$K3$ SURFACES WITH INVOLUTION, EQUIVARIANT ANALYTIC TORSION, AND AUTOMORPHIC FORMS ON THE MODULI SPACE III: THE CASE $r(M) \geq 18$

KEN-ICHI YOSHIKAWA

Abstract. We prove the automorphic property of the invariant of $K3$ surfaces with involution, which we obtained using equivariant analytic torsion, in the case where the dimension of the moduli space is less than or equal to 2.

1. Introduction

Let (X, ι) be a $K3$ surface with anti-symplectic holomorphic involution and let $H^2_+(X, \mathbf{Z})$ be the invariant sublattice of $H^2(X, \mathbf{Z})$ with respect to the ι-action. By Nikulin [12], the topological type of ι is determined by the isometry class of $H^2_+(X, \mathbf{Z})$. Let M be a sublattice of the $K3$-lattice and let $M^\perp$ be the orthogonal complement of M in the $K3$-lattice. The pair (X, ι) is called a 2-elementary $K3$ surface of type M if $H^2_+(X, \mathbf{Z})$ is isometric to M. In this case, M is a primitive, 2-elementary, Lorentzian sublattice of the $K3$-lattice by [11]. Let $\mathcal{M}^o_{M^\perp}$ be the coarse moduli space of 2-elementary $K3$ surfaces of type M. By the global Torelli theorem for $K3$ surfaces, the period map gives an identification between $\mathcal{M}^o_{M^\perp}$ and a Zariski open subset of the modular variety $\Omega^+_{M^\perp}/O^+(M^\perp)$. Here $\Omega^+_{M^\perp}$ is the period domain for 2-elementary $K3$ surfaces of type M, which is isomorphic to a symmetric bounded domain of type IV of dimension $20 - r(M)$, and $O^+(M^\perp) \subset O(M^\perp \otimes \mathbf{R})$ is a certain arithmetic subgroup.

In [17], we introduced a real-valued invariant $\tau_M(X, \iota)$ of (X, ι), which we obtained using equivariant analytic torsion [2] and a Bott–Chern secondary class [3]. (See Sect.2.) Then τ_M gives rise to a function on the coarse moduli space $\mathcal{M}^o_{M^\perp}$.

Let $r(M)$ be the rank of M. When $r(M) \leq 17$, the function τ_M on $\mathcal{M}^o_{M^\perp}$ is expressed as the Petersson norm of an automorphic form on $\Omega^+_{M^\perp}$ characterizing the discriminant locus [17], where the automorphic form takes its values in a certain $O^+(M^\perp)$-equivariant line bundle on $\Omega^+_{M^\perp}$. The purpose of this note is to extend the automorphic property of τ_M to the case $r(M) \geq 18$.

Let X^ι be the set of fixed points of $\iota\colon X \to X$. If $r(M) \geq 18$, X^ι is the disjoint union of finitely many compact Riemann surfaces, whose total genus is determined by M (cf. [12]). Let $g(M)$ be the total genus of X^ι. Then our main result is stated as follows.

Theorem 1.1 (Theorem 5.3). *There exist an integer $\nu \in \mathbf{Z}_{>0}$, an (possibly meromorphic) automorphic form Ψ_M on $\Omega^+_{M^\perp}$ of weight $\nu(r(M)-6)$ and a Siegel modular form S_M on the Siegel upper half space $\mathfrak{S}_{g(M)}$ of weight 4ν such that, for every*

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2-elementary $K3$ surface (X,ι) of type M,

$$\tau_M(X,\iota)=\|\Psi_M(\overline{\varpi}_M(X,\iota))\|^{-1/2\nu}\|S_M(\Omega(X^\iota)\|^{-1/2\nu}.$$

Here $\overline{\varpi}_M(X,\iota)\in\mathcal{M}^o_{M^\perp}$ denotes the period of (X,ι), $\Omega(X^\iota)\in\mathfrak{S}_{g(M)}/Sp_{2g(M)}(\mathbf{Z})$ denotes the period of X^ι, and $\|\cdot\|$ denotes the Petersson norm.

In [19], we shall use Theorem 1.1 to give explicit formulae for Ψ_M and S_M. In fact, Ψ_M is expressed as an explicit Borcherds lift of a certain elliptic modular form and S_M is expressed as the product of all even theta constants.

This note is organized as follows. In Sect.2, we recall the invariant τ_M. In Sect.3, we recall the moduli space of 2-elementary $K3$ surfaces of type M and prove a technical result. In Sect.4, we study the singularity of τ_M. In Sect.5, we prove Theorem 1.1. In Sect.6, we prove a technical result used in the proof of the main theorem for a certain M.

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2. $K3$ surfaces with involution and the invariant τ_M

Let X be a $K3$ surface and let $\iota\colon X\to X$ be a holomorphic involution acting non-trivially on holomorphic 2-forms on X. The pair (X,ι) is called a 2-elementary $K3$ surface. Let $\mathbb{L}_{K3}$ be a fixed even unimodular lattice of signature $(3,19)$, which is called a $K3$-lattice. Then $H^2(X,\mathbf{Z})$ equipped with the cup-product pairing is isometric to $\mathbb{L}_{K3}$. Let $M\subset\mathbb{L}_{K3}$ be a sublattice. The pair (X,ι) is of type M if the invariant part of $H^2(X,\mathbf{Z})$ with respect to the ι-action is isometric to M. By [11], there exists a 2-elementary $K3$ surface of type M if and only if M is a primitive, 2-elementary, Lorentzian sublattice of $\mathbb{L}_{K3}$.

Let (X,ι) be a 2-elementary $K3$ surface of type M. Identify $\mathbf{Z}_2$ with the subgroup of $\mathrm{Aut}(X)$ generated by ι. Let κ be a $\mathbf{Z}_2$-invariant Kähler form on X. Let $\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)$ be the equivariant analytic torsion of the trivial Hermitian line bundle on (X,κ). For the definition and the basic properties of (equivariant) analytic torsion, we refer the reader to [13], [3], [2], [8], [9]. Set $\mathrm{vol}(X,\kappa):=(2\pi)^{-2}\int_X\kappa^2/2!$. Let η be a nowhere vanishing holomorphic 2-form on X. The L^2-norm of η is defined as $\|\eta\|^2_{L^2}:=(2\pi)^{-2}\int_X\eta\wedge\bar{\eta}$.

Let $X^\iota:=\{x\in X;\,\iota(x)=x\}$ be the set of fixed points of ι and let $X^\iota=\sum_i C_i$ be the decomposition into the connected components. By [12], the total genus $g(X^\iota)$ of X^ι depends only on M and hence is denoted by $g(M)$. Set $\mathrm{vol}(C_i,\kappa|_{C_i}):=(2\pi)^{-1}\int_{C_i}\kappa|_{C_i}$. Let $c_1(C_i,\kappa|_{C_i})$ be the Chern form of $(TC_i,\kappa|_{C_i})$ and let $\tau(C_i,\kappa|_{C_i})$ be the analytic torsion of the trivial Hermitian line bundle on $(C_i,\kappa|_{C_i})$.

By [17, Th. 5.7], the real number

$$\begin{aligned}\tau_M(X,\iota):=\mathrm{vol}(X,\kappa)^{\frac{14-r(M)}{4}}\tau_{\mathbf{Z}_2}(X,\kappa)(\iota)\prod_i\mathrm{Vol}(C_i,\kappa|_{C_i})\tau(C_i,\kappa|_{C_i})\\ \times\exp\left[\frac{1}{8}\int_{X^\iota}\log\left(\frac{\eta\wedge\bar{\eta}}{\kappa^2/2!}\cdot\frac{\mathrm{Vol}(X,\kappa)}{\|\eta\|^2_{L^2}}\right)\Bigg|_{X^\iota}c_1(X^\iota,\kappa|_{X^\iota})\right],\end{aligned}$$

is independent of the choice of κ. Hence $\tau_M(X,\iota)$ is a real-valued invariant of (X,ι). We regard τ_M as a function on the moduli space of 2-elementary $K3$ surfaces of type M.

3. The moduli space of 2-elementary $K3$ surfaces

3.1. **The moduli space of 2-elementary $K3$ surfaces.** For a complex vector space V, let $\mathbf{P}(V)$ denote its projectivization. By the global Torelli theorem for $K3$ surfaces, the period domain for 2-elementary $K3$ surfaces of type M is given by the set

$$\Omega_{M^\perp} := \{[\eta] \in \mathbf{P}(M^\perp \otimes \mathbf{C});\ \langle \eta, \eta \rangle = 0, \quad \langle \eta, \overline{\eta} \rangle > 0\},$$

which consists of two connected components $\Omega^+_{M^\perp}$ and $\overline{\Omega^+_{M^\perp}}$. Since $\mathrm{sign}(M^\perp) = (2, 20 - r(M))$, $\Omega^+_{M^\perp}$ is isomorphic to a symmetric bounded domain of type IV of dimension $20 - r(M)$. Let $O(M^\perp)$ be the group of isometries of $M^\perp$, which acts projectively on $\Omega_{M^\perp}$. Let $O^+(M^\perp)$ be the subgroup of $O(M^\perp)$ of index 2, which preserves the connected components of $\Omega_{M^\perp}$. We define

$$\mathcal{M}_{M^\perp} := \Omega^+_{M^\perp}/O^+(M^\perp).$$

The Baily–Borel–Satake compactification of $\mathcal{M}_{M^\perp}$ is denoted by $\mathcal{M}^*_{M^\perp}$, which is a normal projective variety of dimension $20 - r(M)$ with regular part $(\mathcal{M}^*_{M^\perp})_{\mathrm{reg}}$.

Recall that the discriminant locus of $\Omega^+_{M^\perp}$ is the divisor defined as

$$\mathcal{D}_{M^\perp} := \bigcup_{d \in \Delta_{M^\perp}/\pm 1} H_d, \qquad H_d := \{[\eta] \in \Omega^+_{M^\perp};\ \langle d, \eta \rangle = 0\},$$

where $\Delta_{M^\perp} := \{d \in M^\perp;\ \langle d, d \rangle = -2\}$ is the set of roots of $M^\perp$. Let $\overline{\mathcal{D}}_{M^\perp}$ be the divisor of $\mathcal{M}^*_{M^\perp}$ defined as the closure of the image of $\mathcal{D}_{M^\perp}$ by the projection $\Pi_M \colon \Omega^+_{M^\perp} \to \mathcal{M}_{M^\perp}$. By [17, Th. 1.8], the period map induces an isomorphism between the coarse moduli space of 2-elementary $K3$ surfaces of type M and the quasi-projective variety of dimension $20 - r(M)$

$$\mathcal{M}^o_{M^\perp} := (\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp})/O^+(M^\perp) = \mathcal{M}_{M^\perp} \setminus \overline{\mathcal{D}}_{M^\perp}.$$

The boundary locus of $\mathcal{M}^*_{M^\perp}$ is defined as the subvariety:

$$\mathcal{B}_M := \mathcal{M}^*_{M^\perp} \setminus \mathcal{M}_{M^\perp}.$$

Since $\dim \mathcal{B}_M = 1$ if $r(M) \geq 18$ and $\dim \mathcal{B}_M = 0$ if $r(M) = 19$, $\mathcal{B}_M$ is a subvariety of $\mathcal{M}^*_{M^\perp}$ with codimension greater than or equal to 2 when $r(M) \leq 17$ and is a divisor when $r(M) \geq 18$.

3.2. **One parameter families of 2-elementary $K3$ surfaces.** We need a modification of [17, Th. 2.8], which shall be used in Sects.4 and 6.

Theorem 3.1. *Let $C \subset \mathcal{M}^*_{M^\perp}$ be an irreducible projective curve.*

(1) *There exist a smooth projective curve B, a morphism $\varphi \colon B \to \mathcal{M}^*_{M^\perp}$, an irreducible projective threefold X with an involution $\theta \colon X \to X$, and a surjective morphism $f \colon X \to B$ with the following properties:*
   (a) *$\varphi(B) = C$.*
   (b) *The involution $\theta \colon X \to X$ preserves the fibers of $f \colon X \to B$.*
   (c) *There is a non-empty Zariski open subset $B^o \subset B$ such that $(X_b, \theta|_{X_b})$ is a 2-elementary $K3$ surface of type M with period $\varphi(b)$ for $b \in B^o$.*

(2) *Let $p \colon \mathcal{Z} \to \Delta$ be a proper surjective projective morphism from a smooth threefold to the unit disc and let $\iota \colon \mathcal{Z} \to \mathcal{Z}$ be a holomorphic involution preserving the fibers $Z_t = p^{-1}(t)$ of p. Assume that $(Z_t, \iota|_{Z_t})$ is a 2-elementary $K3$ surface for all $t \in \Delta^* := \Delta \setminus \{0\}$ and that the period map for $p \colon (\mathcal{Z}, \iota)|_{\Delta^*} \to \Delta^*$ extends to a non-constant holomorphic map $\gamma \colon \Delta \to C$.*

Let $\nu \in \mathbf{Z}_{\geq 0}$. Then there exist $\varphi\colon B \to C$, $f\colon X \to B$, $\theta\colon X \to X$ as above in (1) and a point $\mathfrak{p} \in \varphi^{-1}(\gamma(0))$ and an isomorphism of germs $\psi\colon (\Delta, 0) \cong (B, \mathfrak{p})$ with the following properties:

(d) $(X_{\mathfrak{p}}, \theta|_{X_{\mathfrak{p}}}) \cong (Z_0, \iota|_{Z_0})$

(e) *The maps of germs $\varphi\colon (B,\mathfrak{p}) \to (C,\gamma(0))$ and $\gamma\colon (\Delta, 0) \to (C, \gamma(0))$ have the same ν-jets: For any $F \in \mathcal{O}_{C,\gamma(0)}$,*

$$F \circ \varphi \circ \psi(t) - F \circ \gamma(t) \in t^{\nu+1}\mathbf{C}\{t\}.$$

(f) *Let $\mathrm{Def}(Z_0)$ be the Kuranishi space of Z_0 and let $\mu_f\colon (B,\mathfrak{p}) \to \mathrm{Def}(Z_0)$ and $\mu_p\colon (\Delta, 0) \to \mathrm{Def}(Z_0)$ be the maps of germs induced by the deformations $f\colon (X, X_{\mathfrak{p}}) \to (B, \mathfrak{p})$ and $p\colon (\mathcal{Z}, Z_0) \to (\Delta, 0)$, respectively. Then μ_f and μ_p have the same ν-jets: For any $F \in \mathcal{O}_{\mathrm{Def}(Z_0)}$,*

$$F \circ \mu_f \circ \psi(t) - F \circ \mu_p(t) \in t^{\nu+1}\mathbf{C}\{t\}.$$

Proof. We follow [17, Th. 2.8]. By the same argument as in [17, Proof of Th. 2.8 (Step 1) and Claim 1], there exist an irreducible projective variety T and a family of projective surfaces with involution $\pi\colon (\mathcal{X}, \mathcal{I}) \to T$ with the following properties:

(i) Let $D \subset T$ be the discriminant locus of $\pi\colon \mathcal{X} \to T$ and define $T^o := T \setminus (\mathrm{Sing}\, T \cup D)$. Then $(\mathcal{X}_t, \mathcal{I}_t)$ is a 2-elementary $K3$ surface of type M for all $t \in T^o$.

(ii) Let $\overline{\varpi}_{T^o}\colon T^o \to \mathcal{M}^o_{M^\perp}$ be the period map for $\pi|_{T^o}\colon (\mathcal{X}|_{T^o}, \mathcal{I}|_{T^o}) \to T^o$. Then $\overline{\varpi}_{T^o}(T^o) \subset C$ and $\overline{\varpi}_{T^o}(T^o)$ contains a non-empty open subset of C.

(iii) The period map $\overline{\varpi}_{T^o}\colon T^o \to C$ extends to a rational map $\overline{\varpi}_T\colon T \dashrightarrow C$.

(iv) In (2), there is a map $c\colon \Delta \to T$ with $c(\Delta^*) \subset T^o$ such that $p\colon (\mathcal{Z}, \iota) \to \Delta$ is induced from $\pi\colon (\mathcal{X}, \mathcal{I}) \to T$ by c.

(1) Let $\Gamma \subset T \times C$ be the closure of the graph of $\overline{\varpi}_{T^o}$. Let B be a smooth projective curve and let $h\colon B \to \Gamma$ be a holomorphic map with $\mathrm{pr}_2(h(B)) = C$. We set $\varphi := \mathrm{pr}_2 \circ h\colon B \to C$. Let $\pi_B\colon (\mathcal{X} \times_T B, \mathcal{I} \times \mathrm{id}_B) \to B$ be the family of algebraic surfaces with involution induced from $\pi\colon (\mathcal{X}, \mathcal{I}) \to T$ by $\mathrm{pr}_1 \circ h\colon B \to T$. Then the period map for $\pi_B\colon (\mathcal{X} \times_T B, \mathcal{I} \times \mathrm{id}_B) \to B$ is given by $\overline{\varpi}_T \circ \mathrm{pr}_1 \circ h$. Since $\Gamma \subset T \times C$ is the closure of the graph of $\overline{\varpi}_{T^o}$, we get $\overline{\varpi}_T \circ \mathrm{pr}_1 \circ h = \mathrm{pr}_2 \circ h = \varphi$. If we set $B^o := \nu^{-1}(B \cap (T^o \times C))$, then $\pi_B\colon (\mathcal{X} \times_T B, \mathcal{I} \times \mathrm{id}_B) \to B$ satisfies (a), (b), (c). This proves (1).

(2) To prove (2), we must choose B more carefully as in [17, Proof of Th. 2.8 Claim 2]. Let $\sigma\colon \Delta \to \Gamma$ be the map defined as $\sigma(t) := (c(t), \gamma(t))$ for $t \in \Delta$. Let $\Sigma\colon \widetilde{\Gamma} \to \Gamma$ be a resolution such that $\widetilde{\Gamma}$ is projective. Since $c(\Delta^*) \subset T \setminus \mathrm{Sing}\, T$ and hence $\sigma(\Delta^*) \subset \Gamma \setminus \mathrm{Sing}\, \Gamma$, σ lifts to a holomorphic map $\widetilde{\sigma}\colon \Delta \to \widetilde{\Gamma}$ such that $\sigma = \Sigma \circ \widetilde{\sigma}$. By [5, Th. 1.1], there exist a pointed smooth projective curve $(B_\nu, \mathfrak{p}_\nu)$, a holomorphic map $\widetilde{h}_\nu\colon B_\nu \to \widetilde{\Gamma}$ and an isomorphism of germs $\psi\colon (\Delta, 0) \cong (B, \mathfrak{p})$ such that for any $G \in \mathcal{O}_{\widetilde{\Gamma}, \widetilde{\sigma}(0)}$,

$$G \circ \widetilde{h}_\nu \circ \psi(t) - G \circ \widetilde{\sigma}(t) \in t^{\nu+1}\mathbf{C}\{t\}. \tag{3.1}$$

Set $B := B_\nu$, $h := \Sigma \circ \widetilde{h}_\nu\colon B \to \Gamma$ and we consider the family of 2-elementary $K3$ surfaces $\pi_B\colon (\mathcal{X} \times_T B, \mathcal{I} \times \mathrm{id}_B) \to B$ of type M. By construction, we get (a), (b), (c), (d). Since

$$\varphi = \mathrm{pr}_2 \circ h = (\mathrm{pr}_2 \circ \Sigma) \circ \widetilde{h}_\nu, \qquad \gamma = \mathrm{pr}_2 \circ \sigma = (\mathrm{pr}_2 \circ \Sigma) \circ \widetilde{\sigma}, \tag{3.2}$$

we get by (3.1), (3.2)

(3.3)
$$F \circ \varphi \circ \psi(t) - F \circ \gamma(t) = (F \circ \mathrm{pr}_2 \circ \Sigma) \circ \widetilde{h}(t) - (F \circ \mathrm{pr}_2 \circ \Sigma) \circ \widetilde{\sigma}(t) \in t^{\nu+1}\mathbf{C}\{t\}.$$

This proves (e). Let $\mu_\pi \colon (T, c(0)) \to \mathrm{Def}(Z_0)$ be the map induced by the deformation $\pi \colon (\mathcal{X}, X_{c(0)}) \to (T, c(0))$. Since

(3.4)
$$\mu_f = \mu_\pi \circ \mathrm{pr}_1 \circ h = (\mu_\pi \circ \mathrm{pr}_1 \circ \Sigma) \circ \widetilde{h}, \qquad \mu_p = \mu_\pi \circ \mathrm{pr}_1 \circ \sigma = (\mu_\pi \circ \mathrm{pr}_1 \circ \Sigma) \circ \widetilde{\sigma},$$

we get by (3.1), (3.4)

(3.5)
$$F \circ \varphi \circ \psi(t) - F \circ \gamma(t) = (F \circ \mathrm{pr}_1 \circ \Sigma) \circ \widetilde{h}(t) - (F \circ \mathrm{pr}_1 \circ \Sigma) \circ \widetilde{\sigma}(t) \in t^{\nu+1}\mathbf{C}\{t\}.$$

This proves (f). This completes the proof of (2). □

4. The singularity of τ_M

We prove the logarithmic divergence of τ_M for any one-parameter degeneration of 2-elementary $K3$ surfaces of type M. For this, we recall the following:

Theorem 4.1. *Let $\pi \colon X \to S$ be a proper surjective holomorphic map from a connected projective algebraic manifold X of dimension $n+1$ to a compact Riemann surface S. Let G be a finite group. Assume that G acts holomorphically on X and trivially on S and that $\pi \colon X \to S$ is G-equivariant. Hence G preserves all the fibers $X_s := \pi^{-1}(s)$, $s \in S$. Let $\Delta := \{s \in S;\, \mathrm{Sing}(X_s) \neq \emptyset\}$ be the discriminant locus.*

Let h_X be a G-invariant Kähler metric on X and set $h_s := h_X|_{X_s}$ for $t \in S \setminus \Delta$. Let $\tau_G(X_s, h_s)(g)$ be the equivariant analytic torsion of the trivial Hermitian line bundle on (X_s, h_s). Let t be a local coordinate of S centered at $0 \in \Delta$. If N is the order of $g \in G$, then there exists $\beta_g(\pi, X_0) \in \sum_{0 \le k < N} \mathbf{Q} \exp(2\pi i k/N)$ such that

$$\log \tau_G(X_t, h_t)(g) = \beta_g(\pi, X_0) \log |t|^2 + O(\log(-\log |t|)) \qquad (t \to 0).$$

Proof. See [20, Th. 1.1 and Cor. 6.10]. We remark that $\Delta \neq S$ by Sard's theorem, since Δ is the analytic subset of S defined as the image of the critical locus of π. □

Theorem 4.2. *Let $(S, 0)$ be a pointed smooth projective curve equipped with a coordinate neighborhood (U, t) centered at 0, let X be a smooth projective threefold equipped with a holomorphic involution $\theta \colon X \to X$, and let $\pi \colon X \to S$ be a surjective holomorphic map. Assume the following:*

1. *the projection $\pi \colon X \to S$ is $\mathbf{Z}_2$-equivariant with respect to the $\mathbf{Z}_2$-action on X induced by θ and with respect to the trivial $\mathbf{Z}_2$-action on S.*
2. *$(X_t, \theta|_{X_t})$ is a 2-elementary $K3$ surfaces of type M for all $t \in U \setminus \{0\}$.*

Then there exists $\alpha \in \mathbf{Q}$ such that

$$\log \tau_M(X_t, \theta|_{X_t}) = \alpha \log |t|^2 + O\left(\log(-\log |t|^2)\right) \qquad (t \to 0).$$

Proof. Set $\theta_t := \theta|_{X_t}$. Let h_X be a $\mathbf{Z}_2$-invariant Kähler metric on X with Kähler form ω_X and set $\omega_t := \omega_X|_{X_t}$. By Theorem 4.1, there exists $\beta \in \mathbf{Q}$ such that

$$\log \tau_{\mathbf{Z}_2}(X_t, \omega_t)(\theta_t) = \beta \log |t|^2 + O\left(\log(-\log |t|^2)\right) \qquad (t \to 0). \tag{4.1}$$

Let X^θ be the set of fixed points of $\theta\colon X \to X$ and let $\Delta \subset S$ be the discriminant locus of $\pi\colon X \to S$. By the $\mathbf{Z}_2$-equivariance of π, we have the decomposition

$$X^\theta = X^\theta_H \amalg X^\theta_V,$$

where $\pi(X^\theta_V) \subset \Delta$ and $\pi|_{X^\theta_H}$ is a surjective map from any component of X^θ_H to S. Set $Y := X^\theta_H$ and $f := \pi|_{X^\theta_H}$. Then Y is a smooth complex surface and $f\colon Y \to S$ is a proper surjective holomorphic map such that $Y_t = X_t^{\theta_t}$ is the disjoint union of compact Riemann surfaces for $t \in U \setminus \{0\}$. It follows from Theorem 4.1 again that there exists $\gamma \in \mathbf{Q}$ with

(4.2)
$$\log\tau(X_t^{\theta_t}, \omega_t|_{X_t^{\theta_t}}) = \log\tau(Y_t, \omega_t|_{Y_t}) = \gamma\,\log|t|^2 + O\left(\log(-\log|t|^2)\right) \qquad (t \to 0).$$

Let $K_{X/S} := \Omega^3_X \otimes (\pi^*\Omega^1_S)^{-1}$ be the relative canonical bundle. Then the direct image sheaf $\pi_*K_{X/S}$ is locally free on S by e.g. [16, Th. 6.10 (iv)]. By assumption (2), $\pi_*K_{X/S}$ has rank one. By shrinking U if necessary, there exists $\Xi \in H^0(\pi^{-1}(U), \Omega^3_X)$ such that $\eta_{X/S} := \Xi \otimes (\pi^*dt)^{-1}$ generates $\pi_*K_{X/S}$ as an $\mathcal{O}_S$-module over U. In particular, we may assume $\eta_{X/S}|_{X_t} \neq 0$ for $t \neq 0$. Since $K_{X/S}|_{X_t}$ is trivial for $t \neq 0$ by (2), this implies that $\eta_{X/S}|_{X_t}$ is nowhere vanishing on X_t, $t \neq 0$. Hence $\mathrm{div}(\Xi) \subset X_0$. We set $\eta_t := \mathrm{Res}_{X_t}[\Xi/(\pi - t)] \in H^0(X_t, \Omega^2_{X_t})$ for $t \in U$. Then $\eta_{X/S}|_{X_t} = \Xi \otimes (\pi^*dt)^{-1}|_{X_t}$ is identified with η_t.

We prove the existence of $\delta \in \mathbf{Q}$ such that as $t \to 0$

(4.3)
$$\int_{X_t^{\theta_t}} \log\left(\frac{\eta_t \wedge \bar{\eta}_t}{\omega_t^2/2!} \cdot \frac{\mathrm{Vol}(X_t, \omega_t)}{\|\eta_t\|^2_{L^2}}\right)\Bigg|_{X_t^{\theta_t}} c_1(X_t^{\theta_t}, \omega_t|_{X_t^{\theta_t}}) = \delta\,\log|t|^2 + O\left(\log(-\log|t|^2)\right).$$

Let $\Sigma_\pi \subset X$ be the critical locus of π and let $TX/S := \ker \pi_*|_{X\setminus\Sigma_\pi}$ be the relative tangent bundle of $\pi\colon X \to S$. Let $h_{X/S} := h_X|_{TX/S}$ be the Hermitian metric on TX/S induced from h_X and let $\omega_{X/S}$ be the $(1,1)$-form on TX/S associated to $h_{X/S}$. We identify $\omega_{X/S}$ with the family of Kähler forms $\{\omega_t\}_{t\in S}$. Let $N^*_{X_t/X}$ be the conormal bundle of X_t in X for $t \in U \setminus \{0\}$. Since $d\pi = \pi^*dt \in H^0(X_t, N^*_{X_t/X})$ generates $N^*_{X_t/X}$ for $t \in U \setminus \{0\}$, $N^*_{X_t/X}$ is trivial in this case. Since the Hermitian metric on $\Omega^1_{X_t}$ is induced from h_X via the C^∞ identification $\Omega^1_{X_t} \cong (N^*_{X_t/X})^\perp$ and since $(\omega^2_{X/S}/2!)|_{X_t}$ is the volume form on X_t, we get on $X \setminus \Sigma_\pi$

$$\frac{\omega_X^3}{3!} = \frac{\omega^2_{X/S}}{2!} \wedge \left(i\,\frac{d\pi}{\|d\pi\|} \wedge \frac{\overline{d\pi}}{\|d\pi\|}\right). \tag{4.4}$$

Since $\Xi|_{X_t} = \eta_t \otimes d\pi$, we get the following equation on $X \setminus \Sigma_\pi$ by (4.4)

(4.5)
$$\frac{\eta_{X/S} \wedge \overline{\eta_{X/S}}}{\omega^2_{X/S}/2!} = \frac{(-1)^3 i^3\,\Xi \wedge \overline{\Xi}}{(\omega^2_{X/S}/2!) \wedge (i\,d\pi \wedge \overline{d\pi})} = \frac{(-1)^3 i^3\,\Xi \wedge \overline{\Xi}}{\omega^3_X/3!} \cdot \frac{1}{\|d\pi\|^2} = \frac{\|\Xi\|^2}{\|d\pi\|^2}.$$

Let $\Sigma_f \subset Y$ be the critical locus of $f\colon Y \to S$ and let $h_{TY/S}$ be the metric on the relative tangent bundle $TY/S := \ker f_*|_{Y\setminus\Sigma_f}$ induced from h_X via the inclusion

$TY/S \subset TY \subset TX|_Y$. Define

(4.6)
$$\mathcal{A}(X/S) := f_*\left[\log\left(\frac{\eta_{X/S}\wedge\overline{\eta_{X/S}}}{\omega_{X/S}^2/2!}\right)\Bigg|_{Y\setminus f^{-1}(\pi(\Sigma_\pi))} c_1(TY/S, h_{TY/S})\right] + \chi(Y_{\rm gen})\log\frac{\mathrm{Vol}(Y_{\rm gen},\omega_X|_{Y_{\rm gen}})}{\|\eta_{X/S}\|_{L^2}^2},$$

where $Y_{\rm gen}$ denotes a general fiber of $f\colon Y\to S$ and $\chi(Y_{\rm gen})$ denotes its topological Euler number. By [20, Th. 6.8], there exists $\epsilon_1\in\mathbf{Q}$ such that

$$\log\|\eta_{X/S}\|_{L^2}^2 = \epsilon_1\,\log|t|^2 + O\left(\log(-\log|t|^2)\right)\qquad(t\to 0). \tag{4.7}$$

Let $\varpi\colon \mathbf{P}(TY)\to Y$ be the projection from the projective tangent bundle of Y to Y. Let $q\colon \widetilde{Y}\to Y$ be the resolution of the indeterminacy of the Gauss map $\nu\colon Y\setminus\Sigma_f\ni y\to[T_yY_{f(y)}]\in\mathbf{P}(TY)$ (cf. [18, Sect. 2]) and set $\widetilde{f}:=f\circ q\colon\widetilde{Y}\to S$ and $\widetilde{\nu}:=\nu\circ q\colon\widetilde{Y}\to\mathbf{P}(TY)$. Then $\widetilde{f}$ and $\widetilde{\nu}$ are holomorphic maps. Let $\mathcal{L}\to\mathbf{P}(TY)$ be the universal line bundle and let $h_{\mathcal{L}}$ be the metric on $\mathcal{L}$ induced from ϖ^*h_Y via the inclusion $\mathcal{L}\subset\varpi^*TY$. Then

$$c_1(TY/S, h_{TY/S}) = \widetilde{\nu}^* c_1(\mathcal{L}, h_{\mathcal{L}}). \tag{4.8}$$

Substituting (4.5) into (4.6), we get

(4.9)
$$\begin{aligned}\mathcal{A}(X/S) &= f_*\left[\log\left(\frac{\|\Xi\|^2}{\|d\pi\|^2}\right)c_1(TY/S,h_{TY/S})\right] - \chi_{\rm top}(Y_{\rm gen})\log\|\eta_{X/S}\|_{L^2}^2 + O(1)\\ &= \widetilde{f}_*\left[\log(q^*\|\Xi\|^2)\,\widetilde{\nu}^*c_1(\mathcal{L},h_{\mathcal{L}})\right] - \widetilde{f}_*\left[\log(q^*\|d\pi\|^2)\,\widetilde{\nu}^*c_1(\mathcal{L},h_{\mathcal{L}})\right]\\ &\quad - \epsilon_1\,\chi_{\rm top}(Y_{\rm gen})\,\log|t|^2 + O\left(\log(-\log|t|^2)\right),\end{aligned}$$

where we used (4.7) and (4.8) to get the second equality. Since $q^*\Xi$ is a holomorphic section of the holomorphic line bundle $q^*\Omega_X^3$ with $\mathrm{div}(q^*\Xi)\subset\widetilde{\pi}^{-1}(0)$, there exists by [18, Lemma 4.4] a constant $\epsilon_2\in\mathbf{Q}$ such that

(4.10)
$$\widetilde{f}_*\left[\log(q^*\|\Xi\|^2)\,\widetilde{\nu}^*c_1(\mathcal{L},h_{\mathcal{L}})\right] = \epsilon_2\log|t|^2 + O(1)\qquad(t\to 0).$$

By [18, Cor. 4.6], there exists $\epsilon_3\in\mathbf{Q}$ such that

(4.11)
$$\widetilde{f}_*\left[\log(q^*\|d\pi\|^2)\,\widetilde{\nu}^*c_1(\mathcal{L},h_{\mathcal{L}})\right] = \epsilon_3\log|t|^2 + O(1)\qquad(t\to 0).$$

Setting $\delta:=\epsilon_2-\epsilon_3-\epsilon_1\,\chi_{\rm top}(Y_{\rm gen})\in\mathbf{Q}$, we get (4.3) by (4.9), (4.10), (4.11).

By the definition of τ_M, the result follows from (4.1), (4.2), (4.3). □

Theorem 4.3. *Let $C\subset\mathcal{M}_{M^\perp}^*$ be an irreducible projective curve intersecting $\overline{\mathcal{D}}_{M^\perp}\cup\mathcal{B}_M$ properly. Let $\mathfrak{b}\in C\cap(\overline{\mathcal{D}}_{M^\perp}\cup\mathcal{B}_M)$ and let $C_{\mathfrak{b}}=\bigcup_{i\in I}C_{\mathfrak{b}}^{(i)}$ be the irreducible decomposition of the set germ $C_{\mathfrak{b}}=(C,\mathfrak{b})$. Let $\nu^{(i)}\colon(\Delta,0)\to C_{\mathfrak{b}}^{(i)}$ be the normalization. Then there exists $\alpha_{\mathfrak{b}}^{(i)}\in\mathbf{Q}$ such that as $t\to 0$,*

$$\log\tau_M(\nu^{(i)}(t)) = \alpha_{\mathfrak{b}}^{(i)}\,\log|t| + O(\log(-\log|t|)).$$

Proof. Let $f\colon (X,\theta) \to B$ be the family of 2-elementary $K3$ surfaces of type M with period map $\varphi\colon B \to C$ as in Theorem 3.1 (1). By [1, Th. 13.4], there exists a resolution of the singularities $\mu\colon \widetilde{X} \to X$ such that θ lifts to an involution $\widetilde{\theta}\colon \widetilde{X} \to \widetilde{X}$. We set $\widetilde{f} := f \circ \mu$. Since μ is an isomorphism outside the singular fibers of f, the period map for $\widetilde{f}\colon (\widetilde{X},\widetilde{\theta}) \to B$ coincides with $\varphi\colon B \to C$. Replacing $f\colon (X,\theta) \to B$ by $\widetilde{f}\colon (\widetilde{X},\widetilde{\theta}) \to B$ if necessary, we may assume that X is smooth.

For $i \in I$, let $\mathfrak{p}^{(i)} \in \varphi^{-1}(\mathfrak{b})$ be such that $\varphi(B_{\mathfrak{p}^{(i)}}) = C_{\mathfrak{b}}^{(i)}$. Let $(V^{(i)}, s)$ be a coordinate neighborhood of $\mathfrak{p}^{(i)}$ in B with $s(\mathfrak{p}^{(i)}) = 0$. Let $\varphi^{(i)}\colon V^{(i)} \to \Delta$ be the holomorphic map such that $\varphi^{(i)} = (\nu^{(i)})^{-1} \circ \varphi$ on $V^{(i)} \setminus \{\mathfrak{p}^{(i)}\}$. There exists $m_i \in \mathbf{Z}_{>0}$ and $\epsilon_i(s) \in \mathbf{C}\{s\}$ such that $t \circ \varphi^{(i)}(s) = s^{m_i}\epsilon_i(s)$ and $\epsilon_i(0) \neq 0$. By Theorem 4.2 applied to the family $f\colon (X,\theta) \to B$, there exists $\alpha_i \in \mathbf{Q}$ such that

$$\log \tau_M(\nu^{(i)} \circ \varphi^{(i)}(s)) = \alpha_i \log|s| + O(\log(-\log|s|)) \qquad (s \to 0).$$

This, together with the relation $t \circ \varphi^{(i)}(s) = s^{m_i}\epsilon_i(s)$, yields the desired estimate with $\alpha_{\mathfrak{b}}^{(i)} = \alpha_i/m_i$. □

5. The automorphic property of τ_M: the case $r(M) \geq 18$

In [17, Main Th.], we proved that τ_M is expressed as the Petersson norm of an automorphic form on the period domain for 2-elementary $K3$ surfaces of type M if $r(M) \leq 17$. In this section, we extend this result when $r(M) \geq 18$. For $n \in \mathbf{Z}$, $\langle n \rangle$ denotes the 1-dimensional lattice $\mathbf{Z}$ equipped with the bilinear form $\langle x, y \rangle = nxy$. We denote by $\mathbb{U}$ the 2-dimensional lattice associated to the matrix $\binom{0\,1}{1\,0}$.

5.1. Automorphic forms on the moduli space.

We fix a vector $l_{M^\perp} \in M^\perp \otimes \mathbf{R}$ with $\langle l_{M^\perp}, l_{M^\perp} \rangle \geq 0$ and set

$$j_{M^\perp}(\gamma, [\eta]) := \frac{\langle \gamma(\eta), l_{M^\perp} \rangle}{\langle \eta, l_{M^\perp} \rangle} \qquad [\eta] \in \Omega^+_{M^\perp}, \quad \gamma \in O^+(M^\perp).$$

Since $\langle l_{M^\perp}, l_{M^\perp} \rangle \geq 0$, $j_{M^\perp}(\gamma, \cdot)$ is a nowhere vanishing holomorphic function on $\Omega^+_{M^\perp}$.

Definition 5.1. A holomorphic function $F \in \mathcal{O}(\Omega^+_{M^\perp})$ is an automorphic form on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight ν if the following two conditions are satisfied:

(i) There exists a unitary character $\chi\colon O^+(M^\perp) \to U(1)$ such that

$$F([\gamma(\eta)]) = \chi(\gamma)\, j_{M^\perp}(\gamma, [\eta])^\nu\, F([\eta]), \qquad [\eta] \in \Omega^+_{M^\perp}, \quad \gamma \in O^+(M^\perp).$$

(ii) Denote by $\|F\|^2 \in C^\infty(\Omega^+_{M^\perp})$ the Petersson norm of F (cf. [17, Def. 3.16] with $p = \nu$, $q = 0$), which is regarded as a C^∞ function on $\mathcal{M}_{M^\perp}$ in the sense of orbifolds. Then $\log\|F\|^2 \in L^1_{\mathrm{loc}}((\mathcal{M}^*_{M^\perp})_{\mathrm{reg}})$ and there exists an effective divisor D on $\mathcal{M}^*_{M^\perp}$ such that

$$-dd^c \log\|F\|^2 = \nu\, \widetilde{\omega}_{M^\perp} - \delta_D$$

as currents on $(\mathcal{M}^*_{M^\perp})_{\mathrm{reg}}$. Here $\widetilde{\omega}_{M^\perp}$ is the current on $(\mathcal{M}^*_{M^\perp})_{\mathrm{reg}}$ defined as the trivial extension of the Kähler form of the Bergman metric, and $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ for a complex manifold.

The notion of meromorphic automorphic form is defined in the same manner.

Since $\mathcal{B}_M$ is a subvariety with codimension greater than or equal to 2 when $r(M) \leq 17$, an automorphic form on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of positive weight extends to a holomorphic section of the corresponding Hodge bundle on $\mathcal{M}^*_{M^\perp}$ by the Koecher principle (cf. [4, p.498]). In particular, the second condition (ii) follows from the first one (i) in this case.

5.2. **The equation satisfied by τ_M on the period domain.** Let $\mathcal{A}_g$ denote the Siegel modular variety of degree g, which is the coarse moduli space of principally polarized Abelian varieties of dimension g. The Petersson norm of a Siegel modular form S on the Siegel upper half space of degree g is denoted by $\|S\|^2$ (cf. [17, Sect. 3.2]), which is a C^∞ function on $\mathcal{A}_g$ in the sense of orbifolds. If k is the weight of S, the $(1,1)$-form $\omega_{\mathcal{A}_g} := -\frac{1}{k}dd^c \log \|S\|^2$ on $\mathcal{A}_g$ in the sense of orbifolds is the Kähler form of the Bergman metric.

As an application of Theorem 4.3, we prove the automorphic property of τ_M when $r(M) \geq 18$. For this, we need an extension of [17, Sect. 7].

Theorem 5.2. *Let $\Pi_M \colon \Omega^+_{M^\perp} \to \mathcal{M}_{M^\perp}$ be the projection and let $\tau_{\Omega^+_{M^\perp}}$ be the $O^+(M^\perp)$-invariant function on $\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp}$ defined as $\tau_{\Omega^+_{M^\perp}} = \Pi^*_M \tau_M$. Then $\tau_{\Omega^+_{M^\perp}}$ lies in $L^1_{\rm loc}(\Omega^+_{M^\perp})$ and satisfies the following equation of currents on $\Omega^+_{M^\perp}$:*

$$dd^c \log \tau_{\Omega^+_{M^\perp}} = \frac{r(M)-6}{4}\,\omega_M + J^*_M \omega_{\mathcal{A}_{g(M)}} - \frac{1}{4}\delta_{\mathcal{D}_{M^\perp}}. \tag{5.1}$$

Proof. Let $O^+(M^\perp)_{[\eta]} \subset O^+(M^\perp)$ be the stabilizer of $[\eta] \in \Omega^+_{M^\perp}$. As in [17], set

$$H^o_d := \{[\eta] \in H_d;\, O^+(M^\perp)_{[\eta]} = \{\pm 1,\, \pm s_d\}\}, \qquad \mathcal{D}^o_{M^\perp} := \bigcup_{d \in \Delta_{M^\perp}} H^o_d$$

and $Z_{M^\perp} := \bigcup_{d\in\Delta_{M^\perp}} H_d \setminus H^o_d$. When $r(M) \leq 18$, $Z_{M^\perp}$ is an analytic subset of $\Omega^+_{M^\perp}$ with codimension greater than or equal to 2 by [17, Prop. 1.9 (2)]. By [17, Sect. (7.1)], $\tau_{\Omega^+_{M^\perp}}$ lies in $L^1_{\rm loc}(\Omega^+_{M^\perp} \setminus Z_{M^\perp})$ and satisfies the following equation of currents on $\Omega^+_{M^\perp} \setminus Z_{M^\perp}$:

$$dd^c \log \tau_{\Omega^+_{M^\perp}} = \frac{r(M)-6}{4}\,\omega_M + J^*_M \omega_{\mathcal{A}_{g(M)}} - \frac{1}{4}\delta_{\mathcal{D}_{M^\perp}}. \tag{5.2}$$

Since $\operatorname{codim} Z_{M^\perp} \geq 2$ when $r(M) \leq 18$, we deduce from (5.2) and [15, p.53, Th. 1] that Eq.(5.1) holds in this case. We consider the case $r(M) \geq 19$. Since $\Omega^+_{M^\perp}$ consists of a unique point when $r(M) = 20$, i.e., $M^\perp \cong \langle 2\rangle \oplus \langle 2\rangle$, the assertion is trivial in this case. It suffices to prove (5.1) when $r(M) = 19$, in which case either $M^\perp \cong \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$ or $M^\perp \cong \mathbb{U} \oplus \langle 2\rangle$ by [12, p.1434, Table 1].

Assume $M^\perp \cong \mathbb{U} \oplus \langle 2\rangle$. By [6, Th. 7.1], there exist isomorphisms $\Omega^+_{M^\perp} \cong \mathfrak{H}$ and $O^+(M^\perp) \cong SL_2(\mathbf{Z})$ such that the $O^+(M^\perp)$-action on $\Omega^+_{M^\perp}$ is identified with the projective action of $SL_2(\mathbf{Z})$ on $\mathfrak{H}$. Let $\mathcal{F} := \{z \in \mathfrak{H};\, |z| \geq 1,\, |\Re z| \leq 1/2\}$ be the fundamental domain for the $PSL_2(\mathbf{Z})$-action on $\mathfrak{H}$. For $\tau \in \mathfrak{H}$, let $SL_2(\mathbf{Z})_\tau \subset SL_2(\mathbf{Z})$ be the stabilizer of τ. Let $d \in \Delta_{M^\perp}$ and let $z \in \mathcal{F}$ be the point corresponding to $[\eta] \in H_d$. Since $O^+(M^\perp)_{[\eta]} \supset \mathbf{Z}_2 \times \mathbf{Z}_2$ and hence $\#\, O^+(M^\perp)_{[\eta]} \geq 4$, we get $\# PSL_2(\mathbf{Z})_z \geq 2$. By e.g. [14], we get $z \in \{i, e^{\pi i/3}, e^{2\pi i/3}\}$. If $z = e^{\pi i/3}$ or $e^{2\pi i/3}$, then $PSL_2(\mathbf{Z})_z \cong \mathbf{Z}_3$. In this case, $SL_2(\mathbf{Z})_z \cong O^+(M^\perp)_{[\eta]}$ does not contain a subgroup of order 4, which contradicts the fact $O^+(M^\perp)_{[\eta]} \supset \{\pm 1, \pm s_d\} = \mathbf{Z}_2 \times \mathbf{Z}_2$. Hence we get $z = i$. Since $\# SL_2(\mathbf{Z})_i = 4$, we get $O^+(M^\perp)_{[\eta]} = \{\pm 1, \pm s_d\}$.

This implies $H_d^o = H_d$ and $Z_M = \emptyset$ when $M \cong \mathbb{U} \oplus \langle 2 \rangle$. This proves (5.1) when $M \cong \mathbb{U} \oplus \langle 2 \rangle$. For the case $M^\perp \cong \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle$, see Sect.6. □

5.3. **The automorphic property of τ_M.**

Theorem 5.3. *There exist an integer $\nu \in \mathbf{Z}_{>0}$ and an (possibly meromorphic) automorphic form Ψ_M on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight $\nu(r(M)-6)$ and a Siegel modular form S_M on $\mathfrak{S}_{g(M)}$ of weight 4ν such that for every 2-elementary $K3$ surface (X,ι) of type M,*

$$\tau_M(X,\iota) = \|\Psi_M(\overline{\varpi}_M(X,\iota))\|^{-1/2\nu} \|S_M(\mathit{\Omega}(X^\iota))\|^{-1/2\nu}.$$

Here $\overline{\varpi}_M(X,\iota) \in \mathcal{M}_{M^\perp}$ denotes the period of (X,ι) and $\mathit{\Omega}(X^\iota) \in \mathcal{A}_{g(M)}$ denotes the period of X^ι.

Proof. Since the assertion was proved when $r(M) \leq 17$ (cf. [17]), we assume $r(M) \geq 18$. Let $\ell \in \mathbf{Z}_{>0}$ be sufficiently large. Let S be a Siegel modular form of weight 4ℓ on $\mathfrak{S}_{g(M)}$ such that the function $\mathcal{M}^o_{M^\perp} \ni (X,\iota) \to \|S(\mathit{\Omega}(X^\iota))\|^2 \in \mathbf{R}_{\geq 0}$ does not vanish identically. Let F be a non-zero automorphic form on $\Omega^+_{M^\perp}$ for $O^+(M^\perp)$ of weight $\ell(r(M)-6)$. Let $J_M^*\omega_{\mathcal{A}_{g(M)}}$ be the current defined as the trivial extension of $(J_M^o)^*\omega_{\mathcal{A}_{g(M)}}$ from $\Omega_{M^\perp} \setminus \mathcal{D}_{M^\perp}$ to $\Omega_{M^\perp}$, where $J_M^o \colon \Omega_{M^\perp} \setminus \mathcal{D}_{M^\perp} \to \mathcal{A}_{g(M)}$ is the holomorphic map defined as $J_M^o(\overline{\varpi}_M(X,\iota)) = \mathit{\Omega}(X^\iota)$ (cf. [17, Sects. 3.1-3.4]). Then the following equations of currents on $\Omega^+_{M^\perp}$ hold:

$$-dd^c \log \|S\|^2 = 4\ell\, J_M^*\omega_{\mathcal{A}_{g(M)}} - \delta_{J_M^*\mathrm{div}(S)}, \tag{5.3}$$

$$-dd^c \log \|F\|^2 = \ell(r(M)-6)\,\omega_M - \delta_{\mathrm{div}(F)}. \tag{5.4}$$

We set

$$\varphi := \tau_{\Omega^+_{M^\perp}} (\|F\| \cdot \|S\|)^{1/2\ell}.$$

By (5.1), (5.3), (5.4), there is an $O^+(M^\perp)$-invariant $\mathbf{Q}$-divisor D on $\Omega^+_{M^\perp}$ satisfying the following equation of currents on $\Omega^+_{M^\perp}$:

$$-dd^c \log \varphi = \delta_D. \tag{5.5}$$

Let $[\eta_0] \in \Omega^+_{M^\perp}$ and let $m \in \mathbf{Z}_{>0}$ be an integer such that mD is an integral divisor on $\Omega^+_{M^\perp}$. We define $G([\eta]) := \exp\left(m\int_{[\eta_0]}^{[\eta]} \partial \log\varphi\right)$. Since the residues of the logarithmic 1-form $m\,\partial\varphi$ on $\Omega^+_{M^\perp}$ are integral, G is a meromorphic function on $\Omega^+_{M^\perp}$ with $\mathrm{div}(G) = m\,D$. By the definition of G and the equality $\overline{\partial\log\varphi} = \bar\partial\log\varphi$, we get

$$|G([\eta])|^2 = \exp\left(m\int_{[\eta_0]}^{[\eta]} \partial\log\varphi + \bar\partial\log\varphi\right) = \varphi([\eta])^m \varphi([\eta_0])^{-m}. \tag{5.6}$$

Let $\gamma \in O^+(M^\perp)$. By the $O^+(M^\perp)$-invariance of φ, we get $\gamma^*\partial\log\varphi = \partial\log\varphi$, which yields that $d\log(\gamma^*G/G) = 0$. Hence there exists a constant $\chi(\gamma) \in \mathbf{C}^*$ with

$$\gamma^* G = \chi(\gamma)\, G. \tag{5.7}$$

Since $(\gamma\gamma')^* = (\gamma')^*\gamma^*$ for $\gamma, \gamma' \in O^+(M^\perp)$, we deduce from (5.7) that $\chi\colon O^+(M^\perp) \to \mathbf{C}^*$ is a character. We see that $|\chi(\gamma)| = 1$. Indeed, by the definition of χ, we get

$$
\begin{aligned}
|\chi(\gamma)|^2 &= \frac{G(\gamma\cdot[\eta])}{G([\eta])}\cdot\overline{\left(\frac{G(\gamma\cdot[\eta])}{G([\eta])}\right)} \\
&= \exp\left(m\int_{[\eta_0]}^{\gamma\cdot[\eta_0]}\partial\log\varphi\right)\cdot\exp\left(m\int_{[\eta_0]}^{\gamma\cdot[\eta_0]}\bar\partial\log\varphi\right) \\
&= \exp\left(m\int_{[\eta_0]}^{\gamma\cdot[\eta_0]}d\log\varphi\right) \\
&= \exp\left[m\,\log\varphi(\gamma\cdot[\eta_0]) - m\,\log\varphi([\eta_0])\right] = 1,
\end{aligned}
\tag{5.8}
$$

where the second equality follows from the fact $\overline{\partial\log\varphi} = \bar\partial\log\varphi$ and the last equality follows from the $O^+(M^\perp)$-invariance of φ. By (5.7) and (5.8), $G^{-4\ell}F^m$ satisfies Definition 5.1 (1).

Set $C := \log\varphi([\eta_0])$. By the definition of φ and (5.6), we get

$$
\tau_{\Omega^+_{M^\perp}} = e^C|G|^{2/m}(\|F\|\cdot\|S\|)^{-1/2\ell} = e^C(\|G^{-4\ell}F^m\|^2\cdot\|S^m\|^2)^{-1/4m\ell}.
\tag{5.9}
$$

We set $\nu := m\ell$, $\Psi_M := G^{-4\ell}F^m$ and $S_M := S^m$. Then

$$
\tau_M = (\|\Psi_M\|^2\|S_M\|^2)^{-1/4\nu}.
$$

Since S_M is a Siegel modular form of weight $4\ell m = 4\nu$ and since Ψ_M is a meromorphic function on $\Omega^+_{M^\perp}$ satisfying the functional equation in Definition 5.1 (1) with weight $(r(M)-6)\ell m = (r(M)-6)\nu$, it suffices to prove that Ψ_M satisfies the regularity condition (2) in Definition 5.1. Since $|G|$ is an $O^+(M^\perp)$-invariant function on $\Omega^+_{M^\perp}\setminus\mathcal{D}_{M^\perp}$ by (5.6), we regard $|G|$ as a function on $\mathcal{M}^o_{M^\perp}$. Since F is an automorphic form on $\Omega^+_{M^\perp}$ and hence satisfies the regularity condition (2) in Definition 5.1, it suffices to prove $\log|G|^2 \in L^1_{\rm loc}((\mathcal{M}^*_{M^\perp})_{\rm reg})$ and the existence of a $\mathbf{Q}$-divisor $\mathfrak{D}$ on $\mathcal{M}^*_{M^\perp}$ satisfying the following equation of currents on $(\mathcal{M}^*_{M^\perp})_{\rm reg}$:

$$
-\,dd^c\log|G|^2 = \delta_{\mathfrak{D}}.
\tag{5.10}
$$

Let Φ be the function on $\mathcal{M}^o_{M^\perp}$ such that $\varphi = \Pi^*_M\Phi$. Let $\overline{D}$ be the closure of $\Pi_M(D)$ in $\mathcal{M}^*_{M^\perp}$. By (5.5), we have the equation of currents on $(\mathcal{M}_{M^\perp})_{\rm reg}$:

$$
-\,dd^c\log\Phi = \delta_{\overline{D}}.
\tag{5.11}
$$

Let $\mathcal{B}_M = \bigcup_{\alpha\in A}\mathcal{B}_{M,\alpha}$ be the irreducible decomposition. Since $r(M)\geq 18$, $\mathcal{B}_M$ is a divisor on $\mathcal{M}^*_{M^\perp}$. Let $C\subset\mathcal{M}^*_{M^\perp}$ be an arbitrary irreducible projective curve intersecting $\mathcal{B}_M$ properly. Let $\mathfrak{b}\in C\cap\mathcal{B}_M$ be an arbitrary point and let $C_{\mathfrak{b}} = \bigcup_{i\in I}C^{(i)}_{\mathfrak{b}}$ be the irreducible decomposition of the set germ $C_{\mathfrak{b}} = (C,\mathfrak{b})$. Let $\nu^{(i)}\colon(\Delta,0)\to C^{(i)}_{\mathfrak{b}}$ be the normalization. By Theorem 4.3, there exists $\alpha^{(i)}_{\mathfrak{b}}\in\mathbf{Q}$ such that as $t\to 0$,

$$
\log\tau_M|_{C^{(i)}_{\mathfrak{b}}}(\nu^{(i)}(t)) = \alpha^{(i)}_{\mathfrak{b}}\,\log|t| + O(\log(-\log|t|)).
\tag{5.12}
$$

Since F and S are automorphic forms on $\Omega^+_{M^\perp}$ and $\mathfrak{S}_{g(M)}$ respectively, there exists $\beta^{(i)}_{\mathfrak{b}}, \gamma^{(i)}_{\mathfrak{b}}\in\mathbf{Z}$ by [10, Th. 3.1] (cf. [17, Prop. 3.12]) such that as $t\to 0$,

$$
(\log\|F\|)|_{C^{(i)}_{\mathfrak{b}}}(\nu^{(i)}(t)) = \beta^{(i)}_{\mathfrak{b}}\,\log|t| + O(\log(-\log|t|)),
\tag{5.13}
$$

$$(\log \|S\|)|_{C^{(i)}_{\mathfrak{b}}}(\nu^{(i)}(t)) = \gamma^{(i)}_{\mathfrak{b}} \log |t| + O(\log(-\log |t|)). \tag{5.14}$$

By (5.12), (5.13), (5.14), there exists $\epsilon^{(i)}_{\mathfrak{b}} \in \mathbf{Q}$ such that

(5.15)
$$\log \Phi|_{C^{(i)}_{\mathfrak{b}}}(\nu^{(i)}(t)) = \epsilon^{(i)}_{\mathfrak{b}} \log |t| + O(\log(-\log |t|)) \qquad (t \to 0).$$

By (5.11), (5.15), there exists $n_{C,\alpha} \in \mathbf{Q}$ such that the following equation of currents on C holds:

$$-\,dd^c \log \Phi|_C = \delta_{\overline{D}\cap C} + \sum_{\alpha\in A} n_{C,\alpha}\,\delta_{\mathcal{B}_{M,\alpha}\cap C}. \tag{5.16}$$

Since $C \subset \mathcal{M}^*_{M^\perp}$ is arbitrary, this implies that $\partial \log \Phi$ is a logarithmic 1-form on $(\mathcal{M}^*_{M^\perp})_{\rm reg}$ and that $n_{C,\alpha}$ is the residue of $\partial \log \Phi$ along the irreducible divisor $\mathcal{B}_{M,\alpha}$ for sufficiently general C. Since $n_{C,\alpha}$ is independent of the choice of sufficiently general C, we write n_α for $n_{C,\alpha}$. By (5.11) and (5.16), we get $\Phi \in L^1_{\rm loc}(\mathcal{M}^*_{M^\perp})$ and the following equation of currents on $(\mathcal{M}^*_{M^\perp})_{\rm reg}$:

$$-\,dd^c \log \Phi = \delta_{\overline{D}} + \sum_{\alpha\in A} n_\alpha\,\delta_{\mathcal{B}_M,\alpha}. \tag{5.17}$$

Set $\mathfrak{D} = m(\overline{D} + \sum_{\alpha\in A} n_\alpha\,\mathcal{B}_{M,\alpha})$. Since $\varphi = e^C|G|^{2/m}$ by (5.6), we get (5.10) from (5.17). This completes the proof. □

Remark 5.4. In fact, one can prove that the boundary divisor $\mathcal{B}_M$ is irreducible when $r(M) \geq 18$. The irreducibility of $\mathcal{B}_M$ plays a crucial role to give an explicit formulae for Ψ_M and S_M, when $r(M) \geq 18$. See [19, Sect. 11.4] for the details.

6. The case $M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$

Throughout Sect.6, we assume

$$M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$$

and prove that (5.1) holds in this case.

6.1. **Preliminaries.** Since $M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$, we get the explicit expression:

$$\Omega^+_{M^\perp} = \{(x:y:z) \in \mathbf{P}^2;\ x^2+y^2-z^2 = 0,\ |x|^2+|y|^2-|z|^2 > 0,\ |x+iy| > |x-iy|\}.$$

The unit disc $\varDelta = \{z \in \mathbf{C};\ |z| < 1\}$ is isomorphic to $\Omega^+_{M^\perp}$ by the map

$$c\colon \varDelta \ni z \to \left(\frac{1+z^2}{2} : \frac{1-z^2}{2i} : z\right) \in \Omega^+_{M^\perp}. \tag{6.1}$$

For $\epsilon \in]0,1[$, we set $\varDelta(\epsilon) := \{z \in \varDelta;\ |z| < \epsilon\}$ and $\Omega^+_{M^\perp}(\epsilon) := c(\varDelta(\epsilon))$. We also set $\delta := (0,0,1) \in \Delta_{M^\perp}$. Then $s_\delta(x:y:z) := (x:y:-z)$ is the reflection on $\Omega^+_{M^\perp}$ associated to δ, and we have

$$H_\delta \cap \Omega^+_{M^\perp} = \{c(0)\} = \{(1:-i:0)\}.$$

Lemma 6.1. *Let $O^+(M^\perp)_{[\eta]}$ be the stabilizer of $[\eta] \in \Omega^+_{M^\perp}$ in $O^+(M^\perp)$. Then $\#O^+(M^\perp)_{[\eta]} = 8$. Moreover, the natural projection $\varPi_M\colon \Omega^+_{M^\perp} \to \mathcal{M}_{M^\perp}$ has ramification index 4 at $c(0) \in \Omega^+_{M^\perp}$.*

Proof. Since

(6.2)
$$O^+(M^\perp)_{c(0)} = \langle -1_{M^\perp}\rangle \times \langle s_\delta\rangle \times \langle \mu\rangle, \qquad s_\delta = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad \mu := \begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix},$$

we get the first assertion. Since $-1_{M^\perp}$, s_δ and μ act on Δ as follows under the identification (6.1):

(6.3)
$$-1_{M^\perp}(z) = z, \qquad s_\delta(z) = -z, \qquad \mu(z) = i\,z,$$

we deduce from (6.2), (6.3) that the projection $\Pi_{M^\perp}\colon \Omega^+_{M^\perp} \to \mathcal{M}_{M^\perp}$ at $c(0) \in \Omega^+_{M^\perp}$ is identified with the map $\mathbf{C} \ni z \to z^4 \in \mathbf{C}$ at $z = 0$. □

We recall the notion of ordinary singular families of 2-elementary $K3$ surfaces. Let $\mathcal{Z}$ be a smooth complex threefold. Let $p\colon \mathcal{Z} \to \Delta$ be a proper surjective holomorphic function without critical points on $\mathcal{Z} \setminus p^{-1}(0)$. Let $\iota\colon \mathcal{Z} \to \mathcal{Z}$ be a holomorphic involution preserving the fibers of p. We set $Z_t = p^{-1}(t)$ and $\iota_t = \iota|_{Z_t}$ for $t \in \Delta$. Then $p\colon (\mathcal{Z}, \iota) \to \Delta$ is called an *ordinary singular family* of 2-elementary $K3$ surfaces of type M if p has a unique, non-degenerate critical point on Z_0 and if (Z_t, ι_t) is a 2-elementary $K3$ surface of type M for all $t \in \Delta^*$. See [17, Sects. 2.2 and 2.3] for more about ordinary singular families of 2-elementary $K3$ surfaces.

Proposition 6.2. *There exist $\epsilon \in]0,1[$ and an ordinary singular family of 2-elementary $K3$ surfaces $p\colon (\mathcal{Z}, \iota) \to \Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle$ of type M with the following properties:*

(1) *The period map for $p\colon (\mathcal{Z}, \iota) \to \Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle$ is given by the projection $\Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle \to \mathcal{M}_{M^\perp}$.*
(2) *The map $p\colon \mathcal{Z} \to \Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle$ is projective.*

Proof. We follow [17, Th. 2.6]. By Lemma 6.1, we get $H^o_\delta = \emptyset$. In particular, [17, Th. 2.6] does not apply at once for $M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$. However, in the proof of [17, Th. 2.6], the fact $c(0) \in H^o_\delta$ was used only to deduce the following (i), (ii):

(i) $s_\delta(c(t)) = c(-t)$ for all $t \in \Delta(\epsilon)$.
(ii) Under the inclusion $M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle \subset \mathbb{L}_{K3}$, set
$$\Delta_{c(0)} := \{d \in \Delta_{\mathbb{L}_{K3}};\ \langle d, (1, -i, 0)\rangle = 0\}.$$
Then there exists $m \in M$ such that $\langle m, m\rangle_M > 0$ and $m^\perp \cap \Delta_{c(0)} = \{\pm\delta\}$.

Once (i), (ii) are verified, the proof of [17, Th. 2.6] for the existence of an ordinary singular family of 2-elementary $K3$ surfaces $p\colon (\mathcal{Z}, \iota) \to \Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle$ with (1), (2) works. (Notice that the condition $r(M) \le 17$ was not used in [17, Th. 2.6].) Hence it suffices to prove (i), (ii). By (6.1), we get (i). By [17, Lemma A.2], it suffices to prove $M^\perp \cap \Delta_{c(0)} = \{\pm\delta\}$. Since $c(0) = (1 : -i : 0)$, we get $M^\perp \cap \Delta_{c(0)} = \{\pm\delta\}$. This proves (ii). □

Proposition 6.3. *Set $\mathfrak{q} := \Pi_{M^\perp}(c(0)) \in \mathcal{M}_{M^\perp}$. Then there exist a pointed smooth projective curve $(B, \mathfrak{p})$, a neighborhood U of $\mathfrak{p}$, a holomorphic map between curves $\varphi\colon (B, \mathfrak{p}) \to (\mathcal{M}^*_{M^\perp}, \mathfrak{q})$, a smooth projective threefold $\mathcal{X}$ with an involution $\theta\colon \mathcal{X} \to \mathcal{X}$, and a surjective holomorphic map $p\colon \mathcal{X} \to B$ with the following properties:*

(1) *$\varphi(B) = \mathcal{M}^*_{M^\perp}$ and the map $f|_U\colon (U, \mathfrak{p}) \to (\varphi(U), \mathfrak{q})$ is a double covering with a unique ramification point $\mathfrak{p}$.*

(2) *The map* $p\colon \mathcal{X} \to B$ *is* $\mathbf{Z}_2$*-equivariant with respect to the* $\mathbf{Z}_2$*-action on* $\mathcal{X}$ *induced by* θ *and with respect to the trivial* $\mathbf{Z}_2$*-action on* B.

(3) *The family of algebraic surfaces with involution* $p|_{p^{-1}(U)}\colon (\mathcal{X},\theta)|_{p^{-1}(U)} \to U$ *is an ordinary singular family of* 2*-elementary* $K3$ *surfaces of type* M *with period map* $\varphi|_U$.

Proof. We follow [17, Th. 2.8]. Since $H^o_\delta = \emptyset$ and hence $\mathcal{D}^o_{M^\perp} = \emptyset$ by Lemma 6.4 below, [17, Th. 2.8] does not apply at once for $M^\perp = \langle 2\rangle \oplus \langle 2\rangle \oplus \langle -2\rangle$. Set $S := \Omega^+_{M^\perp}(\epsilon)/\langle s_\delta\rangle \cong \Delta$. Let $\bar{c}(0) \in S$ be the image of $c(0)$ and let $\gamma\colon (S, \bar{c}(0)) \to (\mathcal{M}_{M^\perp}, \mathfrak{p})$ be the projection induced from $\Pi_{M^\perp}$. By Proposition 6.2, there is an ordinary singular family of 2-elementary $K3$ surfaces $p\colon (\mathcal{Z},\iota) \to S$ of type M with period map γ. We set $C := \mathcal{M}^*_{M^\perp}$. By Theorem 3.1 (2) applied to $p\colon (\mathcal{Z},\iota) \to S$, there exist $\varphi\colon B \to C$, $f\colon X \to B$ and $\theta\colon X \to X$ as in Theorem 3.1 satisfying (a), (b), (c), (d), (e), (f).

We prove that $\operatorname{Sing} X \cap X_\mathfrak{p} = \emptyset$. Since $(X_\mathfrak{p}, \theta|_{X_\mathfrak{p}}) \cong (Z_{\bar{c}(0)}, \iota|_{Z_{\bar{c}(0)}})$ by Theorem 3.1 (d) and since Z_0 has a unique A_1-singularity $o := \operatorname{Sing} Z_0$, the deformations $p\colon (X, X_\mathfrak{p}) \to (B,\mathfrak{p})$ and $p\colon (\mathcal{Z}, Z_0) \to (S, \bar{c}(0))$ induce maps $\rho_f\colon (B,\mathfrak{p}) \to \mathrm{Def}(A_1)$ and $\rho_p\colon (\Delta, 0) \to \mathrm{Def}(A_1)$, where $\mathrm{Def}(A_1) \cong (\mathbf{C}, 0)$ is the Kuranishi space of 2-dimensional A_1-singularity. Since (Z_0, o) is an A_1-singularity, there is an isomorphism of germs $(X, o) \cong (\mathcal{Z}, o)$ if and only if $\mathrm{mult}_\mathfrak{p}\rho_f = \mathrm{mult}_0\rho_p$.

Recall that $\mathrm{Def}(Z_0)$ is the Kuranishi space of Z_0 and let $\rho\colon \mathrm{Def}(Z_0) \to \mathrm{Def}(A_1)$ be the map of germs induced by the local semiuniversal deformation of Z_0 over $\mathrm{Def}(Z_0)$. Recall that $\mu_f\colon (B,\mathfrak{p}) \to \mathrm{Def}(Z_0)$ and $\mu_p\colon (\Delta,0) \to \mathrm{Def}(Z_0)$ are the maps induced by the deformations $f\colon (X, X_\mathfrak{p}) \to (B,\mathfrak{p})$ and $p\colon (\mathcal{Z}, Z_0) \to (\Delta, 0)$, respectively. Since $\rho_f = \rho\circ\mu_f$ and $\rho_p = \rho\circ\mu_p$, there exists by Theorem 3.1 (f) an isomorphism of germs $\psi\colon (\Delta, 0) \cong (B,\mathfrak{p})$ such that $\rho_f\circ\psi(t) - \rho_p(t) \in t^\nu\mathbf{C}\{t\}$. By choosing $\nu \geq 2$ in Theorem 3.1 (2), this implies that $\mathrm{mult}_\mathfrak{p}\rho_f = \mathrm{mult}_0\rho_p$. Since $\mathcal{Z}$ is smooth, we get $\operatorname{Sing} X \cap X_\mathfrak{p} = \emptyset$. Let U be a small neighborhood of $\mathfrak{p}$ in B.

Since the map $\gamma\colon (S, c(0)) \to (C, \mathfrak{q})$ has ramification index 2 by Lemma 6.1, we get (1) by Theorem 3.1 (e). We get (2) by Theorem 3.1 (b). By Theorem 3.1 (c), $(X_b, \theta|_{X_b})$ is a 2-elementary $K3$ surface of type M with period $\varphi(b)$ for $b \in U\setminus\{\mathfrak{p}\}$. Since $p^{-1}(U)$ is smooth, $p|_{p^{-1}(U)}\colon (X,\theta)|_{p^{-1}(U)} \to U$ is an ordinary singular family of 2-elementary $K3$ surfaces of type M with period map $\varphi|_U$. This proves (3).

There is a resolution $\mathcal{X} \to X$ such that θ lifts to an involution $\widetilde{\theta}\colon \mathcal{X} \to \mathcal{X}$ [1, Th. 13.4]. Replace (X,θ) by $(\mathcal{X}, \widetilde{\theta})$. Then (1), (2), (3) are satisfied and $\mathcal{X}$ is smooth. This completes the proof. ☐

Lemma 6.4. *The group* $O^+(M^\perp)$ *acts transitively on* $\Delta_{M^\perp}/\pm 1$.

Proof. Let L be an odd unimodular lattice of signature $(2,1)$ and let $\delta, \delta' \in L$ be vectors with $\langle\delta,\delta\rangle = \langle\delta',\delta'\rangle = -1$. Set $\Lambda := \delta^\perp$ and $\Lambda' := (\delta')^\perp$. Since $\Lambda^\vee/\Lambda \cong (\mathbf{Z}\delta)^\vee/(\mathbf{Z}\delta)$ and $(\Lambda')^\vee/\Lambda' \cong (\mathbf{Z}\delta')^\vee/(\mathbf{Z}\delta')$, the equality $\langle\delta,\delta\rangle = \langle\delta',\delta'\rangle = -1$ implies that Λ and Λ' are positive-definite unimodular lattices of rank 2. It is classical that $\Lambda \cong \Lambda'$. Since $\mathbf{Z}\delta \oplus \Lambda \subset L$ and since both of $\mathbf{Z}\delta\oplus\Lambda$ and L are unimodular, we get $L = \mathbf{Z}\delta\oplus\Lambda$. Similarly, $L = \mathbf{Z}\delta'\oplus\Lambda'$. Let $\varphi\colon \Lambda \to \Lambda'$ be an isometry. Then $g_\pm\colon \mathbf{Z}\delta\oplus\Lambda \ni m\delta+\lambda \to \pm m\delta' + \varphi(\lambda) \in \mathbf{Z}\delta'\oplus\Lambda'$ is an isometry of L with $g_\pm(\delta) = \pm\delta'$ such that either $g_+ \in O^+(L)$ or $g_- \in O^+(L)$. This proves the lemma. ☐

6.2. **Proof of (5.1).** By [17, Th. 5.9], we have the following equation of C^∞ $(1,1)$-forms on $\Omega^+_{M^\perp} \setminus \mathcal{D}_{M^\perp}$

$$dd^c \log \tau_{\Omega^+_{M^\perp}} = \frac{r(M)-6}{4}\,\omega_M + J_M^*\omega_{\mathcal{A}_{g(M)}}. \tag{6.4}$$

By Theorem 4.3 and (6.4), there exists $m_d \in \mathbf{Q}$ for every $d \in \Delta_{M^\perp}$ such that the following equation of currents on $\Omega^+_{M^\perp}$ holds:

(6.5)
$$dd^c \log \tau_{\Omega^+_{M^\perp}} = \frac{r(M)-6}{4}\,\omega_M + J_M^*\omega_{\mathcal{A}_{g(M)}} - \sum_{d\in\Delta_{M^\perp}/\pm} m(d)\,\delta_{H_d}.$$

We compute $m(\delta)$ for $\delta = (0,0,1) \in \Delta_{M^\perp}$. In Proposition 6.3, we may assume that U is equipped with a coordinate function u centered at $\mathfrak{p}$. By [17, Th. 7.5] applied to the ordinary singular family $p|_{p^{-1}(U)} \colon (X,\theta)|_{p^{-1}(U)} \to U$ in Proposition 6.3 (3), we get

(6.6)
$$\log \tau_M(X_u, \theta|_{X_u}) = -\frac{1}{8}\log|u|^2 + O\left(\log(-\log|u|^2)\right) \qquad (u \to 0).$$

Let t be a coordinate function on $f(U)$ centered at $\mathfrak{p}$. By Proposition 6.3 (1), there exists $\epsilon(u) \in \mathcal{O}(U)$ with $\epsilon(0) \neq 0$ such that

$$t \circ f(u) = u^2\epsilon(u). \tag{6.7}$$

By (6.6) and (6.7), we get

$$\begin{aligned} \log \tau_M(f(u)) &= \log \tau_M(X_u, \theta|_{X_u}) \\ &= -\frac{1}{8}\log|u|^2 + O\left(\log(-\log|u|^2)\right) \\ &= -\frac{1}{16}\log|t\circ f(u)|^2 + O\left(\log(-\log|t\circ f(u)|^2)\right). \end{aligned} \tag{6.8}$$

Since the projection $\mathit{\Pi}_{M^\perp} \colon \Omega^+_{M^\perp} \to \mathcal{M}_M$ has ramification index 4 at $c(0)$ by Lemma 6.1, we get by (6.8)

(6.9)
$$\begin{aligned} \tau_{\Omega^+_{M^\perp}}(c(z)) &= -\frac{1}{16}(\mathrm{index}_{c(0)}\mathit{\Pi}_{M^\perp})\,\log|z|^2 + O\left(\log(-\log|z|^2)\right) \\ &= -\frac{1}{4}\log|z|^2 + O\left(\log(-\log|z|^2)\right) \qquad (z \to 0). \end{aligned}$$

By (6.9), we get $m(\delta) = \frac{1}{4}$. Since $\Delta_{M^\perp}/\pm 1 = O^+(M^\perp)\cdot\delta$ by Lemma 6.4 and since $\tau_{\Omega^+_{M^\perp}}$ is $O^+(M^\perp)$-invariant, we get $m(d) = m(\delta) = \frac{1}{4}$ for all $d \in \Delta_{M^\perp}$. Substituting $m(d) = \frac{1}{4}$ into (6.5), we get (5.1). This completes the proof of (5.1). □

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, JAPAN
E-mail address: yosikawa@math.kyoto-u.ac.jp

Korea Institute for Advanced Study, Hoegiro 87, Dongdaemun-gu, Seoul 130-722, KOREA