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SEMI-ALGEBRAIC PARTITION AND BASIS OF BOREL-MOORE HOMOLOGY OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. We describe an explicit semi-algebraic partition for the complement of a real hyperplane arrangement such that each piece is contractible and so that the pieces form a basis of Borel-Moore homology. We also give an explicit correspondence between the de Rham cohomology and the Borel-Moore homology.

1. INTRODUCTION

Semi-algebraic partitions of algebraic sets have been studied in various fields of mathematics from geometry to computational algebra. The general theory says that any algebraic set has a semi-algebraic triangulation. However, not only triangulations, but also other types of "efficient" decompositions are sometimes useful. The following are motivating examples.

Example 1.1. $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^1 \sqcup \{pt\}$.

Example 1.2. Consider $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Put

$$\begin{aligned} S_0 &= \{re^{i\theta} \mid r > 0, 0 < \theta < 2\pi\}, \\ S_1 &= \{z \mid z \in \mathbb{R}, z > 0\}. \end{aligned}$$

Then $\mathbb{C}^* = S_0 \sqcup S_1$.

Both of these decompositions reflect (co)homological structures of the manifolds naturally. More precisely, they are

(i) disjoint unions of contractible semi-algebraic subsets, and
(ii) the closures of the pieces form a basis of Borel-Moore (locally finite) homology $H_*^{BM}(X, \mathbb{Z})$ (or ordinary cohomology $H^*(X, \mathbb{Z})$ via Poincaré duality).

The purpose of this paper is to generalize Example 1.2 to hyperplane arrangements defined over the real number field $\mathbb{R}$.

There is another reason to expect the existence of such partitions. The complement of a complex hyperplane arrangement is known to be a minimal space [1, 2, 6],

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that is, a space homotopy equivalent to a finite CW complex with exactly as many k-cells as the k-th Betti number, for all k. If the arrangement is defined over $\mathbb{R}$, then the real structures (e.g., chambers) are related to the topology of the complexified complement ([7, 12]). With the help of real structures, the minimal CW complex has been explicitly described in [8] and [10]. It is natural to expect that the dual stratification to a minimal CW complex induces a partition satisfying (i) and (ii) above.

2. Preliminary

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an affine hyperplane arrangement in the real vector space $\mathbb{R}^\ell$. Let us fix a defining affine linear form α_i in such a way that $H_i = \{\alpha_i = 0\}$. Let $L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. We denote by $L^p(\mathcal{A})$ the set of all p-dimensional affine subspaces $X \in L(\mathcal{A})$.

For an affine subspace $X \subset \mathbb{R}^\ell$, let us denote by $X_\mathbb{C} = X \otimes_\mathbb{R} \mathbb{C}$ the complexification. Denote by $\mathsf{ch}(\mathcal{A})$ the set of all chambers and by $\mathsf{M}(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H_\mathbb{C}$ the complement to the complexified hyperplanes.

Let $\mathcal{F}$ be a generic flag in $\mathbb{R}^\ell$,

$$\mathcal{F} : \emptyset = \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^\ell = \mathbb{R}^\ell,$$

where each $\mathcal{F}^q$ is a *generic* q-dimensional affine subspace, that is, $\dim \mathcal{F}^q \cap X = q + \dim X - \ell$ for $X \in L(\mathcal{A})$. Let $\{h_1, \dots, h_\ell\}$ be a system of defining equations of $\mathcal{F}$, that is,

$$\mathcal{F}^q = \{h_{q+1} = \cdots = h_\ell = 0\}, \text{ for } q = 0, 1, \dots, \ell - 1,$$

where each h_i is an affine linear form on $\mathbb{R}^\ell$. Define

$$\mathsf{ch}^q_\mathcal{F}(\mathcal{A}) = \{C \in \mathsf{ch}(\mathcal{A}) \mid C \cap \mathcal{F}^q \neq \emptyset \quad \text{and} \quad C \cap \mathcal{F}^{q-1} = \emptyset\}.$$

We assume that $\mathcal{F}$ satisfies the following:

Assumption 2.1. *For $q = 0, \dots, \ell$, $\mathcal{F}^q_{>0}$ denotes*

$$\{h_{q+1} = h_{q+2} = \cdots = h_\ell = 0, h_q > 0\}.$$

(1) *For an arbitrary chamber C, if it belongs to $\mathsf{ch}^q_\mathcal{F}(\mathcal{A})$, then $C \cap \mathcal{F}^q \subset \mathcal{F}^q_{>0}$.*
(2) *For any two X, $X' \in L(\mathcal{A})$ with $\dim X = \dim X' = \ell - q$ (i.e. satisfying $X \cap \mathcal{F}^q = \{1pt\}$ and $X' \cap \mathcal{F}^q = \{1pt\}$), if $X \neq X'$, then*

$$h_q(X \cap \mathcal{F}^q) \neq h_q(X' \cap \mathcal{F}^q).$$

Note that such a flag always exists. Indeed, we first choose a generic hyperplane $\mathcal{F}^{\ell-1}$ in such a way that $\mathcal{F}^{\ell-1}$ does not separate 0-dimensional intersections $L^0(\mathcal{A})$. In a similar fashion, we choose $\mathcal{F}^{\ell-2} \subset \mathcal{F}^{\ell-1}$ inductively.

Let us denote $\mathsf{M}^q := (\mathcal{F}^q_\mathbb{C}) \cap \mathsf{M}(\mathcal{A}) = \mathcal{F}^q_\mathbb{C} \setminus \bigcup_{H\in\mathcal{A}} H_\mathbb{C}$. We have the following propositions (see [5, 11]).

Proposition 2.1. *Let $\mathcal{A}$ be an arrangement and W^q a q-dimensional generic subspace. Let $\mathcal{A} \cap W^q$ be the arrangement on W^q induced by $\mathcal{A}$.*

(1) *Then $L(\mathcal{A} \cap W^q)$ is isomorphic to $L^{\geq \ell - q}(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \dim X \geq \ell - q\}$ as posets.*
(2) *Then the natural inclusion $i : \mathsf{M}(\mathcal{A}) \cap W^q \hookrightarrow \mathsf{M}(\mathcal{A})$ induces isomorphisms*

$$i_k : H_k(\mathsf{M}(\mathcal{A}) \cap W^q, \mathbb{Z}) \xrightarrow{\cong} H_k(\mathsf{M}(\mathcal{A}), \mathbb{Z}),$$

for $k = 0, 1, \dots, q$.

Proposition 2.2. *Let $\mathcal{A}$ be a real arrangement and $\mathcal{F}$ a generic flag. Then $|\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})| = b_q(\mathsf{M}(\mathcal{A}))$.*

Let $X, Y \subset \mathbb{C}^\ell$ be subsets. We denote by $\overline{Y}$ the closure of Y in $\mathbb{C}^\ell$ (with respect to the classical topology) and $\mathrm{cl}_X(Y) = X \cap \overline{Y}$.

For given $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, $\mathrm{cl}_{\mathcal{F}^q}(C) = \overline{C} \cap \mathcal{F}^q \subset \mathbb{R}^\ell$ is a q-dimensional polytope which does not intersect $\mathcal{F}^{q-1}$. By Assumption 2.1, the vertices of $\mathrm{cl}_{\mathcal{F}^q}(C)$ have mutually different and positive heights with respect to h_q (noting that $\mathcal{F}^{q-1} = \mathcal{F}^q \cap \{h_q = 0\}$). There is a unique vertex $p \in \mathrm{cl}_{\mathcal{F}^q}(C)$ at which $h_q|_{\mathrm{cl}_{\mathcal{F}^q}(C)}$ attains the minimum. Then by Proposition 2.1 (1), there exists a unique intersection $X_C \in L(\mathcal{A})$ satisfying $\{p\} = X_C \cap \mathcal{F}^q$. (Note that in case $C \in \mathsf{ch}^0_{\mathcal{F}}(\mathcal{A})$, we consider $X_C = \mathbb{R}^\ell$.)

Definition 2.2. Let $X \subset \mathbb{R}^\ell$ be an affine subspace. Denote by $\tau(X)$ the linear subspace through the origin which is parallel to X and $\dim \tau(X) = \dim X$, and define

$$\mathcal{A}_{[X]} := \{H \in \mathcal{A} \mid \tau(H) \supset \tau(X)\}$$

to be the set of hyperplanes parallel to X. Note that $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\} \subseteq \mathcal{A}_{[X]}$.

Definition 2.3. Let $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$. We denote by $\widetilde{C} \in \mathsf{ch}(\mathcal{A}_{[X_C]})$ the unique chamber with $C \subset \widetilde{C}$.

Using this notation, we shall define a partial ordering $\preceq$ in $\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$.

Definition 2.4. For $C, C' \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, we denote $C \preceq C'$ if and only if $C' \subset \widetilde{C}$.

The following is easy.

Lemma 2.5. *If $C \preceq C'$, then $h_q(X_C \cap \mathcal{F}^q) \leq h_q(X_{C'} \cap \mathcal{F}^q)$.*

3. Minimal partition

In this section, we shall introduce the semialgebraic partition.

Definition 3.1. Let $p_1, p_2 \in \mathbb{R}^\ell$. The set $\mathrm{Sep}(p_1, p_2)$ of separating hyperplanes is defined by

$$\mathrm{Sep}(p_1, p_2) := \{H \in \mathcal{A} \mid [p_1, p_2] \cap H \neq \emptyset\},$$

where $[p_1, p_2]$ is the closed line segment connecting two points p_1 and p_2.

Similarly, we also denote by $\mathrm{Sep}(C_1, C_2)$ the set of separating hyperplanes of two chambers C_1, C_2.

Lemma 3.2. *Let $C, C' \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$. If $X_C = X_{C'}$, then $\mathrm{Sep}(C, C') \subset \mathcal{A}_{X_C}$.*

Proof. Let $H \in \mathrm{Sep}(C, C')$ and choose a defining equation f, i.e., $H = \{f = 0\}$. Since H separates C and C', we may assume $C \subset \{f \geq 0\}$ and $C' \subset \{f \leq 0\}$. Hence $X_C \cap \mathcal{F}^q \subset \mathrm{cl}_{\mathcal{F}^q}(C) \subset \{f \geq 0\}$. Similarly, $X_{C'} \cap \mathcal{F}^q \subset \mathrm{cl}_{\mathcal{F}^q}(C') \subset \{f \leq 0\}$. We have $X_C \cap \mathcal{F}^q \subset \{f \geq 0\} \cap \{f \leq 0\} \cap \mathcal{F}^q = H \cap \mathcal{F}^q$. Then the inclusion $X_C \subset H$ follows from Proposition 2.1 (1). This means that $H \in \mathcal{A}_{X_C}$. □

Lemma 3.3. *Let $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ and $C' \in \mathsf{ch}^{q'}_{\mathcal{F}}(\mathcal{A})$.*

(1) *If $C' \subset \widetilde{C}$, then either $C \preceq C'$ (with $q = q'$) or $q < q'$.*
(2) *If $q = q'$ and $h_q(X_C \cap \mathcal{F}^q) < h_q(X_{C'} \cap \mathcal{F}^q)$, then $\mathcal{A}_{X_{C'}} \cap \mathrm{Sep}(C, C') \neq \emptyset$.*
(3) *If $q < q'$, then $\mathcal{A}_{X_{C'}} \cap \mathrm{Sep}(C, C') \neq \emptyset$.*

Proof. (1) First note that $\widetilde{C} \cap \mathcal{F}^{q-1} = (\widetilde{C} \cap \mathcal{F}^q) \cap \mathcal{F}^{q-1}$. Since $h_q|_{\widetilde{C} \cap \mathcal{F}^q}$ attains the minimum at $X_C \cap \mathcal{F}^q$, $h_q|_{\widetilde{C} \cap \mathcal{F}^q} > 0$. Hence $C' \cap \mathcal{F}^{q-1} = \emptyset$ and we have $q' \geq q$. The assertions thus follow from Definition 2.4.

(2) Suppose that $\mathcal{A}_{X_{C'}} \cap \mathrm{Sep}(C, C') = \emptyset$. Then C and C' are contained in the same chamber $D \in \mathsf{ch}(\mathcal{A}_{X_{C'}})$ of $\mathcal{A}_{X_{C'}}$. Since $h_q|_{\mathrm{cl}_{\mathcal{F}^q}(D)}$ attains the minimum at $X_{C'} \cap \mathcal{F}^q$, we have $h_q(X_C \cap \mathcal{F}^q) \geq h_q(X_{C'} \cap \mathcal{F}^q)$. This contradicts the assumption.

(3) Suppose that $\mathcal{A}_{X_{C'}} \cap \mathrm{Sep}(C, C') = \emptyset$. Then C and C' are contained in the same chamber $D \in \mathsf{ch}(\mathcal{A}_{X_{C'}})$ of $\mathcal{A}_{X_{C'}}$. Obviously $D \cap \mathcal{F}^{q'-1} = \emptyset$ and hence $C \cap \mathcal{F}^q = \emptyset$. This contradicts $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$. □

From now on we fix a base point $p_C \in C \cap \mathcal{F}^q$ for each $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$. It is easily seen that the constructions below do not depend on the choice of p_C.

We can identify $\mathbb{C}^\ell$ with the tangent bundle $T\mathbb{R}^\ell \cong \mathbb{R}^\ell \times \mathbb{R}^\ell$ by

$$\begin{array}{ccc} \mathbb{R}^\ell \times \mathbb{R}^\ell & \longrightarrow & \mathbb{C}^\ell \\ (x, v) & \longmapsto & x + \sqrt{-1}v. \end{array}$$

We also denote x by $\mathrm{Re}(x + \sqrt{-1}v)$.

Now we introduce the main object of this paper.

Definition 3.4. For a chamber $C \in \mathsf{ch}(\mathcal{A})$, we define

$$S(C) = \left\{ x + \sqrt{-1}v \in \mathbb{C}^\ell \;\middle|\; \begin{array}{l} v \in \tau(X_C), x \in \mathbb{R}^\ell \text{ and} \\ v \notin \tau(H), \text{ for } H \in \mathrm{Sep}(p_C, x) \end{array} \right\}.$$

If $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, $S(C)$ is an open subset of $\mathbb{R}^\ell \times \tau(X_C)$, hence a real $(2\ell - q)$-dimensional manifold.

Example 3.5. Let $H = \{0\} \subset \mathbb{R}$ and the arrangement $\mathcal{A} = \{H\}$. Fix a generic flag $\mathcal{F}^0 = \{-1\}$. There are two chambers $C_0 = \mathbb{R}_{<0}$ and $C_1 = \mathbb{R}_{>0}$. Then $\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{C_0\}$ and $\mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{C_1\}$. Then $S(C_0) = S_0$ and $S(C_1) = S_1$ as defined in Example 1.2.

Example 3.6. Let $\mathcal{A} = \{H_1, H_2, H_3\}$ be an arrangement of lines on $\mathbb{R}^2$ and fix a generic flag $\mathcal{F}^\bullet$ as in Figure 1. Then $\mathsf{ch}^0_{\mathcal{F}}(\mathcal{A}) = \{C_0\}, \mathsf{ch}^1_{\mathcal{F}}(\mathcal{A}) = \{C_1, C_2, C_3\}$, $\mathsf{ch}^2_{\mathcal{F}}(\mathcal{A}) = \{C_4, C_5\}$. By definition, we also have $X_{C_0} = \mathbb{R}^2$, $X_{C_1} = H_1, X_{C_2} = H_2, X_{C_3} = H_3$, and $X_{C_4} = H_1 \cap H_3$, $X_{C_5} = H_1 \cap H_2$, $\mathcal{A}_{[X_{C_0}]} = \emptyset$, $\mathcal{A}_{[X_{C_1}]} = \{H_1\}$, $\mathcal{A}_{[X_{C_2}]} = \mathcal{A}_{[X_{C_3}]} = \{H_2, H_3\}$, $\mathcal{A}_{[X_{C_4}]} = \mathcal{A}_{[X_{C_5}]} = \mathcal{A}$, $\widetilde{C_0} = \mathbb{R}^2$, $\widetilde{C_1} = C_1 \cup C_2 \cup C_3$, $\widetilde{C_2} = C_2 \cup C_5$, $\widetilde{C_3} = C_3 \cup C_4$, $\widetilde{C_4} = C_4$, $\widetilde{C_5} = C_5$. Using these data, we can describe $S(C)$. For example, $S(C_4) = C_4$, $S(C_5) = C_5$. Other pieces are shown in Figure 1. (In the figure, a dotted line indicates the direction to which v cannot be directed.)

Remark 3.7. The above example shows that our decomposition is not always a Whitney stratification. Indeed, $\dim S(C_1) = 3$ and $\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C_1)) \setminus S(C_1) = C_2 \cup C_3$. However the subset $C_2 \cup C_3$ is not a union of our 2-dimensional components $S(C_4)$ and $S(C_5)$.

Lemma 3.8. *The real part* $\mathrm{Re}\, S(C) = \{\mathrm{Re}(z) \in \mathbb{R}^\ell \mid z \in S(C)\}$ *of* $S(C)$ *coincides with* $\widetilde{C}$.

Proof. Assume $x \in \mathrm{Re}\, S(C)$. Then there exists $v \in \mathbb{R}^\ell$ such that $x + \sqrt{-1}v \in S(C)$. Let $H \in \mathcal{A}_{[X_C]}$. By definition, $v \in \tau(X_C) \subset \tau(H)$, and $H \notin \mathrm{Sep}(p_C, x)$. Hence

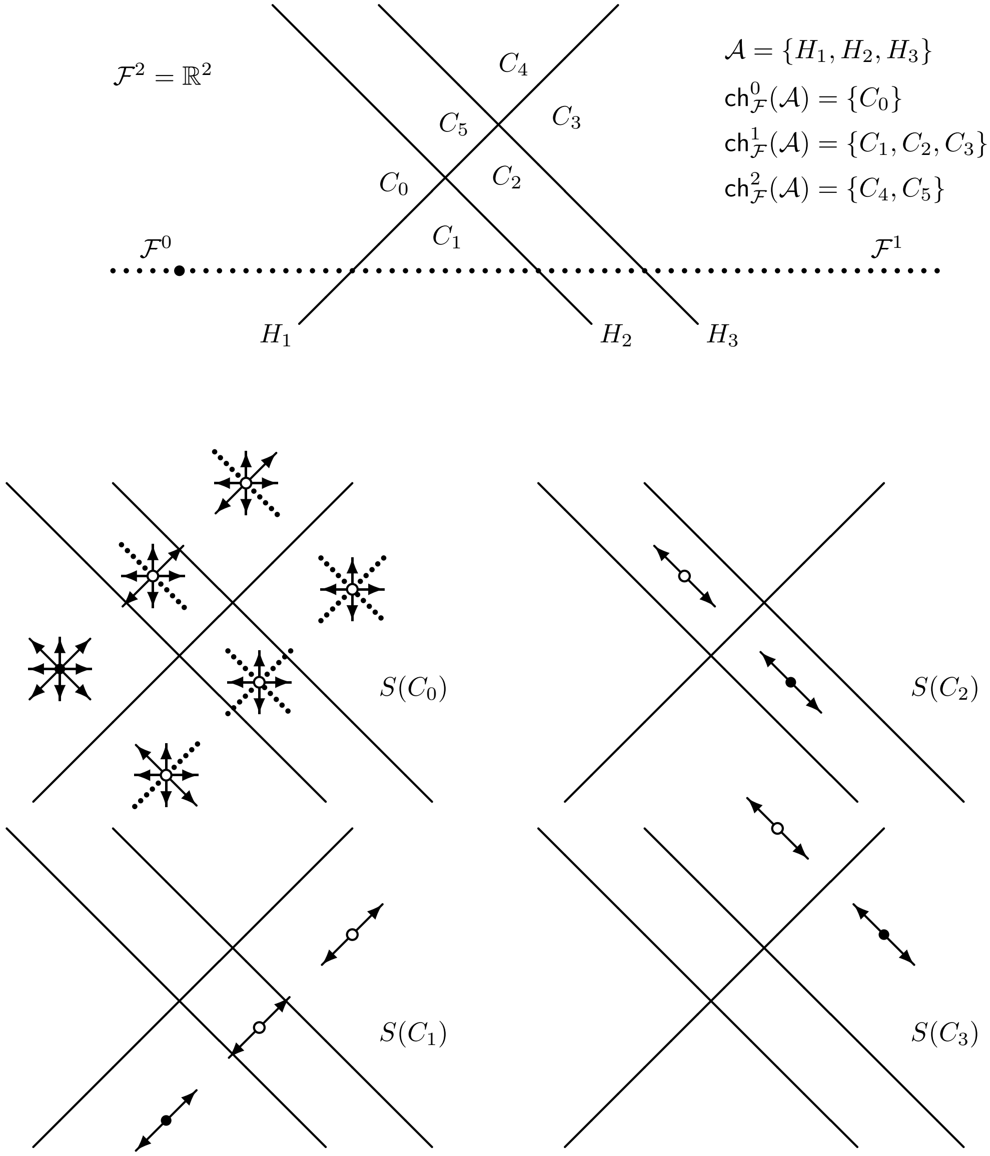

FIGURE 1. Example 3.6

p_C and x are not separated by any hyperplane H belonging to $\mathcal{A}_{[X_C]}$; we have $\operatorname{Re} S(C) \subset \widetilde{C}$.

Conversely, assume $x \in \widetilde{C}$. Since x and p_C are contained in the same chamber of $\mathcal{A}_{[X_C]}$, we have $\operatorname{Sep}(p_C, x) \cap \mathcal{A}_{[X_C]} = \emptyset$. Choose $v \in \tau(X_C) \setminus \bigcup_{H \in \mathcal{A} \setminus \mathcal{A}_{[X_C]}} \tau(H)$. Then $x + \sqrt{-1}v \in S(C)$. □

Lemma 3.9. *If $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, then $S(C)$ is a contractible $(2\ell - q)$-dimensional manifold.*

Proof. Let us prove that $S(C)$ is star-shaped. For a point $x + \sqrt{-1}v \in S(C)$, consider the path $p(t)$ with parameter $0 \le t \le 1$:

$$p(t) = (1-t)p_C + t(x + \sqrt{-1}v) = ((1-t)p_C + tx) + \sqrt{-1}tv.$$

We have $p(1) = x + \sqrt{-1}v$ and $p(0) = p_C$. It suffices to prove that $p(t) \in \mathsf{M}(\mathcal{A})$ for $0 \le t \le 1$. If $\operatorname{Re} p(t) = (1-t)p_C + tx \notin H$, then obviously we have $p(t) \notin H_{\mathbb{C}}$. Suppose $(1-t)p_C + tx \in H$ for some t with $0 < t \le 1$. Then, by assumption, $H \in \operatorname{Sep}(p_C, x)$. By the definition of $S(C)$, v is transverse to H, so is tv, which means $p(t) \in \mathsf{M}(\mathcal{A})$. Hence $S(C)$ is star-shaped. □

Now we have the following:

Theorem 3.10. *The complement* $\mathsf{M}(\mathcal{A})$ *of* $\mathcal{A}$ *is a disjoint union of* $S(C)$*,* $C \in \mathsf{ch}(\mathcal{A})$*, namely,*

$$\mathsf{M}(\mathcal{A}) = \bigsqcup_{C \in \mathsf{ch}(\mathcal{A})} S(C).$$

Proof. First we prove that $S(C) \cap S(C') = \emptyset$ when $C \neq C'$. Suppose this is not the case. Then there exists a point $x + \sqrt{-1}v \in S(C) \cap S(C')$.

(a) If both C, C' are in $\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ and $X_C \neq X_{C'}$, then we may assume $h_q(X_C \cap \mathcal{F}^q) < h_q(X_{C'} \cap \mathcal{F}^q)$. From Lemma 2.5 we have $C' \not\le C$. By Lemma 3.3 (2), there exists $H \in \mathcal{A}_{X_{C'}} \cap \operatorname{Sep}(C, C')$. By definition of $S(C')$, $x + \sqrt{-1}v \in S(C')$ implies that

$$\mathcal{A}_{X_{C'}} \cap \operatorname{Sep}(x, p_{C'}) = \emptyset \tag{3.1}$$

and

$$v \in \tau(X_{C'}). \tag{3.2}$$

It follows from (3.1) that x and p_C are separated by H, and from (3.2) that $v \in \tau(H)$. (Note that $\tau(X_{C'}) \subset \tau(H)$.) Then we have $x + \sqrt{-1}v \notin S(C)$, which contradicts the assumption; this concludes $S(C) \cap S(C') = \emptyset$.

(b) Next we consider the case that C and C' are in $\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ and $X_C = X_{C'}$. By Lemma 3.2, C and C' are separated by a hyperplane $H \in \mathcal{A}_{X_C}$. This implies that H separates $\widetilde{C}$ and $\widetilde{C'}$. By Lemma 3.8, we have $\operatorname{Re} S(C) \cap \operatorname{Re} S(C') = \emptyset$.

(c) Finally, we consider the case $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ and $C' \in \mathsf{ch}^{q'}_{\mathcal{F}}(\mathcal{A})$, with $q < q'$. Then again by Lemma 3.3(3), there exists a hyperplane $H \in \mathcal{A}_{X_{C'}}$ separating C and C'. As in the case (a), we obtain $x + \sqrt{-1}v \notin S(C)$. Therefore $S(C) \cap S(C') = \emptyset$.

Next we prove that

$$\mathsf{M}(\mathcal{A}) = \bigcup_{C \in \mathsf{ch}(\mathcal{A})} S(C).$$

Let $x + \sqrt{-1}v \in \mathsf{M}(\mathcal{A})$. Recall that $\mathcal{A}_{[v]}$ is the set of all hyperplanes parallel to v, namely, $\mathcal{A}_{[v]} = \{H \in \mathcal{A} \mid \tau(H) \ni v\}$. Since v is parallel to hyperplanes in $\mathcal{A}_{[v]}$, x is not contained in $H \in \mathcal{A}_{[v]}$. We can choose a chamber $D \in \mathsf{ch}(\mathcal{A}_{[v]})$ such that $x \in D$. Let $q = \min\{i \mid D \cap \mathcal{F}^i \neq \emptyset\}$. Since the closure $\operatorname{cl}_{\mathcal{F}^q}(D)$ is a convex polytope in $\mathcal{F}^q$ which does not intersect with $\mathcal{F}^{q-1}$, there exists a unique point $p \in \operatorname{cl}_{\mathcal{F}^q}(D)$ of the minimum with respect to h_q. We can choose $X \in L(\mathcal{A})$ such that $p = X \cap \mathcal{F}^q$.

Note that $X = \bigcap_{H \in \mathcal{A}_p} H$ and then $v \in \tau(X)$. There exists $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ satisfying $X_C = X$ and $C \subset D$. We prove that

$$x + \sqrt{-1}v \in S(C).$$

It is enough to prove that $v \notin \tau(H)$ for $H \in \mathrm{Sep}(x, p_C)$. Note that x and p_C are contained in the same chamber $D \in \mathsf{ch}(\mathcal{A}_{[v]})$. Hence if $H \in \mathrm{Sep}(x, p_C)$, then $H \notin \mathcal{A}_{[v]}$. By definition of $\mathcal{A}_{[v]}$, $v \notin \tau(H)$. Therefore v is transverse to H, which means that $x + \sqrt{-1}v \in S(C)$. □

4. Basis of BM-homology

In this section, we shall prove that the closures $\{\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))\}_{C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})}$ form a basis of $H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}), \mathbb{Z})$. In §4.1, we determine orientations on our spaces. In §4.2, we recall the constructions of a basis $\{[\sigma_C] \mid C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})\}$ of $H_q(\mathsf{M}(\mathcal{A}), \mathbb{Z})$ from [10]. By computing intersection numbers of $\mathrm{cl}_{\mathsf{M}}(S(C))$ and $[\sigma_{C'}]$, in §4.2, we prove the main result.

4.1. **Orientations.** In this section, we shall define orientations for $\mathcal{F}^q$, X_C and $S(C)$ by choosing an ordered basis of the tangent spaces. (See chapter 3 of [3] for generalities of orientations and intersections of manifolds.)

Recall that the subspace $\mathcal{F}^q$ is defined by $\{x \in \mathbb{R}^\ell \mid h_{q+1}(x) = \cdots = h_\ell(x) = 0\}$, where h_i $(i = 1, \ldots, \ell)$ are linear forms. Hence $(h_1, \ldots, h_q)$ forms a coordinate of the space $\mathcal{F}^q$. We consider the orientation defined by the ordered basis $(\partial_{h_1}, \ldots, \partial_{h_q})$ of $T_x\mathcal{F}^q = \tau(\mathcal{F}^q)$. In particular, the orientation of $\mathbb{R}^\ell$ is determined by the ordered basis $(\partial_{h_1}, \ldots, \partial_{h_\ell})$. If C belongs to $\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, then X_C is an affine subspace complemental to $\mathcal{F}^q$. So $(h_{q+1}, \ldots, h_\ell)$ forms a coordinate of X_C, and we consider the orientation determined by the dual basis $(\partial_{h_{q+1}}, \ldots, \partial_{h_\ell})$ with an order. Note that the intersection number $\mathcal{F}^q \cdot X_C = (-1)^{q(\ell-q)} \cdot X_C \cdot \mathcal{F}^q$ equals $+1$.

Next we consider the orientation of $S(C)$. By definition, the tangent space of $S(C)$ at p_C is expressed as

$$T_{p_C}S(C) \simeq T_{p_C}C \oplus \sqrt{-1} \cdot \tau(X_C).$$

Thus we define the orientation by $(\partial_{h_1}, \partial_{h_2}, \ldots, \partial_{h_\ell}, \sqrt{-1}\partial_{h_{q+1}}, \ldots, \sqrt{-1}\partial_{h_\ell})$. The case $q = 0$ defines an orientation on $\mathbb{C}^\ell$ by $(\partial_{h_1}, \ldots, \partial_{h_\ell}, \sqrt{-1}\partial_{h_1}, \ldots, \sqrt{-1}\partial_{h_\ell})$. We should note that this orientation is different from the usual one defined by $(\partial_{h_1}, \sqrt{-1}\partial_{h_1}, \partial_{h_2}, \sqrt{-1}\partial_{h_2}, \ldots, \partial_{h_\ell}, \sqrt{-1}\partial_{h_\ell})$.

The rest will be used in §5. Let $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$ be an ordered subset of q indices, $\mathcal{A}(I) := \{H_{i_1}, \ldots, H_{i_q}\}$ be a subarrangement consisting of q hyperplanes. Assume $H_{i_1}, \ldots, H_{i_q}$ are independent, that is, $d\alpha_{i_1} \wedge \cdots \wedge d\alpha_{i_\ell} \neq 0$ or equivalently the intersection $X(I) := H_{i_1} \cap \cdots \cap H_{i_q}$ is a nonempty subspace of codimension q.

Definition-Lemma 4.1. The set of chambers $\mathsf{ch}(\mathcal{A}(I))$ consists of 2^q chambers. There is a unique chamber, denoted by $C_0(I) \in \mathsf{ch}(\mathcal{A}(I))$, which satisfies $C_0(I) \cap \mathcal{F}^{q-1} = \emptyset$.

Proof. The Poincaré polynomial of $\mathbb{C}^\ell \setminus \bigcup_{i \in I} H_{i,\mathbb{C}}$ is $(1+t)^q$. In particular, $b_q = 1$. Hence by Proposition 2.2, $|\mathsf{ch}^q_{\mathcal{F}}(\mathcal{A}(I))| = 1$. □

Choose a normal vector $w_{i_k} \perp H_{i_k}$ for each H_{i_k} such that $C_0(I)$ is contained in the half-space $H_{i_k} + \mathbb{R}_{>0} \cdot w_{i_k}$. Suppose $H_{i_1}, \ldots, H_{i_q}$ are independent (i.e., the intersection $X(I) = H_{i_1} \cap \cdots \cap H_{i_q}$ has codimension q with $q \le \ell$). Since $\mathcal{F}^q$ is generic, $\mathcal{F}^q \cap X(I)$ is 0-dimensional. Thus by the identification $\mathbb{R}^\ell / X(I) \simeq \mathcal{F}^q$, the normal vectors $w_{i_1}, \dots, w_{i_q}$ induce a basis of $\mathcal{F}^q$.

Definition 4.2. For an ordered q-tuple $I = \{i_1, \dots, i_q\} \subset \{1, \dots, n\}$, define $\varepsilon(I)$ by

$$\varepsilon(I) = \begin{cases} 0 & \text{if } H_{i_1}, \dots, H_{i_q} \text{ are dependent,} \\ 1 & \text{if } (w_{i_1}, \dots, w_{i_q}) \text{ induces a positive basis of } \mathcal{F}^q, \\ -1 & \text{if } (w_{i_1}, \dots, w_{i_q}) \text{ induces a negative basis of } \mathcal{F}^q. \end{cases}$$

4.2. **Minimal CW-decomposition.** Here we recall results from [10, §5.2]. For each $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$, there exists a continuous map, unique up to homotopy,

$$\sigma_C : (D^q, \partial D^q) \longrightarrow (\mathsf{M}^q, \mathsf{M}^{q-1}),$$

from the q-dimensional disk to the complement $\mathsf{M}^q = \mathsf{M}(\mathcal{A}) \cap \mathcal{F}^q_{\mathbb{C}}$ such that

(Transversality) $\sigma_C(0) = p_C \in C \cap \mathcal{F}^q$, and $\sigma_C(D^q)$ intersects $C \cap \mathcal{F}^q$ transversally in $\mathcal{F}^q_{\mathbb{C}}$ at the point; $\sigma_C(D^q) \pitchfork C = \{p_C\}$, and (Non-intersecting) $\sigma_C(D^q) \cap C' = \emptyset$ for $C' \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A}) \setminus \{C\}$.

These properties guarantee the following homotopy equivalence ([10, 4.3.1]):

$$\mathsf{M}^q \simeq \mathsf{M}^{q-1} \cup_{(\partial \sigma_C)} \left(\bigsqcup_{C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})} D^q \right), \tag{4.1}$$

where the right-hand side is obtained by attaching q-dimensional disks to M^{q-1} along $\partial \sigma_C : \partial D^q \to \mathsf{M}^{q-1}$ for $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$.

Recall that $T_{p_C} \mathsf{M}^q \simeq \tau(\mathcal{F}^q) \oplus \sqrt{-1} \cdot \tau(\mathcal{F}^q)$. We introduce an orientation on σ_C by identifying $T_{p_C} \sigma_C(D^q)$ with $\sqrt{-1} \cdot \tau(\mathcal{F}^q)$, equivalently, by an ordered basis $(\sqrt{-1}\partial_{h_1}, \dots, \sqrt{-1}\partial_{h_q})$.

Proposition 4.1 ([10])**.** (1) $[\sigma_C] \in H_q(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathbb{Z}), (C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A}))$ *forms a basis.*
(2) $H_q(\mathsf{M}^q, \mathbb{Z}) \simeq H_q(\mathsf{M}^q, \mathsf{M}^{q-1}; \mathbb{Z}) \simeq H_q(\mathsf{M}(\mathcal{A}), \mathbb{Z})$.

We construct the basis of $H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}), \mathbb{Z})$. Let $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$. Lemma 3.8 indicates that

$$\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C)) = (\widetilde{C} \times \sqrt{-1} \cdot \tau(X_C)) \cap \mathsf{M}(\mathcal{A}), \tag{4.2}$$

which is a closed oriented $(2\ell - q)$-dimensional submanifold of $\mathsf{M}(\mathcal{A})$ because $\dim X_C = \ell - q$. The closed submanifold $\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))$ determines a cycle $[\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))] \in H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}), \mathbb{Z})$.

Theorem 4.3. *The classes* $\{[\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))]\}_{C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})}$ *form a basis of the* $(2\ell - q)$*-th Borel-Moore homology group* $H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}), \mathbb{Z})$.

Proof. We compute the intersection number of $[\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))] \in H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}))$ and $[\sigma(C')] \in H_q(\mathsf{M}(\mathcal{A}))$, and show that the intersection matrix

$$I([\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))], [\sigma(C')])_{C, C' \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})}$$

is a triangular matrix with each diagonal entry $(-1)^{q(\ell-q)}$.

We fix an ordering on $\{C_1, \dots, C_b\} = \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ in such a way that $C_i \preceq C_j \Longrightarrow i < j$ (e.g. choose an ordering with $h_q(X_{C_1} \cap \mathcal{F}^q) \le h_q(X_{C_2} \cap \mathcal{F}^q) \le \cdots \le h_q(X_{C_b} \cap \mathcal{F}^q)$).

Since $\mathcal{F}^q$ and X_C are mutually complementary in $\mathbb{R}^\ell$, the tangent space $T_{p_C}\mathbb{C}^\ell$ can be expressed as

$$T_{p_C}\mathbb{C}^\ell = T_{p_C}\mathbb{R}^\ell \oplus \sqrt{-1}\cdot T_{p_C}\mathcal{F}^q \oplus \sqrt{-1}\cdot \tau(X_C).$$

The above-mentioned properties and (4.2) imply that $\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C))$ intersects transversally to $\sigma_{C'}$ if and only if $p_{C'} \in \widetilde{C}$. In fact, we have $T_{p_C}\,\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(C)) = \mathbb{R}^\ell \oplus \sqrt{-1}\cdot\tau(X_C)$ and $T_{p_{C'}}\sigma_{C'}(D^q) = \sqrt{-1}\cdot T_{p_{C'}}\mathcal{F}^q$, which implies the transversality and its intersection number is $(-1)^{q(\ell-q)}$. □

5. Relations with OS-type generators

As is mentioned in §1, there is a canonical isomorphism $\varphi : H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \xrightarrow{\cong} H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ between cohomology and Borel-Moore homology of $\mathsf{M}(\mathcal{A})$. In this section, we describe φ explicitly by using the basis introduced in the previous sections.

First note that both $H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ and $H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ are dual to the homology group $H_q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$. The pairing $H^{BM}_{2\ell-q}(\mathsf{M}(\mathcal{A}),\mathbb{Z})\times H_q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \to \mathbb{Z}$ is defined by the intersection $I(\cdot,\cdot)$, and $H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \times H_q(\mathsf{M}(\mathcal{A}),\mathbb{Z}) \to \mathbb{Z}$ is defined by the cap product $\cap$ (or the integration if we consider de Rham cohomology).

The structure of the cohomology ring $H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ is well studied (see e.g. [4]), and especially, by Arnold-Brieskorn's result, it is generated by logarithmic forms

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\frac{d\alpha_i}{\alpha_i},$$

for $i = 1,\dots,n$. The q-th cohomology group $H^q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$ is spanned by $\omega_{i_1,\dots,i_q} = \omega_{i_1}\wedge\omega_{i_2}\wedge\cdots\wedge\omega_{i_q}$ with $H_{i_1},\dots,H_{i_q}$ linearly independent.

Theorem 5.1. *Let $I = \{i_1,\dots,i_q\} \subseteq \{1,\dots,n\}$ be an ordered index (see* §4.1 *for notation). Then*

$$\varphi(\omega_I) = (-1)^{q(\ell-q)}\varepsilon(I)\cdot\sum_C [\mathrm{cl}_{\mathsf{M}}(S(C))],$$

where C runs over all chambers $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ satisfying $C \subset C_0(I)$ and $\tau(X_C) = \tau(X(I))$.

Proof. Let us define $S(I) \subset \mathbb{C}^\ell$ to be

$$S(I) = C_0(I) \oplus \sqrt{-1}\cdot\tau(X(I)).$$

Then $\mathrm{cl}_{\mathsf{M}(\mathcal{A})}(S(I))$ is a disjoint union of $\mathrm{cl}_{\mathsf{M}}(S(C))$'s with C running over all chambers $C \in \mathsf{ch}^q_{\mathcal{F}}(\mathcal{A})$ satisfying $C \subset C_0(I)$ and $\tau(X_C) = \tau(X(I))$. It is enough to show that $\varphi(\omega_I) = (-1)^{q(\ell-q)}\varepsilon(I)\cdot\mathrm{cl}_{\mathsf{M}}(S(I))$. To do this, we shall consider the pairing with the homology class $[\sigma_{C'}] \in H_q(\mathsf{M}^q,\mathsf{M}^{q-1},\mathbb{Z}) \cong H_q(\mathsf{M}(\mathcal{A}),\mathbb{Z})$.

First we compute $\int_{[\sigma_{C'}]}\omega_I$. The complement $\mathbb{C}^\ell\setminus\bigcup_{i\in I}H_{i,\mathbb{C}}$ is homotopy equivalent to $(\mathbb{C}^*)^q \simeq (S^1)^q$. The top homology $H_q(\mathbb{C}^\ell\setminus\bigcup_{i\in I}H_{i,\mathbb{C}},\mathbb{Z}) \cong \mathbb{Z}$ is rank one. If $C' \subset C_0(I)$, then $[\sigma_{C'}]$ is transverse to $C_0(I)$. By applying Proposition 4.1 to the arrangement $\mathcal{A}(I) = \{H_{i_1},\dots,H_{i_q}\}$, we obtain the fact that $[\sigma_{C'}]$ is a generator of $H_q(\mathbb{C}^\ell\setminus\bigcup_{i\in I}H_{i,\mathbb{C}},\mathbb{Z})$. Similarly, if $C' \not\subset C_0(I)$, then $[\sigma_{C'}] = 0$. We have

$$\int_{[\sigma_{C'}]}\omega_I = \begin{cases} \varepsilon(I) & \text{if } C' \subset C_0(I),\\ 0 & \text{else.}\end{cases}$$

By a computation similar to the proof of Theorem 4.3, we have

$$I([S(I)],[\sigma_{C'}]) = \begin{cases} (-1)^{q(\ell-q)} & \text{if } C' \subset C_0(I), \\ 0 & \text{else.} \end{cases}$$

This completes the proof. □

Remark 5.2. The correspondences between chambers and de Rham cohomology groups were investigated by Varchenko and Gel′fand in [9]. Indeed, the cycle $S(I)$ appeared in their paper.

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