,k)$ and $W_A^n(i,k)$ converge to $V(i,k)$ and $W(i,k)$, respectively, as $n \to \infty$ by Condition 2.2, we have the following lemma.

Lemma 6.4

(1) $V_A(i,k) \leq \lambda_0/\alpha$.

(2) $V_A(i+1,k) - V_A(i,k) \geq -1$ *and* $W_A(i+1,k) - W_A(i,k) \geq -1$.

(3) $V_A(i,k+1) \leq V_A(i,k)$ *and* $W_A(i,k+1) \leq W_A(i,k)$.

Furthermore, if $\lambda_0 \leq \mu$ holds, the following lemma can be proved.

Lemma 6.5
If $\lambda_0 \leq \mu$, *then* $W_A(i,k) \geq M_A(i)$ *for* $i \geq 1$.

Proof.
Since $V_A(i,k) \geq M_A(i)$ holds by Eq. (6.3),

$$\begin{aligned}
W_A(i,k) &\geq \frac{1}{\Lambda}[\lambda_k M_A(i) + \mu M_A(i-1) + (\Gamma + \lambda - \lambda_k)M_A(i)] \\
&= \frac{1}{\Lambda}[(\lambda+\mu+\Gamma)(-i+hV_A(0,0)) + \mu] \\
&= -i + hV(0,0) + \frac{1}{\Lambda}[-\alpha(-i+hV_A(0,0)) + \mu] \\
&\geq M_A(i) + \frac{1}{\Lambda}[-\alpha(-i+h\lambda_0/\alpha) + \mu] \\
&\geq M_A(i) + \frac{1}{\Lambda}[-\lambda_0 + \mu] \\
&\geq M_A(i).
\end{aligned}$$

Therefore, $W_A(i,k) \geq M_A(i)$ holds for $i \geq 1$, if $\lambda_0 \leq \mu$ holds. This completes the proof. □

We use Lemma 6.4 to prove the switch curve structure of the optimal policy, and use Lemma 6.5 to restrict the switch curve structure.

6.3.3 Structure of the optimal policy

By the lemmas shown in the previous subsection, we show the structure of the optimal policy in this subsection.

The following theorem shows a monotone property of $D_A(i,k)$.

Theorem 6.1
If $D_A(i,l) = 2$, *then* $D_A(j,k) = 2$ *for all* $k \le l$ *and* $j \ge i$.

Proof.

$$W_A(j,k) \ge W_A(j,l) \ge W_A(i,l) - j + i \ge M_A(i) - j + i = M_A(j).$$

The first inequality holds by Lemma 6.4(3), and we obtain the second inequality by applying Lemma 6.4(2) repeatedly. This completes the proof. □

Theorem 6.1 shows that if it is optimal to continue service for some state, it is also optimal to continue service for the states with longer queue and better server state.

The next theorem is a direct result of Lemma 6.5.

Theorem 6.2
If $\lambda_0 \le \mu$, *then* $D_A(i,k) = 2$ *for* $i \ge 1$.

This theorem gives a sufficient condition under which we should not perform maintenance whenever there are customers in the system.

6.3.4 Numerical example

We give a numerical example here.

Let us consider a problem with server state set $\{0, 1, \ldots, 6\}$, service rate $\mu = 12.0$ and system capacity $N = 20$. Arrival rate and transition rate are shown below.

$\lambda_0 = 18.0$, $\lambda_1 = 12.0$, $\lambda_2 = 11.0$,
$\lambda_3 = 7.0$, $\lambda_4 = 6.0$, $\lambda_5 = 5.0$, $\lambda_6 = 4.0$,
$\gamma_{00} = 0.0098$, $\gamma_{01} = 0.0002$, $\gamma_{11} = 0.0095$, $\gamma_{12} = 0.0005$,
$\gamma_{22} = 0.0090$, $\gamma_{23} = 0.0010$, $\gamma_{33} = 0.0075$, $\gamma_{34} = 0.0025$,
$\gamma_{44} = 0.0060$, $\gamma_{45} = 0.0040$, $\gamma_{55} = 0.0010$, $\gamma_{56} = 0.0090$, $\gamma_{66} = 0.0100$.

The discount factor α is 0.01 and the maintenance time is deterministic, and is $-100 \log 0.1$, which means $h = 0.1$.

The optimal policy is derived by value iteration method as in Section 5.6, and we obtain the following optimal policy.

Server state k	Optimal policy $D_A(i,k)$																			
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2
2	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	0					5					10					15					20
	Queue length i																				

The result fits Theorem 6.1.

The result for Theorem 6.2 is obtained by setting $\mu = \lambda_0 = 18.0$.

Server state k	Optimal policy $D_A(i,k)$																			
6	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	0					5					10					15					20
	Queue length i																				

This optimal policy shows that maintenance is not optimal when there are any customers in the system.

6.4 Model with fees upon departures

In this section we deal with the system which receives a unit fee from each customer upon his/her departure. Upon the start of maintenance, if there are any customers in the system, we can clear them without any cost. We discuss properties of the optimal policy for this model in the following subsections.

6.4.1 Formulation

We define $V_B(i,k)$, $M_B(i,k)$, $W_B(i,k)$ and $D_B(i,k)$ in a similar way as done in Section 6.3. As we see in the following that $M_B(i,k)$ is a constant in Model B, it is denoted as M_B. For these functions, the following optimality equation holds.

$$W_B(0,k) = \frac{1}{\Lambda}\Big[\lambda_k V_B(1,k) + \mu V_B(0,k) + \sum_{l=0}^{s}\gamma_{kl}V_B(0,l) + (\lambda - \lambda_k)V_B(0,k)\Big], \tag{6.9}$$

$$W_B(i,k) = \frac{1}{\Lambda}\Big[\lambda_k V_B(i+1,k) + \mu(V_B(i-1,k)+1) + \sum_{l=0}^{s}\gamma_{kl}V_B(i,l) + (\lambda - \lambda_k)V_B(i,k)\Big] \quad (1 \le i \le N-1), \tag{6.10}$$

$$W_B(N,k) = \frac{1}{\Lambda}\Big[\lambda_k V_B(N,k) + \mu(V_B(N-1,k)+1) + \sum_{l=0}^{s}\gamma_{kl}V_B(N,l) + (\lambda - \lambda_k)V_B(N,k)\Big], \tag{6.11}$$

$$M_B = hV_B(0,0), \tag{6.12}$$

$$V_B(i,k) = \max\{M_B, W_B(i,k)\}. \tag{6.13}$$

6.4.2 Analysis

We consider the value iteration method to obtain the solution of the equations (6.9)–(6.13), and define $V_B^n(i,k)$, M_B^n and $W_B^n(i,k)$ in the same way as that in Section 6.3. We also assume Condition 6.1 in this subsection.

Lemma 6.6

(1) $V_B^n(i,k) \le \mu/\alpha$.

(2) $V_B^n(i,k)$ *and* $W_B^n(i,k)$ *are increasing in* i.

(3) $V_B^n(i,k)$ *and* $W_B^n(i,k)$ *are decreasing in* k.

The proofs of Lemma 6.6(1) and Lemma 6.6(2) can be done in the same way as in Lemma 6.1, and therefore we omit it.

Lemma 6.6(3) is proved by induction. It is obvious for $V_B^0(i,k)$. Next, we prove that $W_B^{n+1}(i,k+1) \le W_B^{n+1}(i,k)$, using the inductive hypothesis $V_B^n(i,k+1) \le V_B^n(i,k)$. For $1 \le i \le N-1$,

$$\begin{aligned}
&W_B^{n+1}(i,k+1) - W_B^{n+1}(i,k)\\
&= \frac{1}{\Lambda}\Big[\lambda_{k+1}V_B^n(i+1,k+1) - \lambda_k V_B^n(i+1,k)\\
&\qquad + \mu(V_B^n(i-1,k+1) - V_B^n(i-1,k))\\
&\qquad + \sum_{l=0}^{s}\gamma_{k+1\,l}V_B^n(i,l) - \sum_{l=0}^{s}\gamma_{kl}V_B^n(i,l)\\
&\qquad + (\lambda - \lambda_{k+1})V_B^n(i,k+1) - (\lambda-\lambda_k)V_B^n(i,k)\Big]\\
&\le \frac{1}{\Lambda}\Big[\lambda_{k+1}V_B^n(i+1,k+1) - \lambda_k V_B^n(i+1,k)\\
&\qquad + (\lambda - \lambda_{k+1})V_B^n(i,k+1) - (\lambda-\lambda_k)V_B^n(i,k)\Big]\\
&\le \frac{1}{\Lambda}\Big[\lambda_{k+1}V_B^n(i+1,k+1) - \lambda_k V_B^n(i+1,k+1)\\
&\qquad + (\lambda - \lambda_{k+1})V_B^n(i,k+1) - (\lambda-\lambda_k)V_B^n(i,k+1))\Big]\\
&= \frac{1}{\Lambda}(\lambda_{k+1} - \lambda_k)(V_B^n(i+1,k+1) - V_B^n(i,k+1))\\
&\le 0.
\end{aligned}$$

The first inequality follows from the inductive hypothesis and Lemma 4.1(2), and the last inequality holds by Condition 6.1(1) and Lemma 6.6(2).

For $i=0$ or $i=N$, the proofs are similar and omitted. This completes the proof. □

Thus the following lemma is obtained for the same reason as in Lemma 6.4.

Lemma 6.7

(1) *$V_B(i,k) \leq \mu/\alpha$.*

(2) *$V_B(i,k)$ and $W_B(i,k)$ are increasing in i.*

(3) *$V_B(i,k)$ and $W_B(i,k)$ are decreasing in k.*

Though Lemma 6.7 indicates that the optimal policy for Model B has the same structure as that of Theorem 6.1, the following lemma shows that the structure of Theorem 6.2 holds in Model B without the condition $\lambda_0 \leq \mu$.

Lemma 6.8
For $i \geq 1$, $W_B(i,k) \geq M_B$ holds.

Proof.

$$\begin{aligned} W_B(i,k) &\geq \frac{1}{\Lambda}[\lambda_k M_B + \mu(M_B+1) + \Gamma M_B + (\lambda - \lambda_k)M_B] \\ &= \frac{1}{\Lambda}[(\lambda + \mu + \Gamma + \alpha)M_B - \alpha M_B + \mu] \\ &\geq M_B. \end{aligned}$$

The last inequality follows from $M_B = hV(0,0) \leq V(0,0) \leq \mu/\alpha$. This completes the proof. □

By Lemma 6.7(3) and Lemma 6.8, we obtain the following structure of the optimal policy for Model B.

Theorem 6.3

(1) If $D_B(0,k) = 1$, then $D_B(0,l) = 1$ for $l \geq k$.

(2) $D_B(i,k) = 2$ for $i \geq 1$.

Proof.
It holds that $M_B > W_B(0,k)$ by $D_B(0,k) = 1$. Then, by Lemma 6.7(3), it holds that $M_B > W_B(0,l)$. Theorem 6.3(2) is obvious from Lemma 6.8. This completes the proof. □

This theorem means that we should perform maintenance only when there is no customer in the system and the server state is higher than some specified state.

6.4.3 Numerical example

Here we supply a numerical example for Model B. The parameters are the same as the first case of Model A; that is, $\mu = 12.0 < \lambda_0$. In Model A, the optimal policy had a switch curve structure. However, in Model B, we have the following optimal policy, which shows that the maintenance should be undertaken only when the system is empty.

Server state k	Optimal policy $D_B(i,k)$																				
6	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	0					5					10					15					20
	Queue length i																				

6.5 Conclusion

We discussed the optimal maintenance policy for a server with decreasing arrival rate. We dealt with the two models, and proved a switch curve structure of the optimal policy under some conditions. We also studied a sufficient condition under which we should perform maintenance only when the system is empty. Our results also hold for the following extended models.

(1) The system capacity could be infinite.

(2) The constant maintenance cost R is incurred for each maintenance. The proofs are done in the same way, only by replacing the terms $hV(0,0)$ by $-R + hV(0,0)$.

(3) Though we assumed the server state becomes 0 with probability 1 after the maintenance, we can consider a model in which the server becomes state l ($l = 0, 1, \ldots, s$) with probability p_l. This means that the server does not necessarily become new by the maintenance. The proofs are done in the same way, only by replacing the terms $hV_\cdot(0,0)$ by $h\sum_{j=0}^{s} p_l V_\cdot(0,l)$.

Chapter 7

A Queue with Decreasing Service Rate and Non-cancelable Customer

7.1 Introduction

In a production system, a system which produces a new product tends to receive more orders and earns more money than other systems. As the product becomes less popular, the orders will decrease and the manager will decide to change the product to a new one. The order in a production system is considered as the customer in a queueing system, if the production starts after we have orders, that is, no inventory system. Though we dealt with a model of this kind in Chapter 6, we assumed that the customers in the queue can be cancelled. In this chapter, we deal with the system where the customers in the queue can not be cancelled. In this system, we perform maintenance in the following procedure.

(1) Decide to perform maintenance.

(2) Reject all arriving customers and serve the customer in the system.

(3) Start maintenance after the system becomes empty.

Thus, the total time needed to the maintenance is the sum of the time needed to serve the customers in the queue and the maintenance time.

We consider the same situation as in Chapter 6 in other settings. The server state changes in a Markov process. The customers arrive at the system according to Poisson process whose arrival rate depends on the server state. The server state becomes higher as the time goes, and the arrival rate decreases. The customer is served at the server in exponential service time. A unit fee is received upon arrival in Model A, and received upon departure in Model B. A semi-Markov decision process is formulated to find the optimal policy that maximizes the income from customers.

7.2 Model

The system in this chapter has Poisson arrivals and an exponential server. The system capacity is N, and the service rate is μ. The server has $s+1$ state which are numbered from 0 to s. The number indicates the popularity of the server. In state 0 the server is most popular, and becomes less popular as the number becomes large. Therefore, λ_k is assumed to be decreasing in k. The transitions of the server state are governed by a Markov process and the rate from state k to l is γ_{kl}. The server state can be recovered by maintenance. We assume that once the customers enter the system, they must be served. Thus, the manager can complete a maintenance procedure, according to the following steps.

(1) Close the system and reject all the arrivals until the maintenance is completed.

(2) Serve all the customers in the system.

(3) Start maintenance when the system becomes empty.

(4) Open the system again after maintenance.

The distribution function of the maintenance time is denoted by $H(x)$. After the maintenance, the server recovers completely. The manager decides whether to perform maintenance, observing the queue length i and server state k. We consider two models, Model A and Model B. In Model A, the system receives a unit fee from customer upon arrival and in Model B upon departure.

We assume that the manager wants to maximize the total expected discounted reward. We derive a structure of the optimal policy under some conditions.

7.3 Model with fees upon arrivals

In this section we deal with Model A, i.e., the system that receives a unit fee from each customer upon arrival. We assume the following conditions to prove a structure of the optimal policy.

Condition 7.1

(1) $\lambda_k \geq \lambda_l$ for $k \leq l$.

(2) $\mu \geq \lambda_0$.

(3) $\sum_{l=u}^{s} \gamma_{kl}$ is increasing in k for all u.

Here we again assume that $\sum_{l=0}^{s} \gamma_{kl} = \Gamma$ holds for all k. These conditions are almost the same as Condition 6.1, except that we add Condition 7.1(2) in this section.

7.3.1 Formulation

The decision process is defined in a way similar to Chapter 6. The sets of actions and system states are the same.

$V_A(i,k)$: The optimal reward function for state (i,k).

$M_A(i,k)$: The reward function for the policy of performing maintenance upon transition to (i,k) and taking optimal behavior thereafter. Since this function depends only on i, it will be denoted by $M_A(i)$ in the following.

$W_A(i,k)$: The reward function for the policy of continuing service upon transition to (i,k) and taking optimal behavior thereafter.

$D_A(i,k)$: The optimal action for state (i,k), where

$$D_A(i,k) = \begin{cases} 1 \text{ if it is optimal to perform maintenance } (M_A(i) > W_A(i,k)), \\ 2 \text{ if it is optimal to continue service } (M_A(i) \le W_A(i,k)). \end{cases}$$

We obtain the following optimality equation.

$$\begin{aligned} W_A(i,k) &= \frac{1}{\Lambda}\Big[\lambda_k(V_A(i+1,k)+1) + \mu V_A(i-1,k) + \sum_{l=0}^{s}\gamma_{kl}V_A(i,l) \\ &\quad + (\lambda-\lambda_k)V_A(i,k)\Big] \ (0 \le i \le N), \\ M_A(i) &= \left(\frac{\mu}{\mu+\alpha}\right)^i hV_A(0,0), \\ V_A(i,k) &= \max\{M_A(i), W_A(i,k)\}, \end{aligned}$$

where

$$\begin{aligned} \lambda &\equiv \max_k \lambda_k, \\ \Lambda &\equiv \lambda+\mu+\Gamma+\alpha, \\ h &\equiv \int_0^\infty e^{-\alpha t}dH(t), \\ V_A(-1,k) &\equiv V_A(0,k) \text{ and } V_A(N+1,k) \equiv V_A(N,k)-1. \end{aligned}$$

The problem in this chapter is different from other problems in this thesis, in that the cost functions are not linear with respect to i.

The value iteration method is defined in a similar manner, and we define $V_A^n(i,k)$, $M_A^n(i)$ and $W_A^n(i,k)$ accordingly. With initial value $V_A^0(i,k) := 0$, they are calculated as follows.

$$\begin{aligned} W_A^{n+1}(i,k) &:= \frac{1}{\Lambda}\Big[\lambda_k(V_A^n(i+1,k)+1) + \mu V_A^n(i-1,k) \\ &\quad + \sum_{l=0}^{s}\gamma_{kl}V_A^n(i,l) + (\lambda-\lambda_k)V_A^n(i,k)\Big] \ (0 \le i \le N), \quad (7.1) \\ M_A^{n+1}(i) &:= \left(\frac{\mu}{\mu+\alpha}\right)^i hV_A^n(0,0), \quad (7.2) \\ V_A^{n+1}(i,k) &:= \max\{M_A^n(i), W_A^n(i,k)\}, \quad (7.3) \end{aligned}$$

where $V_A^n(-1,k)$ and $V_A^n(N+1,k)$ are defined by $V_A^n(-1,k) \equiv V_A^n(0,k)$ and $V_A^n(N+1,k) \equiv V_A^n(N,k)-1$, respectively.

Before studying the structure of the optimal policy, we prove some properties of these functions.

7.3.2 Analysis

First, we have the next lemma.

Lemma 7.1

(1) $V_A(i,k) \le \mu/\alpha$.

(2) $V_A(i+1,k) - V_A(i,k) \ge -1$ *and* $W_A(i+1,k) - W_A(i,k) \ge -1$.

(3) $W_A(i,k+1) \le W_A(i,k)$ *and* $V_A(i,k+1) \le V_A(i,k)$.

Proof of (1).
This proof is similar to the proof of Lemma 6.1, and is omitted.

Proof of (2).
The inequality for $V_A^0(i,k)$ is obvious. We prove $W_A^{n+1}(i+1,k) - W_A^{n+1}(i,k) \ge -1$ and $M_A^{n+1}(i+1) - M_A^{n+1}(i) \ge -1$ from inductive hypothesis $V_A^n(i+1,k) - V_A^n(i,k) \ge -1$. For $0 \le i \le N-1$,

$$\begin{aligned} &W_A^{n+1}(i+1,k) - W_A^{n+1}(i,k) \\ &= \frac{1}{\Lambda}\Big[\lambda_k(V_A^n(i+2,k) - V_A^n(i+1,k)) + \mu(V_A^n(i,k) - V_A^n(i-1,k)) \\ &\qquad + \sum_{l=0}^{s}\gamma_{kl}(V_A(i+1,l) - V_A(i,l)) \\ &\qquad + (\lambda-\lambda_k)(V_A^n(i+1,k) - V_A^n(i,k))\Big] \\ &\ge \frac{1}{\Lambda}\Big[-\lambda_k - \mu - \Gamma - (\lambda-\lambda_k)\Big] \ge -1, \end{aligned}$$

$$\begin{aligned} &M_A^{n+1}(i+1) - M_A^{n+1}(i) \\ &= \left(\frac{\mu}{\mu+\alpha}\right)^{i+1} hV_A^n(0,0) - \left(\frac{\mu}{\mu+\alpha}\right)^i hV_A^n(0,0) \end{aligned}$$

$$= \left(\frac{\mu}{\mu+\alpha}\right)^i \left(\frac{-\alpha}{\mu+\alpha}\right) hV_A^n(0,0) \geq -1.$$

By above two inequalities, the inequality $V_A^{n+1}(i+1,k) - V_A^{n+1}(i,k) \geq -1$ is obvious from

$$\max\{x,y\} - \max\{a,b\} \geq \min\{x-a, y-b\}.$$

Lemma 7.1(2) is then proved, for the same reason as in Lemma 6.4.

Proof of (3).

$V_A^0(i,k+1) \leq V_A^0(i,k)$ is obvious. We show that $V_A^n(i,k+1) \leq V_A^n(i,k)$ implies $W_A^{n+1}(i,k+1) \leq W_A^{n+1}(i,k)$.

$$\begin{aligned}
&W_A^{n+1}(i,k+1) - W_A^{n+1}(i,k) \\
&= \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k)+1) \\
&\quad + \mu(V_A^n(i-1,k+1) - V_A^n(i-1,k)) \\
&\quad + \sum_{l=0}^{s}\gamma_{k+1\,l}V_A^n(i,l) - \sum_{l=0}^{s}\gamma_{kl}V_A^n(i,l) \\
&\quad + (\lambda-\lambda_{k+1})V_A^n(i,k+1) - (\lambda-\lambda_k)V_A^n(i,k)\Big] \\
&\leq \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k+1)+1) \\
&\qquad + (\lambda-\lambda_{k+1})V_A^n(i,k+1) - (\lambda-\lambda_k)V_A^n(i,k+1)\Big] \\
&= \frac{1}{\Lambda}\Big[\lambda_{k+1}(V_A^n(i+1,k+1)+1) - \lambda_k(V_A^n(i+1,k+1)+1) \\
&\qquad - \lambda_{k+1}V_A^n(i,k+1) + \lambda_k V_A^n(i,k+1)\Big] \\
&= \frac{1}{\Lambda}(\lambda_{k+1}-\lambda_k)(V_A^n(i+1,k+1) - V_A^n(i,k+1)+1) \leq 0
\end{aligned}$$

The first inequality holds by Condition 7.1(3) and $V_A^n(i,k+1) \leq V_A^n(i,k)$. The last inequality holds by Lemma 7.1(2).

It is obvious that $V_A^{n+1}(i,k+1) \leq V_A^{n+1}(i,k)$ from $W_A^{n+1}(i,k+1) \leq W_A^{n+1}(i,k)$. Thus, Lemma 7.1(3) is proved, for the same reason as in Lemma 6.4. This completes the proof. □

Lemma 7.2

(1) $W_A(i,k) \geq M_A(i)$ *for* $i \geq 1$.

(2) $W_A(0,k) \geq M_A(i)$ *for* k *such that* $\lambda_k \geq h\mu(\mu+\alpha)/(\mu+\alpha-h\mu)$.

Proof of (1).

For $1 \leq i \leq N-1$, we have

$$\begin{aligned}
W_A(i,k) &= \frac{1}{\Lambda}\Big[\lambda_k(V_A(i+1,k)+1) + \mu V_A(i-1,k) \\
&\qquad + \sum_{l=0}^{s}\gamma_{kl}V_A(i,l) + (\lambda-\lambda_k)V_A(i,k)\Big] \\
&\geq \frac{1}{\Lambda}\Big[\lambda_k(M_A(i+1)+1) + \mu M_A(i-1) + \\
&\qquad \Gamma M_A(i) + (\lambda-\lambda_k)M_A(i)\Big] \\
&= \frac{1}{\Lambda}\Big[\lambda_k\left(\frac{\mu}{\mu+\alpha}\right)M_A(i) + \lambda_k + \mu\left(\frac{\mu+\alpha}{\mu}\right)M_A(i) \\
&\qquad + \Gamma M_A(i) + (\lambda-\lambda_k)M_A(i)\Big] + \frac{\lambda_k}{\Lambda} \\
&= M_A(i) - \frac{\lambda_k}{\Lambda}\left(\frac{\alpha}{\mu+\alpha}\right)M_A(i) + \frac{\lambda_k}{\Lambda} \geq M_A(i).
\end{aligned}$$

For $i = N$, the proof is similar, and is omitted.

Proof of (2).

First we can derive the following inequality by $V_A(i,k) \geq M_A(i)$.

$$\begin{aligned}
W_A(0,k) &\geq \frac{1}{\Lambda}\Big[\lambda_k(M_A(1)+1) + \mu M_A(0) + \Gamma M_A(0) + (\lambda-\lambda_k)M_A(0)\Big] \\
&= \frac{1}{\Lambda}\Big[\frac{\lambda_k\mu}{\mu+\alpha}M_A(0) + \lambda_k + \mu M_A(0) + \Gamma M_A(0) + (\lambda-\lambda_k)M_A(0)\Big] \\
&= M_A(0) + \frac{1}{\Lambda}\left\{-\left(\frac{\alpha\lambda_k}{\mu+\alpha}+\alpha\right)M_A(0) + \lambda_k\right\} \\
&\geq M_A(0) + \frac{1}{\Lambda}\left\{-\left(\frac{h\mu\lambda_k}{\mu+\alpha}+h\mu\right) + \lambda_k\right\}.
\end{aligned}$$

Thus, if $\lambda_k - \left(\frac{h\mu\lambda_k}{\mu+\alpha}+h\mu\right) \geq 0$ i.e., $\lambda_k \geq \frac{h\mu(\mu+\alpha)}{\mu+\alpha-h\mu}$ holds, then $W_A(0,k) \geq M_A(0)$ holds. This completes the proof. □

We obtain a structure of the optimal policy from Lemma 7.1 and 7.2.

Theorem 7.1

(1) *If* $D_A(0,k) = 1$, *then* $D_A(0,l) = 1$ *for* $l \geq k$.

(2) $D_A(0,k) = 2$ *for* k *such that* $\lambda_k \geq h\mu(\mu+\alpha)/(\mu+\alpha-h\mu)$.

(3) $D_A(i,k) = 2$ *for* $i \geq 1$.

Proofs are omitted because it is direct from Lemma 7.1 and 7.2.

7.3.3 Numerical example

We confirm our result by a numerical example in this section. We calculate value functions for a system with capacity $N = 10$, server state $\{0,\ldots,7\}$, discount factor $\alpha = 1.0$ and $\mu = 10.0$. The distribution function of maintenance time is assumed to be

$$H(x) = \begin{cases} 0 & x < 1.0 \\ 1 & x \geq 1.0, \end{cases}$$

that is, $h = 1/e$.

The arrival rates for server states are

$$\lambda_0 = 9.0, \quad \lambda_1 = 8.0, \quad \lambda_2 = 7.0, \quad \lambda_3 = 6.0,$$
$$\lambda_4 = 4.0, \quad \lambda_5 = 3.0, \quad \lambda_6 = 1.0, \quad \lambda_7 = 0.5.$$

The transition rates of server states are

$$\begin{aligned} &\gamma_{00} = 1.5, \quad \gamma_{01} = 0.1, \quad \gamma_{07} = 0.4, \\ &\gamma_{11} = 1.4, \quad \gamma_{12} = 0.1, \quad \gamma_{17} = 0.5, \\ &\gamma_{22} = 0.7, \quad \gamma_{23} = 0.1, \quad \gamma_{27} = 1.2, \\ &\gamma_{33} = 0.6, \quad \gamma_{34} = 0.2, \quad \gamma_{37} = 1.2, \\ &\gamma_{44} = 0.4, \quad \gamma_{45} = 0.3, \quad \gamma_{47} = 1.3, \\ &\gamma_{55} = 0.3, \quad \gamma_{56} = 0.4, \quad \gamma_{57} = 1.3, \\ &\gamma_{66} = 0.2, \quad \gamma_{67} = 1.8, \quad \gamma_{77} = 2.0. \end{aligned}$$

Since these parameters satisfy Condition 7.1, we should continue service if the state is (0, 0), (0, 1), (0, 2) or (0, 3) by Theorem 7.1(2), because

$$h\mu(\mu+\alpha)/(\mu+\alpha-h\mu) = 5.527 \leq \lambda_3 \text{ holds.}$$

Since $\lambda_0 \leq \mu$, $D_A(i,k) = 2$ holds for $i \geq 1$ by Theorem 7.1(3). The optimal policy is calculated by value iteration method as follows.

Server State k	Optimal policy $D_A(i,k)$										
7	1	2	2	2	2	2	2	2	2	2	2
6	1	2	2	2	2	2	2	2	2	2	2
5	2	2	2	2	2	2	2	2	2	2	2
4	2	2	2	2	2	2	2	2	2	2	2
3	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
1	2	2	2	2	2	2	2	2	2	2	2
0	2	2	2	2	2	2	2	2	2	2	2
	0	1	2	3	4	5	6	7	8	9	10
	Queue length i										

7.4 Model with fees upon departures

We examine in this section the case in which the customer pays upon departure. $V_B(i,k)$, $W_B(i,k)$, $M_B(i)$ and $D_B(i,k)$ are defined in a similar manner as in the previous section.

7.4.1 Formulation

The optimality equation is obtained as follows:

$$\begin{aligned} W_B(i,k) = \frac{1}{\Lambda}\Big[&\lambda_k V_B(i+1,k) + \mu(V_B(i-1,k)+1) \\ &+ \sum_{l=0}^{s} \gamma_{kl} V_B(i,l) + (\lambda-\lambda_k)V_B(i,k)\Big], \end{aligned} \tag{7.4}$$

$$M_B(i) = \frac{\mu}{\alpha}\left[1-\left(\frac{\mu}{\mu+\alpha}\right)^i\right] + \left(\frac{\mu}{\mu+\alpha}\right)^i hV_B(0,0), \tag{7.5}$$

$$V_B(i,k) = \max\{M_B(i), W_B(i,k)\}, \tag{7.6}$$

$M_B(i)$ is obtained by the following relation

$$M_B(i+1) = \int_0^\infty \mu e^{-(\mu+\alpha)x}(M_B(i)+1)dx = \frac{\mu}{\mu+\alpha}(M_B(i)+1),$$

where $M_B(0) = hV_B(0,0)$.

We define the value iteration method with initial value $V_B^0(i,k) := 0$ for all (i,k).

$$W_B^{n+1}(i,k) = \frac{1}{\Lambda}\Big[\lambda_k V_B^n(i+1,k) + \mu(V_B^n(i-1,k)+1) + \sum_{l=0}^{s}\gamma_{kl}V_B^n(i,l) + (\lambda-\lambda_k)V_B^n(i,k)\Big], \tag{7.7}$$

$$M_B^{n+1}(i) = \frac{\mu}{\alpha}\left[1-\left(\frac{\mu}{\mu+\alpha}\right)^i\right] + \left(\frac{\mu}{\mu+\alpha}\right)^i hV_B^n(0,0), \tag{7.8}$$

$$V_B^{n+1}(i,k) = \max\{M_B^{n+1}(i), W_B^{n+1}(i,k)\}. \tag{7.9}$$

In Model B, we assume the following.

Condition 7.2

(1) $\lambda_k \geq \lambda_l$ for $k \leq l$,

(2) $\gamma_{kl} = 0$ for $k \geq l$ and $\sum_{l=u}^{s}\gamma_{kl}$ is increasing in k for all u.

7.4.2 Analysis

To prove the properties of the optimal policy, we prove the following lemma in this subsection.

Lemma 7.3

(1) $V_B(i,k) \leq \mu/\alpha$.

(2) *$W_B(i+1,k) \geq W_B(i,k)$ and $V_B(i+1,k) \geq V_B(i,k)$.*

(3) *$W_B(i,k+1) \leq W_B(i,k)$ and $V_B(i,k+1) \leq V_B(i,k)$.*

(4) *For $i \geq 1$, $W_B(i,k) \geq M_B(i)$.*

(5) *$W_B(0,k) \geq M_B(0)$ for k such that $\lambda_k \geq h\mu(\mu+\alpha)/(\mu+\alpha-h\mu)$.*

Proof of (1).
Lemma 7.3(1) is easily proved, and therefore omitted.

Proof of (2).
We prove $W_B^{n+1}(i+1,k) \geq W_B^{n+1}(i,k)$ and $M_B^{n+1}(i+1) \geq M_B^{n+1}(i)$ from inductive hypothesis $V_B^n(i+1,l) \geq V_B^n(i,l)$ as follows.

$$\begin{aligned} & W_B^{n+1}(i+1,k) - W_B^{n+1}(i,k) \\ &= \frac{1}{\Lambda}\Big[\lambda_k(V_B^n(i+2,k) - V_B^n(i+1,k)) + \mu(V_B^n(i,k) - V_B^n(i-1,k)) \\ &\quad + \sum_{l=0}^{s}\gamma_{kl}(V_B^n(i+1,l) - V_B^n(i,l)) \\ &\quad + (\lambda-\lambda_k)(V_B^n(i+1,k) - V_B^n(i,k))\Big] \\ &\geq 0. \end{aligned} \tag{7.10}$$

$$\begin{aligned} & M_B^{n+1}(i+1) - M_B^{n+1}(i) \\ &= \frac{\mu}{\mu+\alpha}(M_B^n(i)+1) - M_B^n(i) \\ &= \frac{-\alpha}{\mu+\alpha}M_B^n(i) + \frac{\mu}{\mu+\alpha} \geq 0. \end{aligned} \tag{7.11}$$

The last inequality holds by Lemma 7.3(1). $V_B^{n+1}(i+1,k) \geq V_B^{n+1}(i,k)$ is obvious from inequalities (7.10) and (7.11). Then Lemma 7.3(2) is shown for the same reason as in Lemma 6.4.

Proof of (3).
The inequalities are shown as follows.

$$\begin{aligned} & W_B^{n+1}(i,k+1) - W_B^{n+1}(i,k) \\ &\leq \frac{1}{\Lambda}\Big[\lambda_{k+1}V_B^n(i+1,k+1) - \lambda_k V_B^n(i+1,k)) \end{aligned}$$

$$
\begin{aligned}
&+ (\lambda - \lambda_{k+1})V_B^n(i, k+1) - (\lambda - \lambda_k)V_B^n(i, k))\Big] \\
\le\ & \frac{1}{\Lambda}\Big[\lambda_{k+1}V_B^n(i+1, k+1) - \lambda_k V_B^n(i+1, k+1)) \\
&+ (\lambda - \lambda_{k+1})V_B^n(i, k+1) - (\lambda - \lambda_k)V_B^n(i, k+1))\Big] \\
=\ & \frac{1}{\Lambda}(\lambda_{k+1} - \lambda_k)[V_B^n(i+1, k+1) - V_B^n(i, k+1)] \\
\le\ & 0 \qquad (7.12)
\end{aligned}
$$

The first and second inequalities follow from Condition 7.2(2) and inductive hypothesis $V_B^n(i, k+1) \le V_B^n(i, k)$. The last inequality holds by Condition 7.2(1) and Lemma 7.3(2). It is obvious from (7.12) that $V_B^{n+1}(i, k+1) \le V_B^{n+1}(i, k)$ holds. Then Lemma 7.3(3) is shown for the same reason as in Lemma 6.4.

Proof of (4).

We can derive the following relations.

$$
\begin{aligned}
W_B(i,k) &= \frac{1}{\Lambda}\Big[\lambda_k V_B(i+1, k) + \mu(V_B(i-1, k) + 1) \\
&\quad + \sum_{l=0}^{s} \gamma_{kl} V_B(i, l) + (\lambda - \lambda_k)V_B(i, k)\Big] \\
&\ge \frac{1}{\Lambda}\Big[\lambda_k M_B(i+1) + \mu(M_B(i-1) + 1) \\
&\quad + \Gamma M_B(i) + (\lambda - \lambda_k)M_B(i)\Big] \\
&= \frac{1}{\Lambda}\Big[\lambda_k \left(\frac{\mu}{\mu+\alpha}\right)(M_B(i) + 1) + \mu\left(\frac{\mu+\alpha}{\mu}\right)M_B(i) \\
&\quad + \Gamma M_B(i) + (\lambda - \lambda_k)M_B(i)\Big] \\
&= M_B(i) - \frac{\lambda_k}{\Lambda}\left(\frac{\alpha}{\mu+\alpha}\right)M_B(i) + \frac{\lambda_k}{\Lambda}\frac{\mu}{\mu+\alpha} \\
&\ge M_B(i).
\end{aligned}
$$

The last inequality holds by Lemma 7.3(1).

Proof of (5).

For $i = 0$, we have the following inequality.

$$
W_B(0,k) \ge \frac{1}{\Lambda}\Big[\lambda_k M_B(1) + \mu M_B(0) + \Gamma M_B(0) + (\lambda - \lambda_k)M_B(0)\Big]
$$

$$
\begin{aligned}
&= \frac{1}{\Lambda}\Big[\frac{\lambda_k \mu}{\mu+\alpha}M_B(0) + \lambda_k + \mu M_B(0) + \Gamma M_B(0) \\
&\quad + (\lambda - \lambda_k)M_B(0)\Big] \\
&= M_B(0) + \frac{1}{\Lambda}\left\{-\left(\frac{\alpha\lambda_k}{\mu+\alpha} + \alpha\right)M_B(0) + \lambda_k\right\} \\
&\ge M_B(0) + \frac{1}{\Lambda}\left\{-\left(\frac{h\mu\lambda_k}{\mu+\alpha} + h\mu\right) + \lambda_k\right\}.
\end{aligned}
$$

This indicates that if $\lambda_k \ge h\mu(\mu+\alpha)/(\mu+\alpha-h\mu)$, then $W_B(0,k) \ge M_B(0)$ holds. This completes the proof. □

From this lemma we have the following theorem.

Theorem 7.2

(1) *If* $D_B(0,k) = 1$, *then* $D_B(0,l) = 1$ *for* $l \ge k$.

(2) $D_B(0,k) = 2$ *for* k *such that* $\lambda_k \ge h\mu(\mu+\alpha)/(\mu+\alpha-h\mu)$.

(3) *For* $i \ge 1$, $D_B(i,k) = 2$.

Since the result is similar to that in Model A, numerical example is omitted.

7.5 Conclusion

We discussed the optimal maintenance policy for the server with decreasing arrival rate under the assumption that the customers in the queue must be served. Therefore, the start of maintenance is postponed until we serve all the customers remaining in the system if we decide the maintenance when there are any customer in the system. After we decide the maintenance, the arrivals are lost while we serve the customers and take the subsequent maintenance. We dealt with two models, which are different as to when to receive the fee. We showed a structural theorem of the optimal policy under some conditions. The structure shows that the maintenance action should not be taken when there are any customers in the queue. We also showed that the maintenance should be taken when the system is empty and the deterioration level exceeds a certain number, which is easily calculated from given parameters.

Chapter 8

Conclusion

8.1 Summary of the thesis

This thesis dealt with various preventive maintenance problems that arise in queueing systems. Many systems need maintenance, but customers usually suffer from inconvenience caused by the maintenance. In reliability theory, the inconvenience is evaluated by the cost which is incurred during maintenance. Since the cost of maintenance usually depends on only the deterioration level of the system, the optimal maintenance policy also depends only on the deterioration level, as we reviewed in Section 1.2. In queueing systems, which involve scheduling problems, maintenance problems have been also studied. As we reviewed in Section 1.3, the inconvenience is due to the delay or holding cost caused by the failure. Though we usually consider the preventive maintenance in reliability theory, no preventive maintenance is considered in such queueing systems because the failure time distribution is exponential. In these problems, the optimal policy determines which maintenance action should be taken upon failure, if there are several maintenance actions with different costs and times, or which job should be processed before the failure in scheduling problems.

In this thesis, we assume that the failure time of the queueing system is not exponential. The failure time is expressed by a general distribution function in Chapter 3, and by the time before the server state becomes the failure state in a Markov process in Chapters 4 –7. As the optimality criterion, we considered the number of lost customers caused by maintenance and failure in Chapters 3, 4 and 5, and the reward which is taken from the customers in Chapters 6 and 7.

In Chapter 3, we considered an $M/M/1$ queueing system, in which the server may fail according to a failure distribution function. This is a basic combination of queueing theory and reliability theory, because the $M/M/1$ queueing system is a fundamental model in queueing theory and the failure distribution function is a basic element in reliability theory. Under the assumption that the server is checked periodically, we proposed a maintenance policy which is based on the observation times and queue length. The state space becomes two-dimensional, where coordinates indicate the number of observations and queue length. For this state space, we proved that the optimal policy has a switch curve structure under some appropriate conditions on the failure time distribution and the maintenance time.

As another basic model, we can consider an $M/M/1$ queueing system with Markovian degradation process. In Chapter 4, the system was further extended for the $M/G/1$ queue with Markovian degradation process, which is an extension of $M/M/1$ queue with Markovian degradation process, because the service time distribution is allowed to be general. Again, the state space becomes two-dimensional, where coordinates indicate the queue length and the server state. We proved a switch curve structure of the optimal policy for this problem under some conditions on the service time distribution, the transition rate of the server state and the maintenance time.

Chapter 5 dealt with a queueing system which has N terminals to serve customers and all the terminals are controlled by one server. In a network system, a server may be considered as a printer or a file server. The server deteriorates according to a Markovian degradation process. As the server deteriorates, it is assumed that the service at a terminal becomes slow. Since the capacity of the system is restricted, the customers who arrive while the system is full are also counted as loss. For this system, we proved that the optimal policy has a switch curve structure, and restricted the region of the switching curve by giving the bound of the preventive maintenance area.

In Chapters 6 and 7, two queueing systems were studied, where arrival rate decreases as a result of the deterioration of the server. The objective is to maximize the total amount of fees which is collected from customers. As this type of system, we can give an example of a production system. In a production system, the number of orders will decrease as the production becomes out-of-date, where deterioration is considered as the age of the production. In Chapter 6, we assumed that the maintenance starts whenever we decide to do so. If there are customers in the system, they are removed by paying the cancel cost. In Chapter 7, we assumed that the customers in the system cannot be cancelled, and the maintenance starts after serving all these customers. Both systems has almost the same properties of the optimal policy. This indicates that the maintenance action while the customers are in the system is not optimal, and the maintenance action should be taken when the system is empty and the deterioration level becomes higher than a specified state.

8.2 Future study

In this thesis, we studied basic maintenance problems of queueing systems. We assumed several conditions in order to obtain theoretical results of the optimal policy. We may try to extend the results for the following models as future studies.

(1) We assumed that the maintenance time or cost is the same in all states except the failure state. This assumption could be true if the maintenance is to replace the old machine to the new one without salvage. However, in general, the time or cost depends on the deterioration level of the system, because the maintenance for the older machine is usually more expensive. If we assume that the maintenance time depends on the deterioration level, it will be difficult to prove a switch curve structure of the optimal policy, because the proof of the monotone property for the deterioration level becomes difficult.

(2) In Chapters 5, 6 and 7, it was considered that the deterioration process affects the queue length process. But we have not dealt with the system where the queue length process affects the deterioration process. In a real system, it may be more appropriate to consider that the system which has served more customers is more likely to fail. This model, however, needs a continuous state space to express the time during which the server serves the customers. The continuous state space also makes it difficult to analyze the optimal policy.

(3) We consider that the customers are lost when the server fails, and the cost is incurred for the lost customers. As another cost, we may consider the system in which the customers are kept waiting during the maintenance, and the holding cost is incurred for the waiting customers in this model.

Though we can consider other models by combining a queueing system with a reliability model, the systems including two general distributions will be theoretically difficult to analyze, and we probably need to develop numerical methods to solve the problems associated with such systems.

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