STABILITY PROBLEMS OF LARGE SCALE SYSTEMS

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PREFACE

Stability is one of the most important concepts in the theory of automatic control systems. Once a control system in equilibrium state is perturbed by some external disturbances, the system will be in the tansient state from the time disturbances are removed. In the case where this transient state dies out after a sufficiently long time and the system returns to the original equilibrium state, the system is called asymptotically stable.

Investigations on the conditions for stability of control systems originated with E.J.Routh(1877) and A.Hurwitz(1895). They independently obtained the stability conditions of the systems, whose characteristics were represented by linear ordinary differential equations with constant coneffícients, in an algebraic form. Henceforth, many studies on stability of the systems characterized by linear equations have been promoted until now. Especially frequency domain analysis and synthesis of closed loop control systems proposed by H.Nyquist(1932) marked the first milestone in constructing a systematic method of analyzing and synthesizing the control systems.

On the other hand, on systems which have nonlinear characteristics the developement of studies has been less fruitful than that of linear systems. One of the most representative work on the analysis of nonlinear systems is "the absolute stability problem" (Lur'e problem) first formulated by A.I.Lur'e (1944). Many reseachers have been reported about this problem. As is the case with linear systems, there are two ways to solve this problem, one is time domain method, i.e., the second method of Lyapunov, the other is frequency domain method developed by V.M.Popov(1959). In either case, for linear or nonlinear systems, the main interest of the stability problems was taken in the following two points : to extend the classes of the systems to which the stability theorems are applicable, and to obtain the better stability conditions using system parameters.

Recently according as the scales of the system dealt by control theory are getting larger and the structures of the systems are becoming much more complex, the stability analyses of large scale systems are attracting much attention among stability problems of control systems. This thesis is devoted to the stability analysis of large scale systems. Generally speaking, large dimensionality and complexity of the systems may necessarily cause many difficulties in treating these systems. In this thesis, the assumption that the systems are decomposed into several subsystems is employed throughout. The method for analizing the systems on this assumption is often called the decomposition method. The essential feature of the method is the reduction of complexity which comes from decomposing the overall systems into some subsystems with appropriate size. The procedures of applying the method to large scale systems are as follows.

First, the comparison equations are derived from the properties of subsystems and the relations among them. Then the "imaginary systems", whose characteristics are described by the comparison equations, is supposed. The imaginary systems don not formulate objective physical systems directly, but it corresponds to mathematical expression, i.e., the comparison equations. Next, it is shown that the stability conditions of the overall systems can be obtained by examining the properties of the imaginary systems. Finally, the stability properties of the imaginary systems are examined by some well-known analyzing methods. Thus the stability criteria for some classes of large scale systems can be obtained with reduced dimensional difficulties.

This thesis uses two different methods to get the stability conditions of the imaginary systems. One is a time domain method, the other is a frequency domain one. Two chapters are devoted to the discussions for each case. In another part of this thesis the conditions of positive definiteness and positive semidefiniteness of real rational matrix for any value of its argument is considered. By using these conditions, the relations between the results obtained by the two methods above mentioned are investigated. In the last part of this thesis, the stability and instability conditions of large scale systems that have unstable subsystems are derived using the decomposition method. The method in this case, however, is different slightly from that using the comparison equations.

CONTENTS

Chapter 1. Introduction

Section 1.1 Stability studies of Large Scale Systems

In this section studies on the stability of large scale control systems already reported are summarized briefly. The control systems dealt with here have been called, in addition to the above-mentioned "large scale systems", multi-input multi-output systems, control systems with many nonlinearities and so on. Recently according as the "decomposition" method utilized in this thesis has been introduced in analyzing control systems with large dimensionalities, these systems get to be called interconnected systems, composite systems and the like. Because these systems are looked upon as consisting of some subsystems which have comparatively independent properties with each other. Throughout this thesis "large scale systems" is adopted.

There are various kinds of definitions of stability, but roughly speaking they are classified into following two classes. The first class of stability concepts is intrinsic stability which is related to the intrinsic states of systems. The second class of stability considerations of systems deals with input-output stability which is considered by aiming only at input-output relations of systems. In this thesis, the former definitions, especially asymptotic stability in the large (henceforth abbreviated to ASIL), are considered. Investigations on input-output stability of large scale systems have been developed actively in parallel with those of intrinsic stability, but here they are not touched on.

Studies on stability of large scale systems started with extending the methods for single-input single-outputsystems so far obtained to systems with many nonlinearities or to multi-input multi-output systems at the beginning. As are known well there are two main methods of stability investigation for systems with a single nonlinearity. One is the second method of Lyapunov and the other is frequency domain method stated by V.M.Popov. These two methods were extended and generalized to be applicable to control systems with many nonlinearities. Sultanov(61), Tokumaru and Saito (62), employed the former way, Jury and Lee (25), Yakubovich (69), Partovi and Nahi (51), Tokumaru and Saito (62), Lee Xùn-Jing (32) applied the latter. However, these two methods get to possess some defects due to extending to multi-variable or multi-input multi-output case. Araki (2) pointed out these defects were as in the following :

(1) Stability theorems contain too many arbitrary parameters.
(2) Computations required for checking whether systems satisfy the assumptions of the theorem or not become increasingly difficult as dimensions of systems increase.

Thereafter the direction to remedy these defects was showed by Bailey (7) and other researchers. They employed the method called the "decomposition" method. Fundamental procedures of the method are as follows.

(1) to decompose the systems into some subsystems
(2) to obtain the stability conditions of the overall systems from properties of subsystems and relations among them.

The most advantageous point of this method is that it introduces hierarchical point of view in system analysis and gives a systematic way of practical computation for obtaining the stability conditions. Bailey assumed the decomposed systems and utilized the vector Lyapunov function method, which was a sort of the comparison method requires that there exists an appropriate Lyapunov function for each subsystem. The stability conditions of the overall systems can be derived from examining the properties of solutions of auxiliary equations (comparison equations), whose dimensions are as many as the number of subsystems and whose variable is the Lyapunov function corresponding to each subsystem. As well, Tokumaru, et. al. (67) defined an analogous stability concept with respect to input-output relations and derived the conditions of stability using comparison equations. However, most of the researchers of stability of large scale systems after Bailey have adopted slightly defferent methods from the comparison method. They employed the second method of Lyapunov using the sum of scalar-multiplied Lyapunov function corresponding to each subsystem as a candidate of Lyapunov function for the overall systems without constructing the comparison equations. This method was employed by Araki and Kondo (2), (6), Michel (41), (42) and others. Particularly, Araki and Kondo showed that an M-matrix, which has been used hitherto in numerical analysis and theoretical economics, was an effective tool for analysis of large scale systems by the "decomposition" method and derived the stability conditions of the systems superior to Bailey's theorem.

The most principal feature of the above-mentioned "decomposition" method is that it has made a detailed information on systems available. Generally speaking, the more information on systems are utilized, the more superior conditions of stability can be obtained. With respect to this point, the "decomposition" method however is still unsatisfactory, because it is based on merely the "absolute value" of the state variables of systems and neglects phase relations among subsystems. Therefore, it is needed to provide completely with the better stability theorems by virtue of utilizing the more minute properties of systems.

Now, the reports on stability problems of large scale systems after Bailey were mostly based on the "decomposition" methods, which were methods in time domain. The other method, i.e. frequency domain method for single-input single-output systems equips following merits, even if it were extended to multi-input multi-output systems.

(1) It can be applied to systems with transfer functions obtained experimentally.

(2) It is applicable to systems which involve dead time elements.

(3) It can be stated without restriction on the order of systems expressed by a certain input-output relation.

(4) Various modifications of this method have been proposed until now making use of the properties of systems.

The reason why the method is nevertheless less useful for large scale systems is that computational difficulties have not been overcome so far. However recently by Rosenbrock and other English researchers (33), (36), (53), (56), though mostly for linear case, frequency domain theories of analysis and synthesis for large scale systems (multivariable systems) have been steadily developed from practical viewpoints. The central idea of these theories is to surmount the burden owing to increase of dimensions by utilizing computers with graphic display terminal. In these theories, under a certain condition, whose satisfaction is able to be checked easily by computers with display, analytic and synthetic techniques for large scale systems were developed in parallel with the already perfected classical control theory for systems with single input-output relation.

However these theories are not seemed to be fully completed, for namely, nonlinear characteristics of systems are not fully accepted by these theories. Therefore it appears that there is a room for constructing a better frequency domain stability criterion. It is certain that availability of computers with graphic terminal is practically a powerful tool of the stability investigations of large scale systems in frequency domain. So the developement of stability theory of large scale systems aided effectively by computers with appropriate diaplay device is hoped henceforth.

Section 1.2 Summary of the Contents

In the following chapters, stability theory of large scale control systems is established, making the most of knowledge on properties of the systems. Stability theorems are described in two different forms : one is a time domain criterion, and the other a frequency domain criterion. In both cases, the "decomposition" method is adopted throughout. Relationship between these two kinds of stability theorems is discussed using the positive definiteness conditions of rational matrices.

In chapter 2, stability theorems of large scale systems with time-varying interconnections between subsystems are derived using the vector Lyapunov function method. It is also shown that the better stability conditions can be obtained when much more information on systems, such as periodicity condition for interconnections, are available. In chapter 3, frequency domain stability criteria for large scale systems are given on the assumption that the systems are decomposed into subsystems. The comparison equations and Li Xun-Jing's theorems, viz. Popov-type stability theorems extended to multi-input multi-output systems, are basic tools for obtaining frequency domain criteria by the "decomposition" method. In chapter 4, conditions for positive definiteness of real rational matrices are discussed. It is shown that, under a appropriate condition, sufficient conditions or necessary and sufficient conditions for positive definiteness of the matrices are expressed in a simple algebraic form. An analogous results are given for positive semidefiniteness of the matrices. Using the results obtained in this chapter, relations between time domain criteria obtained up to now and frequency domain criteria developed in the preceding chapter are discussed in chapter 5. In chapter 6, stability and instability theorems of large scale systems with stable and unstable subsystems are studied without employing the comparison method and the second method of Lyapunov. In chapter 7, some concluding remarks are given.

Section 1.3 List of Notation

Throughout this thesis, unless otherwise specified, the following symbol conventions are utilized. Here, generally scalars are denoted by lightface, lower case Roman and Greek letters, vectors by boldface, lower case Roman, and matrices by lightface, upper case Roman and Greek.

notations

$\mathbb{R}$: field of real numbers

$\mathbb{C}$: fields of complex numbers

$\mathbb{C}^n(\mathbb{R}^n)$: linear space of ordered n-tuples in $\mathbb{C}$ ($\mathbb{R}$)

$\mathbb{C}^{n\times m}(\mathbb{R}^{n\times m})$: ring of matrices with n rows and m columnswith element in $\mathbb{C}$ ($\mathbb{R}$)

x_i : n_i dimensional state vector of the i-th subsystem, $x_i \in \mathbb{R}^{n_i}$

m : the number of subsystems which constitute the large scale system

x : the state vector of the large scale system, $x \in \mathbb{R}^n$, where $n = \sum_{i=1}^{m} n_i$

0_{n_i} : n_i dimensional null vector, $0_{n_i} \in \mathbb{R}^{n_i}$

$|y|$: Euclidean norm of vector y

$\|P\|$: matrix norm of a matrix P compatible with the vector norm defined above

$y'(P')$: transpose of a vector y (matrix P)

$\bar{y}(\bar{P})$: conjugate of a vector y (matrix P)

$y^*(P^*)$: conjugate transposeof a vector y (matrix P)

∇v_i : gradient vector of a scalar function v_i with respect to x_i
$\triangleq \left(\frac{\partial v_i}{\partial x_1}, \frac{\partial v_i}{x_2}, \cdots, \frac{\partial v_i}{\partial x_{n_i}}\right)'$

M : set of natural numbers, $\triangleq \{1,2,\cdots,m\}$

$x' \cdot y$: inner product of vectors x and y

$A = a_{ij}$:- matrix having a_{ij} as element in row i, column j

A^{-1}	: inverse of a nonsingular square matrix A
$\lvert A \rvert$	: determinant of A (detA)
$\mathrm{diag}(a_{ii})$	: square diagonal matrix having a_{ii} as the i-th diagonal element
$\mathrm{offdiag}(a_{ij})$	: square matrix whose diagonal elements are equal to zero and offdiagonal (i,j) element a_{ij}
$\mathbb{K}$	: a set of M-matrices, i.e. matrices with non-positive offdiagonal elements and positive principal minors
$A \geq B$	: $a_{ij} \geq b_{ij}$, for all possible i,j
$x \geq y$	: $x_i \geq y_i$, for all $i \in M$, $x, y \in \mathbb{R}^m$
p.d(p.s.d)	: a class of positive definite(positive semidefinite) matrices
$\mathcal{D}(\mathcal{D}_0)$	: a class of diagonal matrices, real parts of whose elements are all positive(non-negative)
I	: unit matrix, i.e. diag(1)
τ	: time retardation of the system
$\phi_i(r)$	: scalar non-decreasing function satisfying, $\phi_i(0) = 0$, $\phi_i(r) \to \infty \; (r \to \infty)$
sgn k	: signum function $\triangleq \begin{cases} +1 & , k > 0 \\ -1 & , k < 0 \\ 0 & , k = 0 \end{cases}$
i,j,k	: the number for identifying the subsystem, unless otherwise specified, they take values $1,2,\cdots,m$
$\Longrightarrow$	: implies
$\Longleftrightarrow$	: if and only if

Section 1.4 System Description

In this section the equations of large scale systems dealt with through this thesis are given in a general form. Let us consider a large scale system, which is composed of m subsystems as shown in Fig. 1.1. Each subsystem S_i is described by the vector differential equation,

$$\dot{x}_i = f_i(\ x_i,\ t\ ) + u_i \tag{1-1}$$

Here x_i is a real n_i vector, i.e. $x_i \in R^{n_i}$ and $(\cdot)$ denotes differentiation with respect to time t . The first term in the right hand side of (1-1) satifies a following equation,

$$f_i(\ 0_{n_i}\ ,\ t\ ) \equiv 0_{n_i} \tag{1-2}$$

Furthermore, the function f_i satisfies a gloabal Lipshitz condition so that the solution $x_i(\ t\ ;\ t_0,\ x_{i0})$ of (1-1) exists and is unique and continuous for all initial conditions and t. The second term in the right hand side of (1-1) expresses an input to the i-th subsystem and consists of an interconnection function g_i and an outer-input vector u_{i0} such that

$$u_i = g_i(\ x_1,\ x_2,\ \cdots,x_m,\ t\ ) + u_{i0} \tag{1-3}$$

where the function g_i satisfies

$$g_i(\ 0_{n_1},\ 0_{n_2},\ \cdots,0_{n_m},\ t\ ) \equiv 0_{n_i} \tag{1-4}$$

When $u_i \equiv 0_{n_i}$, (1-1) becomes

$$\dot{x}_i = f_i(\ x_i,\ t\ ) \tag{1-5}$$

The above equation is considered to be the characteristics of the unforced subsystem separated from each other. So henceforth the system described by (1-5) will be called the i-th isolated subsystem.

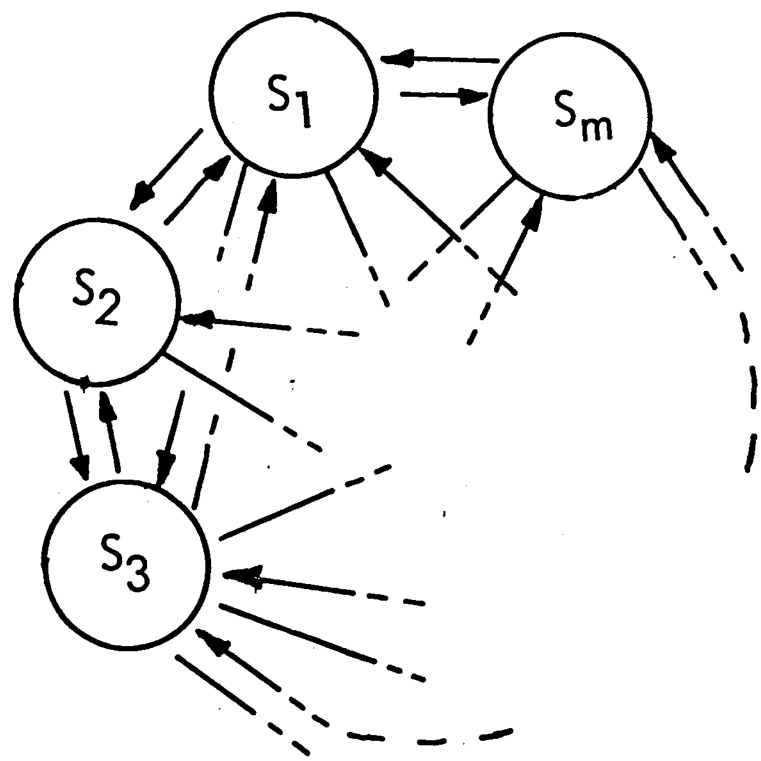


Fig. 1.1 Configuration of Large Scale System

Equation (1-1), combined with (1-3) describes the large scale system in question as a whole. They are rewritten as follows

$$\begin{cases} \dot{x}_i = f_i(\ x_i\ ,\ t) + u_i \\ u_i = g_i(\ x_1,\ x_2,\ \cdots,\ x_m,\ t\ ) + u_{io} \end{cases} \tag{1-6}$$

Hereafter we call the large scale system expressed by (1-6) system Σ . The state vector of Σ is the direct sum of that of each subsystem and written as

$$x = (\ x_1',\ x_2',\ \cdots\ ,\ x_m'\ )' \tag{1-7}$$

By (1-7), the connecting function $g_i(\ x_1, x_2, \cdots, x_m,\ t)$ in (1-3) may be replaced simply as $g_i(\ x\ ,\ t\ )$. When $u_{io} \equiv 0$, the sole equilibrium state of (1-6) is $x = 0_n$ by (1-2) and (1-4). We say that the system Σ is ASIL (Asymptotically Stable In the Large), if its trivial solution $x = 0_n$ is ASIL. In this thesis the condition for ASIL of the system Σ where $u_{io} \equiv 0_n$ is chiefly discussed.

Another frequently encountered defintion of stability is ESIL (Exponentially Stable In the Large). Exponential stablility of systems means that the absolute values of the system states decay exponentially as an increase of time. In the light of (26, Krasovskii), the system(1-5) is ESIL if and only if there are a positive definite function $v_i(\ x_i, t)$ and four positive cinstants c_{ij} (j=1,2,3,4) such that

$$\begin{aligned} &\text{(a)} \quad c_{i1}|x_i|^2 \le v_i(\ x_i, t) \le c_{i2}|x_i|^2 \\ &\text{(b)} \quad \dot{v}_i(\ x_i, t)\Big|_{(1-5)} \le -c_{i3}|x_i|^2 \\ &\text{(c)} \quad |\nabla v_i| \le c_{i4}|x_i| \end{aligned} \tag{1-8}$$

where $\dot{v}_i\Big|_{(1-5)} \triangleq \frac{\partial}{\partial t} v_i + (\nabla v_i)' \cdot f_i$ is the total time derivative of $\mathcal{V}_i(\ \mathcal{X}_i,\ t\ )$ along the solution of (1-5). It is assumed hereafter that those quantities are already found for the exponentially stable systems. Note that in (1-8) when $|\mathcal{X}_i|$ is replaced by any non-decreasing functions $\phi_i(|\mathcal{X}_i|)$ satisfying:

$$\phi_i(0) = 0\ , \qquad \lim_{r \to \infty} \phi_i(r) = \infty \quad ,$$

system (1-5) is concluded to be ASIL, not ESIL.

Chapter 2 Stability Criteria of Large Scale Systems with Time-varying Interconnecting Relations

Section 2.1 Introduction

A systematic method for analyzing large scale systems was first proposed by F.N.Bailey (7). He assumed that the system was decomposed into several subsystems and derived the conditions for stability of the system utilizing the properties of subsystems and relations of interconnection among them. This method has been called the "decomposition" method. Many reports about the stability conditions of large scale systems have been published after Bailey, using the "decomposition" method. However, most of them assumed that the conditions corresponding to "sector" cinditions, which were supposed frequently in the absolute stability problems of nonlinear feedback control systems, were satisfied for the interconnecting relations among subsystems.

In this chapter, stability of a large scale system with time-varying interconnecting relations is analyzed, but the "sector" conditions are not necessarily assumed for the system dealt with here. To obtain the stability conditions of the system, the weighted vector Lyapunov function method is introduced. The principal feature of the method is that the elements of a vector Lyapunov function are multiplied by appropriate scalar weighting functions. The results obtained is shown to coincide with Bailey's theorem when the interconnecting relations are time-invariant. It is also shown with an illustrative example that by making use of much more infromation on system properties, such as periodicity conditions of interconnecting relations, improved conditions of stability can be obtained.

Section 2.2 System Equations and Some Preliminary Lemmas

Let us study the system Σ described by the equations of the more specified form than (1-6) as follows

$$\begin{cases} \dot{x}_i = f_i(x_i, t) + u_i \\ u_i = \sum_{\substack{j=1 \\ \neq i}}^{m} C_{ij}(t) x_j + K_i u \end{cases} \tag{2-1}$$

where $C_{ij}(t)$ is a matrix-valued function of a real variable t, K_i a real constant matrix of an appropriate size, and u an outer-input vector to Σ. The equations (2-1) are considered to be a special form of (1-6) where $g_i(x , t) = \sum C_{ij}(t) x_j$ and to describe a large scale system with time--varying linear interconnecting elements. Putting $u \equiv 0$ in (2-1), we will consider the stability properties of Σ with zero outer-input of the form

$$\dot{x}_i = f_i(x_i, t) + \sum_{\substack{j=1 \\ \neq i}}^{m} C_{ij}(t) x_j \tag{2-2}$$

We assume that each subsystem described by (1-5) is ESIL. As mentioned in Chapter 1, it means the existence of the positive definite function $w_i(x_i, t)$ and four positive constants c_{ij}(j=1,2,3,4) satisfying

$$\begin{aligned} &\text{(a)} \quad c_{i1} |x_i|^2 \leq w_i(x_i, t) \leq c_{i2} |x_i|^2 \\ &\text{(b)} \quad \dot{w}_i(x_i, t)\Big|_{(1-5)} \leq -c_{i3} |x_i|^2 \\ &\text{(c)} \quad |\nabla w_i(x_i, t)| \leq c_{i4} |x_i| \end{aligned} \tag{2-3}$$

In order to derive the stability conditions of Σ by the "decomposition" method, some preliminary lemmas will be shown in the next place. The first lemma concerns the condition under which the solutions of first order differential ineaualities are bounded by those of differential equations having the same right hand side of the differentail inequalities.

The comparison theorems for more general types of equations and inequalities are listed in Appendix (B).

Lemma 2.1 [15]

The solution $\mathcal{X}$ of the differential equation of the form $\dot{\mathcal{X}} = (A + B(t))\mathcal{X}$, $\mathcal{X}(0) = \mathcal{X}_0$ and the solution of the differential inequality $\dot{\mathcal{Y}} \leq (A + B(t))\mathcal{Y}$, $\mathcal{Y}(0) = \mathcal{X}_0$ satisfies the relation $\mathcal{Y} \leq \mathcal{X}$, $\forall t \geq 0$, if the following inequalities hold.

$$a_{ij} + b_{ij}(t) \geq 0 \ (i \neq j) \ , \quad \forall t \geq 0$$

where $A = \{a_{ij}\}$ and $B = \{b_{ij}(t)\}$ are a real constant matrix and a matrix-valued continuous function, respectively.

The next two lemmas deal with the stability conditions of the first order linear differential equations.

Lemma 2.2

Let the solution of the differential equation $\dot{\mathcal{Y}} = A\mathcal{Y}$ be ASIL (i.e. in this case ESIL). If for the equation $\dot{\mathcal{Z}} = (A + B(t))\mathcal{Z}$ having the same initial conditions as above equation at time t = 0, the condition ;

$$\|B(t)\| \leq c \ , \quad \forall t \geq 0$$

where c is a positive constant relating to the property of A, is satisfied, then the solution of $\dot{\mathcal{Z}} = (A + B(t))\mathcal{Z}$ is also ASIL (ESIL).

Though the proof of this lemma was shown in (10), the outline will be given for subsequent discussions.

The proof of Lemma 2.2

The solution of the differential equation $\dot{\mathcal{Z}} = (A + B(t))\mathcal{Z}$ is

$$\mathcal{Z} = \mathcal{Y} + \int_0^t Y(t-t_1)B(t_1)\mathcal{Z}(t_1)\,dt_1$$

where Υ is the solution of matrix equation $\dot{\Upsilon} = A\Upsilon$, $\Upsilon(0) = I$ and $y = \Upsilon\, z(0)$. From the assumptions, there exist positive constants c_1 , c_2 and a satisfying $|y| \leq c_1 e^{-at}$, $\|\Upsilon(t)\| \leq c_2 e^{-at}$. Using these inequalities $|z|$ is evaluated as:

$$|z| \leq c_1 e^{-at} + c_2 \int_0^t e^{-a(t-t_1)} \|B(t_1)\| \, |z(t_1)| \, dt_1$$

Then it follows:

$$|z| \leq c_1 + cc_2 \int_0^t e^{at_1} |z(t_1)| \, dt_1$$

Bellman-Grownwall's inequality says that $u \leq c_1 \int_0^t uv\,dt_1$ implies $u \leq c_1 \exp(\int_0^t v\,dt_1)$, where u, v are non-negative scalar functions, c_1 a positive constant. According to this inequality, we get the relation

$$|z|\, e^{at} \leq c_1\, e^{cc_2 t}$$

Thus if we choose c as $cc_2 \leq a$, $|z|$ goes to 0 as t to ∞ .

Lemma 2.3[14]

Without changing the conclusion of Lemma 2.2, the condition with respect to $\|B(t)\|$ of the above lemma can be replaced by the following condition

$$B(t) \to 0 \quad \text{(Null Matrix)}, \quad t \to \infty$$

Section 2.3 The Weighted Vector Lyapunov Function

In this section, some differential inequalities used to obtain the stability conditions of Σ are derived, making use of the functions $\mathcal{W}_i(x_i,t)$ which satisfy (2-3).

Let scalar functions $\alpha_i(t)$ be such that

$$\alpha_i(t) \geq m_i > 0 \quad , \quad t \geq t_0 \tag{2-4}$$

where, m_i is a constant. We constitute a following m dimensional vector $\mathcal{V}$ using above defined $\alpha_i(t)$ and $\mathcal{W}_i(x_i,t)$ of (2-3).

$$\begin{cases} \mathcal{V} \triangleq (\mathcal{V}_1(x_1,t),\ \mathcal{V}_2(x_2,t),\ \cdots,\ \mathcal{V}_m(x_m,t))' \\ \mathcal{V}_i(x_i,t) \triangleq \alpha_i(t)\ \mathcal{W}_i(x_i,t) \end{cases} \tag{2-5}$$

Now, the differentiation of $\mathcal{V}_i$ with respect to t along the solution of Σ is

$$\begin{aligned} \dot{\mathcal{V}}_i(x_i,t)\big|_{(2\text{-}2)} &= \dot{\alpha}_i(t)\ \mathcal{W}_i(x_i,t) + \alpha_i(t)\ \dot{\mathcal{W}}_i(x_i,t)\big|_{(2\text{-}2)} \\ &= \dot{\alpha}_i(t)\ \mathcal{W}_i(x_i,t) + \alpha_i(t)\left(\frac{\partial \mathcal{W}_i}{\partial t} + (\nabla \mathcal{W}_i)' \cdot f_i\right) \\ &\quad + \alpha_i(t)\left\{(\nabla \mathcal{W}_i)' \cdot \left(\sum_{\substack{j=1 \\ \neq i}}^{m} c_{ij}(t) x_j\right)\right\} \end{aligned} \tag{2-6}$$

The second term in the right hand side of the above equations can be evaluated, by the assumptions, as

$$\alpha_i(t)\left(\partial \mathcal{W}_i/\partial t + (\nabla \mathcal{W}_i)' \cdot f_i\right) \leq -c_{i3}\ \alpha_i(t)\, |x_i|^2$$

and the third term as

$$\begin{aligned} \alpha_i(t)\left\{(\nabla \mathcal{W}_i)' \cdot \left(\sum_{\substack{j=1 \\ \neq i}}^{m} c_{ij}(t) x_j\right)\right\} &\leq \alpha_i(t)\, |\nabla \mathcal{W}_i|\, \Big|\sum_{\substack{j=1 \\ \neq i}}^{m} c_{ij}(t) x_j\Big| \\ &\leq c_{i4}\, \alpha_i(t)\left(\sum_{\substack{j=1 \\ \neq i}}^{m} |c_{ij}(t) x_j|\right) |x_i| \end{aligned}$$

Using again the equations (2-3), (2-5), scalar inequality $-az^2+bz \leq -az^2/2+b^2/2a$ $(a>0,\ b>0)$ and Schwartz's inequality, we obtain

$$\dot{v}_i(x_i,t)\Big|_{(2-2)} \leq \left(\frac{\dot{\alpha}_i(t)}{\alpha_i(t)} - \frac{c_{i3}}{2c_{i2}} \right) v_i + \frac{c_{i4}^2}{2c_{i3}} \left(\sum_{\substack{k=1\\ \neq i}}^{m} \|C_{ik}(t)\|^2 \right) \left(\sum_{\substack{j=1\\ \neq i}}^{m} \frac{\alpha_i(t)}{c_{j1}\alpha_j(t)} v_j \right) \qquad (2\text{-}7)$$

This inequality can be rewritten as in the following vector form

$$\dot{v} \leq (A + B(t))\, v \qquad (2\text{-}8)$$

where v is defined by (2-5) and both A and B(t) are m dimensional square matrices defined respectively as follows

$$A = \{a_{ij}\} \triangleq \mathrm{diag}\left(-\frac{c_{i3}}{2c_{i2}} \right)$$

$$B(t) = \{b_{ij}(t)\} \triangleq \begin{cases} \dfrac{\dot{\alpha}_i(t)}{\alpha_i(t)}, & \text{for } i=j \\ \dfrac{c_{i4}^2}{2c_{i3}c_{j1}} \left(\displaystyle\sum_{\substack{k=1\\ \neq i}}^{m} \|C_{ik}(t)\|^2 \right) \dfrac{\alpha_i(t)}{\alpha_j(t)}, & \text{for } i \neq j \end{cases} \qquad (2\text{-}9)$$

Now, in order to express (2-7) by another form , we assume the existence of real constants M_i and non-negative constants N_{ij} satisfying

$$\frac{\dot{\alpha}_i(t)}{\alpha_i(t)} = M_i + \delta_i(t), \quad \delta_i(t) \geq 0, \quad \lim_{t\to\infty} \delta_i(t) = 0 \qquad (2\text{-}10)$$

$$\begin{cases} \dfrac{c_{i4}^2}{2c_{i3}c_{j1}} \left(\displaystyle\sum_{\substack{k=1\\ \neq i}}^{m} \|C_{ik}(t)\|^2 \right) \dfrac{\alpha_i(t)}{\alpha_j(t)} = N_{ij} + \delta_{ij}(t), \quad i \neq j \\ \delta_{ij}(t) \geq 0, \quad t \geq t_0; \qquad \lim_{t\to\infty} \delta_{ij}(t) = 0 \end{cases} \qquad (2\text{-}11)$$

where $\delta_i(t)$ and $\delta_{ij}(t)$ are continuous functions of time t .
Under these assumptions (2-7) is written as follows

$$\dot{v}_i(x_i,t)\Big|_{(2-2)} \leq \left(M_i - \frac{c_{i3}}{2c_{i2}} \right) v_i + \sum_{\substack{j=1\\ \neq i}}^{m} N_{ij} v_j + \left(\frac{\dot{\alpha}_i(t)}{\alpha_i(t)} - M_i \right) v_i + \sum_{\substack{j=1\\ \neq i}}^{m} \left\{ \frac{c_{i4}^2}{2c_{i3}} \left(\sum_{\substack{k=1\\ \neq i}}^{m} \|C_{ik}(t)\|^2 \right) \frac{\alpha_i(t)}{c_{j1}\alpha_j(t)} - N_{ij} \right\} v_j \qquad (2\text{-}12)$$

The vector form of (2-12) are

$$\dot{v} \leq (A_0 + B_0(t)) v \qquad (2\text{-}13)$$

where A_0 , $B_0(t)$ are defined respectively as follows

$$A_0 \doteq \left\{ a^{\circ}_{ij} \right\} \triangleq \begin{cases} M_i - \dfrac{c_{i3}}{2c_{i2}} , & \text{for } i=j \\ N_{ij} , & \text{for } i \neq j \end{cases}$$

$$B_0(t) = \left\{ b^{\circ}_{ij}(t) \right\} \triangleq \begin{cases} \dfrac{\dot{\alpha}_i(t)}{\alpha_i(t)} - M_i , & \text{for } i=j \\ \dfrac{c_{i4}^2}{2c_{i3}c_{j1}} \Big(\sum\limits_{\substack{k=1 \\ \neq i}}^{m} \| C_{ik}(t) \|^2 \Big) \dfrac{\alpha_i(t)}{\alpha_j(t)} - N_{ij} , & \text{for } i \neq j \end{cases} \qquad (2\text{-}14)$$

Section 2.4 Stability Theorems

Two theorems are derived on the basis of the results developed in the preceding sections.

<u>Theorem 2.1</u>

If the following conditions (i) and (ii) are satisfied, the system Σ_1 is ASIL.

(i) There exist m continuously differentiable scalar functions $\alpha_i(t)$ satisfying

$$\alpha_i(t) \geq m_i > 0 , \qquad t \in [t_0 , \infty)$$

where m_i is a constant.

(ii) For the matrix given by

$$B(t) = \left\{ b_{ij}(t) \right\} \triangleq \begin{cases} \dfrac{\dot{\alpha}_i(t)}{\alpha_i(t)} , & \text{for } i=j \\ \dfrac{c_{i4}^2}{2c_{i3}c_{j1}} \Big(\sum\limits_{\substack{k=1 \\ \neq i}}^{m} \| C_{ik}(t) \|^2 \Big) \dfrac{\alpha_i(t)}{\alpha_j(t)} , & \text{for } i \neq j \end{cases}$$

the following relation is established.

$$\| B(t) \| < \min_i \frac{c_{i3}}{2c_{i2}} , \qquad t \in [t_0 , \infty)$$

Theorem 2.2

If in addition to the condition (i) of Theorem 2.1 the following conditions (ii), (iii) are satisfied, then the system is ASIL.

(ii) There exist real numbers M_i, non-negative numbers N_{ij}, and non-negative continuous functions $\delta_i(t)$ and $\delta_{ij}(t)$ satisfying the conditions of (2-10) and (2-11), respectively.

(iii) The matrix given by

$$A_0 = \{ a^{\circ}_{ij} \} \triangleq \begin{cases} M_i - \dfrac{c_{i3}}{2c_{i2}} , & \text{for } i=j \\ N_{ij} , & \text{for } i \neq j \end{cases}$$

is a stable matrix.

Proof of Theorem 2.1

Consider the differential equation $\dot{Z} = (A + B(t)) Z$ where matrices A and B(t) are defined by (2-9). By Lemma 2.2, if the relation such that

$$\| B(t) \| \leq c < a/c_2 \qquad (2\text{-}15)$$

is satisfied, then the solution of the differrential equation is ESIL. The constants a and c_2 in (2-15) should be chosen so as to satisfy $\| e^{At} \| < c_2 e^{-at}$. As in the present case matrix e^{At} is given by

$$e^{At} = \mathrm{diag} \left[\exp(- \frac{c_{i3}}{2c_{i2}} t) \right] ,$$

we choose a, c_2 such that

$$a = \min_i \frac{c_{i3}}{2c_{i2}} \quad , \quad c_2 = 1 \quad ,$$

which give the bounds for $\| B(t) \|$ in the condition (ii) of the theorem. Here, as the matrix norm $\|\cdot\|$, the induced one from the Euclidean vector norm is adopted. Moreover, by the way of construction, the offdiagonal elements of the matrix $B(t)$ are all non-negative, so we get by Lemma 2.1 an inequality of the form,

$$\mathcal{V}(t) \leq \mathcal{Z}(t) \quad , \quad t \geq t_0 \tag{2-16}$$

whence we have

$$|\mathcal{V}(t)| \leq |\mathcal{Z}(t)| \quad , \quad t \geq t_0 \tag{2-17}$$

Because $\mathcal{Z}(t)$ is ESIL, there exist positive constants ε and ν such that

$$|\mathcal{V}(t)| \leq \nu |\mathcal{V}(t_0)| \; e^{-2\varepsilon(t-t_0)} \tag{2-18}$$

From this inequality we have

$$\sum_{i=1}^{m} v_i \leq \sqrt{m} \, |\mathcal{V}(t_0)| \; e^{-2\varepsilon(t-t_0)}$$

Here, $v_i(x_i, t)$ is denoted simply as v_i.

Meanwhile, from (2-3), (2-4) and (2-5), we obtain

$$\frac{v_i}{c_{i2}\,\alpha_i(t)} \leq |x_i|^2 \leq \frac{v_i}{c_{i1}\, m_i}$$

Whence if we put $\underline{c_{i1}\, m_i} = \min_i c_{i1}\, m_i$ and $\overline{c_{i2}\,\alpha_i(t)} = \max_i c_{i2}\,\alpha_i(t)$, we have

$$\frac{1}{\overline{c_{i2}\,\alpha_i(t)}} \sum_{i=1}^{m} v_i \leq |x_i|^2 \leq \frac{1}{\underline{c_{i1}\, m_i}} \sum_{i=1}^{m} v_i \tag{2-19}$$

If we also put $v_i(x_{i0}, t_0) = v_i^{\circ}$, we get

$$|\mathcal{V}(t_0)| \leq \sum_{i=1}^{m} v_i \tag{2-20}$$

From (2-18), (2-19) and (2-20), $|x|$ is evaluated as:

$$|x| \leq \tau \, |x_0| \, e^{-\varepsilon(t-t_0)}$$

where $\tau = \dfrac{\sqrt{m} \cdot \nu \cdot \overline{c_{i2} \alpha_{i0}}}{\underline{c_{i1} \; m_i}}$, $\overset{o}{\alpha}_i = \alpha_i(t_0)$, $x_0 = x(t_0)$.

Thus ASIL of Σ is proved.

Q.E.D.

Proof of Theorem 2.2

Consider the differetial equation $\dot{z} = (A_0 + B_0(t))z$, where A_0 and B_0 are given by (2-14). By the condition (ii) it can readily be shown that the offdiagonal elements of matrix A_0 and $B_0(t)$ are all non-negative and $B_0(t) \to 0$(Null Matrix) as $t \to \infty$. Therefore the conditions of Lemma 2.1 are satisfied and we have $|v(t)| \leq |z(t)|$. Furthermore, if the condition (iii) is assumed, then the solution of $\dot{r} = A_0 r$ is ASIL(ESIL). Hence the conditions of Lemma 2.3 are satisfied and z , the solution of $\dot{z} = (A_0 + B_0(t))z$, tends to 0_m as t increases infinitely. The discussion to be followed is the same as that of Theorem 2.1 and is omitted.

Q.E.D.

Section 2.5 Consideration on Theorems

In section 2.3, m of the differential inequality with respect to $\mathcal{V}_i(\mathcal{X}_i,t)$, which was the weighted Lyapunov function for the i-th subsystem, were combined into two different forms of vector differential inequalities. In either form the coefficient matrix of the inequality was partitioned into two portions ; time-invariant part and time-varying one, so as to obtain the stability conditions. The reason why these partitioning were adopted is that the analysis was carried out from the inference that the system would be stable, if the relative "value" of time-varying interactions was small enough in contrast to the "degree" of stability for all subsystems. The obtained results validate this inference. The condition of Theorem 2.1 shows a limit to which relative "value" of interactions (i.e., norms of interconnecting functions) should be increased in comparison with the "degree" of stability for each subsystem to assure stability of the overall system. Also, in the case when the interconnecting relations among subsystems have constant limits such as "DC-bias", Theorem 2.2 shows the relation among the limits and the "degree" of stability of each subsystem to make overall system stable.

Now, the obtained theorems can be considered to be the extension of Bailey's theorem. Because, Bailey's theorem is applicable only to the system Σ , in which the interconnecting relations are time-invariant, i.e. $C_{ij}(t) \equiv C_{ij}(\text{const.})$. Let us examine our theorem, restricted to this case, in comparison with Bailey's. Bailey's theorem is outlined in Appendix (A). If we put $C_{ij}(t) \equiv C_{ij}(\text{const.})$ in Theorem 2.1 and

Theorem 2.2, all the time-varying components of the elements of the matrix B and B_o are those which involve only $\alpha_i(t)$. Therefore it is sufficient to choose $\alpha_i(t)$ as time-invariant function. If we choose $\alpha_1(t) = \alpha_2(t) = \cdots = \alpha_m(t) \equiv 1$, the diagonal elements of the matrix A of Theorem 2.1 become those of $\bar{A}$, where $\bar{A}$ is a matrix defined in Appendix (A). Namely, the matrices A and B(t) in Theorem 2.1 are no more than a decomposed form of $\bar{A}$ in this case. On the other hand, in Theorem 2.2 under the conditions $\alpha_1(t) = \alpha_2(t) = \cdots = \alpha_m(t) \equiv 1$ we have

$$M_i = \delta_i(t) = \delta_{ij}(t) = 0$$

$$N_{ij} = \frac{c_{i4}}{2c_{i3}c_{j1}} \cdot \sum_{\substack{k=1 \\ \neq i}}^{m} \|c_{ik}\|^2$$

Then the matrices A_0 and $B_0(t)$ become A and Q(null matrix), respectively. In this case Theorem 2.2 coincide with Bailey's theorem completely. Araki pointed out the conclusion of Bailey's theorem could be replaced by ESIL instead of ASIL. Therefore, we can also conclude ESIL of in this case.

Now, let us show an illustrative examples to compare the condition of Theorem 2.1 with that of Bailey's. Consider a large scale system composed of two exponentially stable subsystems, which are mutually connected by linear time-invariant elements as shown in Fig. 2.1. Application of Bailey's theorem for this system gives following stability condition

$$\|c_{12}\| \|c_{21}\| < \left[\frac{c_{11}c_{21}}{c_{12}c_{32}}\right]^{\frac{1}{2}} \left[\frac{c_{13}c_{23}}{c_{14}c_{24}}\right] \qquad (2\text{-}21)$$

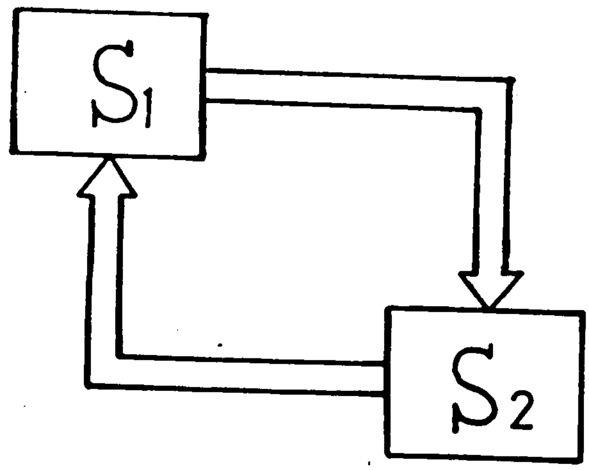


Fig. 2.1 An Example of Large Scale System (m = 2)

On the other hand, by choosing $\alpha_i(t)$ $(i=1,2)$ as $\alpha_1(t) = \alpha_2(t) = 1$, Theorem 2.1 gives us ;

$$\max \left(\frac{c_{14}^2}{c_{13}\, c_{21}} \|C_{12}\|^2 , \quad \frac{c_{24}}{c_{23} c_{11}} \|C_{21}\|^2 \right) < \min \left(\frac{c_{13}}{c_{12}} , \frac{c_{23}}{c_{22}} \right) \quad \cdots\cdots\cdots \quad (2\text{-}22)$$

Since condition (2-22) implies (2-21) and the converse is not true as readily understood, Bailey's theorem provides the better stability condition for systems with time-invariant interconnecting relations than Theorem 2.1 does. This is owing to the following fact : The condition of Bailey's theorem is a necessary and sufficient condition for ASIL of the solution of the "comparison" equation $\dot{r} = A r$, while the condition of Theorem 2.1 is a sufficient condition for ASIL of the equation $\dot{y} = (A+B(t))y$.

Section 2.6 Examples

In applying Theorem 2.1 or Theorem 2.2 to large scale systems, arbitrary scalar functions $\alpha_i(t)$ satisfying (2-4) should be chosen at first so as to satisfy the assumptions of each theorem. Namely, matrix norm of $B(t)$ in Theorem 2.1 should be able to be calculated or the existence of M_i, N_i, $\delta_i(t)$, $\delta_{ij}(t)$ satisfying the condition (ii) in Theorem 2.2 must be assured. Moreover, as the arbitrary function $\alpha_i(t)$ itself has several arbitrary parameters, we must take care of reduction of the number of these parameters. However, in many cases interconnecting matrices $C_{ij}(t)$ are given in a definite form and in some cases any constraints are placed to them. We can make use of such a knowledge of $C_{ij}(t)$ in choosing $\alpha_i(t)$. In the following, we will give a few exemplary applications.

Example 1

Consider the case in which the sum of the norm of the matrices which interconnect the i-th subsystem with the other is bounded as follows

$$\sum_{\substack{j=1\\ \neq i}}^{m} \|C_{ij}(t)\|^2 \leq K_i \; e^{-k_i t} \quad , \quad t \geq t_0 \qquad (2\text{-}23)$$

where K_i and k_i are scalar constant and $K_i > 0$. If the signs of k_i's are all negative, the stability condition cannot be obtained by the theorems in section 2.4. However, in case where the only one k_i is negative and the absolute value of the k_i is not larger than the other, we can get the stability condition by Theorem 2.1. Without loss of generality, we can rearrange the order of subsystems such that

$$\begin{cases} \sum_{\substack{j=1\\ \neq i}}^{m} \|C_{ij}(t)\|^2 \leq K_i \; e^{-k_i t} , \quad t \geq t_0 , \quad i=1,2,\cdots,m-1 \\ \sum_{k=1}^{m-1} \|C_{mk}(t)\|^2 \leq K_m \; e^{k_m t} , \quad t \geq t_0 \end{cases} \qquad (2\text{-}24)$$

where $K_i > 0$, $i=1,2,\cdots,m$ and $k_1 \geq k_2 \geq \cdots \geq k_{m-1} \geq k_m > 0$. Inequalities (2-24) shows that unbouded "gain" elements may exist between the m-th subsystem and the other. We choose arbitrary function $\alpha_i(t)$ in Theorem 2.1 as follows

$$\alpha_i(t) = e^{\gamma_i t} \quad , \quad \gamma_i \geq 0 \quad , \quad t \geq t_0 \qquad (2\text{-}25)$$

Here,

$$\begin{cases} \gamma_i = k_i \quad , \quad i = 1,2,\cdots,m-1 \\ \gamma_m = k_{m-1} - k_m \end{cases}$$

Then we can calculate the upper bound of the norm of the matrix B(t) in condition (ii) of Theorem 2.1. In case of m = 2 we obtain the following stability condition

$$\|B(t)\| \leq \left\| \begin{matrix} k_1 & \dfrac{c_{14}^2 K_1}{2c_{13}\, c_{21}} \\ \dfrac{c_{24}^2 K_2}{2c_{23}\, c_{11}} & k_1 - k_2 \end{matrix} \right\| < \min \left(\frac{c_{13}}{2c_{12}} , \frac{c_{23}}{2c_{22}} \right) \quad (2\text{-}26)$$

Note that we adopted so far the matrix norm which was induced from the vector norm, however none the less we may choose any norm which gives the best possible condition so far as it is compatible with the vector norm.

Example 2

Let us consider the system Σ in which the interconnecting relations are given

$$\sum_{\substack{k=1 \\ \neq i}}^{m} \|c_{ik}(t)\|^2 = K_i' \, e^{k_i t} (1 + K_i \, e^{-k_p t}) \qquad (2\text{-}27)$$

where k_i, K_i and K_i' are scalar constants such that $k_i < 0$, $K_i > 0$, $K_i' > 0$ $(i=1,2,\cdots,m)$; $k_p = \max_j k_j$ $(j \neq i)$.

If we choose $\alpha_i(t) \equiv e^{-k_i t}$, by Theorem 2.2 we have

$$M_i = - k_i$$

$$N_{ij} = \begin{cases} N_{ip} = \dfrac{c_{i4}^2 K_i K_i'}{2c_{i3} c_{j1}} \\ N_{ij} = 0 \quad (j \neq p) \end{cases}$$

The matrix A_0 defined by (2-14) is as follows

$$A_0 = \left\{a^{\circ}_{ij}\right\} = \begin{cases} -k_i - \dfrac{c_{i3}}{2c_{i2}} & , \ i=j \\ \dfrac{c^2_{i4}K_iK'_i}{2c_{i3}c_{j1}} & , \ i\neq j \ , \ j=p \\ 0 & , \ i\neq j \ , \ j\neq p. \end{cases}$$

The stability condition of $\sum$ is obtained as the condition that the above A_0 is a stable matrix. In case of m=2, we get the following conditions .

$$k_1 + k_2 > -\frac{1}{2}\left(\frac{c_{13}}{c_{12}} + \frac{c_{23}}{c_{22}}\right)$$
$$K_1K_2K'_1K'_2 < \frac{4c_{11}c_{13}c_{21}c_{23}(2c_{12}k_1+c_{13})(2c_{22}k_2+c_{23})}{c^2_{14}c^2_{24}} \tag{2-28}$$

Example 3

Assume m=2 and $\|C_{ij}(t)\|^2$ is given by

$$\begin{cases} \|C_{12}(t)\|^2 = \dfrac{K_1}{t^2 - ta + 2a^2} & , \ (a \geq \frac{1}{\sqrt{2}}) \\ \|C_{21}(t)\|^2 = K_2(t^2 + 1) \end{cases} \tag{2-29}$$

We choose $\alpha_i(t)$ as follows

$$\alpha_1(t) = t^2 + 2a^2 \quad , \quad \alpha_2(t) \equiv 1$$

Then, the matrix B(t) of Theorem 2.1 is calculated as

$$B(t) = \begin{bmatrix} \dfrac{2t}{t^2 + 2a^2} & \dfrac{c^2_{14}K_1}{2c_{13}c_{21}} \cdot \dfrac{t^2 + 2a^2}{t^2 - ta + 2a} \\ \dfrac{c^2_{24}K_2}{2c_{23}c_{11}} \cdot \dfrac{t^2 + 1}{t^2 + 2a^2} & 0 \end{bmatrix} \tag{2-30}$$

If we adopt the matrix norm of the form $\| A \| = m \max_{i,j} a_{ij}$, we obtain the stability condition by Theorem 2.1 as follows

$$\max\left(\frac{1}{\sqrt{2}\alpha} , \ \frac{\sqrt{3}+4}{7} \cdot \frac{c_{14}^2 K_1}{2c_{13}c_{21}} \cdot \frac{c_{24}^2 K_2}{2c_{23}c_{11}} \right) < \min\left(\frac{c_{13}}{4c_{12}} , \frac{c_{23}}{4c_{22}} \right) \quad (2\text{-}31)$$

As guessed by the above examples, the main weak point is that we have no systematic means to choose $\alpha_i(t)$ satisfying the conditions of the theorems and that the obtained stability conditions are no better than that of Bailey's. These defects should be overcome hereafter. On the other hand, the usefulness of our theorems are due mainly to its applicability to more general classes of large scale systems. By choosing $\alpha_i(t)$ appropriately, the theorems will be effective tools for analyzing the large scale systems with wider classes of interconnecting relations.

Section 2.7 Improvement of the Stability Condition

As discussed in the previous sections, the theorems derived in this chapter does not give wider ranges for stability in parameter space than that obtained by Bailey's. However, it can intuitively considered that if we are given more information on systems, the improved stability condition could be obtained by effective use of this information. In this section, it is shown that the improved stability condition of Σ_1 can be established under the assumption that the interconnecting matrices are periodic.

First, in the system Σ_1 , assume that there exists a positive number T, a period of interconnecting matrices, satisfying

$$C_{ij}(t+T) = C_{ij}(t) \quad , \quad t \geq t_0 \quad , \quad i \neq j \tag{2-32}$$

We choose an arbitrary scalar function $\alpha_i(t)$ in the same way as in the previous discussions and add another assumption that $\alpha_i(t)$ is also periodic with period T such that

$$\alpha_i(t+T) = \alpha_i(t) \quad , \quad t \geq t_0 \tag{2-33}$$

From this equation, we have

$$\dot{\alpha}_i(t+T) = \dot{\alpha}_i(t) \quad , \quad t \geq t_0 \tag{2-34}$$

Using these $\alpha_i(t)$'s we constitute a matrix B(t) in the same way as in Theorem 2.1. Then from (2-32) and (2-34), it is straightforward to show

$$B(t+T) = B(t) \quad , \quad t \geq t_0 \tag{2-35}$$

In parallel with the discussion in the proof of Theorem 2.1, it can be easily verified that if the solution of the equation

$$\begin{cases} \dot{V} = (A + B(t))V \\ A = \mathrm{diag}\left(-\frac{c_{13}}{2c_{12}} , -\frac{c_{23}}{2c_{22}} , \cdots , -\frac{c_{m3}}{2c_{m2}} \right) \end{cases} \tag{2-36}$$

is ASIL, then the system $\sum$ with periodic interconnections is also ASIL. Now, we will introduce a preparatory lemma to derive the stability condition of $\sum$.

Lemma 2.4 [23]

Let C be a real constant matrix, the real parts of whose eigenvalues are all negative, and D(t) be a periodic matrix with period T. If the quantity $\int_0^T \|D(t)-C\| dt$ is small enough, then the solution of the equation $\dot{r} = D(t)r$ is ASIL.

In the above lemma, "small enough" means that the following inequality is satisfied.

By the assumption, there exist positive constant K, β such that

$$\|\exp(Ct)\| \leq Ke^{-\beta t} \tag{2-37}$$

and for these values the relation

$$\int_0^T \|D(t)-C\| dt < \beta T/K \tag{2-38}$$

holds.

If we put D(t)=A+B(t) and C=A, these matrices are shown to satisfy the assumptions of the above lemma. In this case the constants K, β satisfying (2-37) is calculated respectively as K=1 and $\beta = \min_i (c_{i3}/2c_{i2})$. By (2-38) we can obtain a stability condition of the solution of the equation (2-36) as follows ;

$$\frac{1}{T}\int_0^T \|B(t)\| dt < \min_i \left(\frac{c_{i3}}{2c_{i2}} \right) \tag{2-39}$$

The above discussion leads us to the following theorem.

Theorem 2.3

If the inequality (2-39) is satisfied for the matrix B(t) of (2-9), then the system Σ with linear periodic interconnecting elements with period T is ASIL. Here, the function $\alpha_i(t)$ in the elements of B(t) should be chosen so as to be perodic of period T.

Apparently, the condition (2-39) is implied by the condition for $\|B(t)\|$ in Theorem 2.1. Though Theorem 2.1 can be, of course, applied to systems with periodic interconnections, it could be replaced by the better result, i.e. Theorem 2.3, if only periodic $\alpha_i(t)$ could be chosen. Intuitively speaking, the condition (2-39) shows that an average of the absolute "value" of coupling matrices over a period T should be less than the "degree" of stability for each subsystem for stability of the overall system.

Now, we will give an exemplary application.

Example

Let us consider the large scale system composed of two subsystems. We assume that the norm of the interconnecting matrices are given by

$$\|C_{12}(t)\| = (2+\sin t)\, K \qquad\qquad K > 0$$
$$\|C_{21}(t)\| = (2+\cos t)\, K$$

We assume further that $\mathcal{W}_i = \frac{1}{2}|x_i|^2 \, (i=1,2)$ is the function which satisfies the assumptions of (2-3) for each subsystem. Now, let us try to obtain the stability regions in (c_{13}, c_{23}) plane. Both c_{13} and c_{23} are considered to express the "degree" of stability for each subsystem. In this case, we have $c_{i1} = c_{i2} = 1/2$, $c_{i4} = 1 (i=1,2)$. If we choose $\alpha_1(t) = \alpha_2(t) \equiv 1$, B(t) can be calculated as follows

$$B(t) = \begin{bmatrix} 0 & \dfrac{(2+\sin t)^2}{c_{13}} K^2 \\ \dfrac{(2+\cos t)^2}{c_{23}} K^2 & 0 \end{bmatrix}$$

Whence,

$$\frac{1}{2\pi}\int_0^{2\pi}\|B(t)\|\,dt$$

$$\leq \frac{1}{2\pi}\max\left\{(2+\sin t)^2,\ (2+\cos t)^2\right\} dt \cdot \max\left(\frac{K^2}{c_{13}},\frac{K^2}{c_{23}}\right)$$

$$= \left(\frac{9}{2}+\frac{4\sqrt{2}}{\pi}\right)\cdot\max\left(\frac{K^2}{c_{13}},\frac{K^2}{c_{23}}\right)$$

From (2-39) we obtain the stability condition as

$$\min\left(c_{13}^2,\ c_{23}^2\right) > \left(\frac{9}{2}+\frac{4\sqrt{2}}{\pi}\right)K^2 \fallingdotseq 6.3\,K^2 \tag{2-40}$$

When Araki's theorem (cf. Appendix(A)) is applied to the system, the following condition is derived.

$$c_{13}c_{23} > 9\,K^2 \tag{2-41}$$

On the other hand, the application of Theorem 2.1 gives the following condition

$$\min\left(c_{13},\ c_{23}\right) > 3\,K \tag{2-42}$$

The regions for condition (2-40) to (2-42) are illustrated in Fig. 2.2. As clarified by this figure, the conditions (2-40) and (2-41) are not included by each other. That is, in some case Theorem 2.3 gives stability regions that has not been obtained by other theorems.

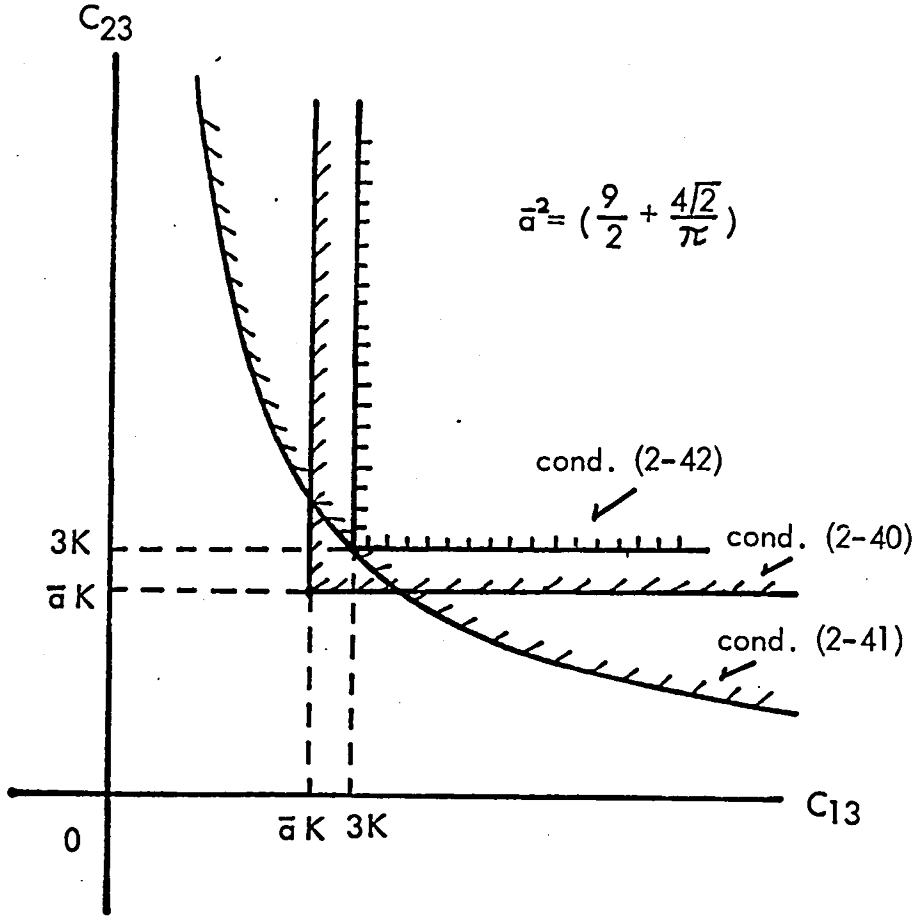


Fig. 2.2 Stability Regions for Condition (2-40), (2-41), (2-42)

Chapter 3 Frequency Domain Criteria of Large Scale Systems

Section 3.1 Introduction

In the preceding chapter, several stability theorems of large scale systems were derived by the weighted vector Lyapunov function method, which was a kind of the "comparison" methods. As mentioned previously, the principal idea of these "comparison" methods is that the stability conditions of the overall systems can be established by examining the stability properties of the solution of a comparison equation. The comparison equation is derived from the properties of subsystems and the interconnecting relations. Hence, various classes of large scale systems lead to various types of comparison equations. Therefore, to analyze a variety of large scale systems, we should provide the stability criteria that are applicable to wider class of systems characterized by the comparison equations. As are known well, frequency domain methods have been widely developed up to now to investigate a various kinds of systems. From this point, we will choose frequency domain stability criteria to examine the properties of the comparison equations. Hereafter, we will call the system whose characteristics are subject to the comparison equation an imaginary system.

In this chapter, various imaginary systems are analyzed by the extended Popov-type stability criteria given by Li Xun-Jing to obtain the stability conditions of corresponding large scale systems. The obtained conditions are also expressed as the form of multiple input-output frequency domain condition. Generally speaking, in contrast to the effectiveness of frequency domain method for single-input single-output systems, the method for multiple input-output systems is less useful because of troublesome work required for its application. This difficulty is considerablly reduced by utilizing a computer with a graphical display terminal. The obtained theorems are applied to some large scale systems, whose subsystems have dead time elements in their own feedback loop.

Section 3.2 Comparison Equations and Imaginary Systems

In this chapter, we deal with the system described by (1-6), though in which the function $g_i(x,t)$ is assumed to have still more specified forms or to be placed some restrictions. First, we assume the following relations for the vector-valued function $g_i(x,t)$

$$\nabla w_i' \cdot g_i(x,t) \leq \sum_{j=1}^{m} r_{ij} f_j(w_j) \quad , \quad r_{ij} \geq 0 \ (i \neq j) \qquad (3\text{-}1)$$

where w_i is a scalar function for each subsystem satisfying (2-3) and $f_j(w_j)$ is a non-decreasing scalar function satisfying $f_j(0)=0$. As a particular case of (3-1), we also consider the large scale system, the "magnitude" of whose interconnecting relations between subsystems is evaluated by linear combination of the norm of the state for each subsystem such as

$$\left|g_i(x,t)\right| \leq \sum_{\substack{j=1 \\ \neq i}}^{m} \gamma_{ij} \left|x_j\right| \quad , \qquad \gamma_{ij} \geq 0 \ (i \neq j) \qquad (3\text{-}2)$$

We call the system represented by (1-6) Σ , Σ_L , if it satisfies (3-1), (3-2), respectively for convenience sake in this chapter. Apparantly, Σ_L is a subclass of Σ and Bailey discussed the system Σ_L , where γ_{ij} in (3-2) was given as $\gamma_{ij} = \| C_{ij} \|$. In this case, (3-1) is satisfied by putting r_{ij} as

$$r_{ij} = \gamma_{ij} \frac{c_{i4}}{c_{j1}} \qquad (i \neq j) \tag{3-3}$$

Now, we differentiate the function w_i along the solution of Σ with respect to time t and evaluate the derivative using (2-3) and (3-1). Then, we have

$$\dot{w}_i \leq - \frac{c_{i3}}{c_{i2}} w_i + \sum_{j=1}^{m} r_{ij} f_j(w_j)$$

If we put $W \triangleq (w_1, \ w_2, \cdots, \ w_m)'$, the above inequalities can be written in the vector form of

$$\dot{W} \leq AW + RF(W)$$

with

$$A \triangleq \text{diag} \left(- \frac{c_{i3}}{c_{i2}} \right) \tag{3-4}$$

$$R \triangleq \{ r_{ij} \}$$

$$F(W) \triangleq \text{diag} \left(f_i(w_i) \right)$$

The comparison principles (cf. Appendix(B)) tell that the solutions of the above inequalities are bounded above by those of corresponding differential equations such as

$$\dot{V} = AV + RF(V) \tag{3-5}$$

Therefore, ASIL of Σ can be proved by checking whether the system described by (3-5) has the same properties or not. Hereafter, the system whose characteristics are given by (3-5) will be called the imaginary system and be written as I . In case of the system Σ_L , (3-5) is expressed as

$$\dot{V} = AV + BV \tag{3-6}$$

where

$$B \triangleq \text{offdiag} \left(\gamma_{ij} \frac{c_{i4}}{c_{j1}} \right)$$

Similarly, we call the imaginary system corresponding to Σ_L , I_L.

Next, we consider the case where each subsystem described by (1-5) has a dead time element in their own feedback loop. We assume that dead time of all the loops are given by a same value $\tau > 0$. These systems are described by the following equations

$$\begin{cases} \dot{x}_i = f_i(x_i, t) + u_i \\ u_i = g_i(x_i, x_i(t-\tau), t) + h_i(x, t) \end{cases} \tag{3-7}$$

where x_i and x denote $x_i(t)$ and $x(t)$, respectively for the sake of simplicity and $h_i(x, t)$ satisfies the relation such as

$$\nabla w_i' \cdot h_i(x, t) \leq \sum_{j=1}^{m} r_{ij} f_j(w_j) \tag{3-8}$$

where $r_{ij} \geq 0$ $(i \neq j)$ and $f_j(w_j)$ is a scalar non-decreasing function. Moreover, we assume that $g_i(x_i, x_i(t-\tau), t)$ satisfies the following relations

$$\begin{cases} g_i(0_{n_i}, 0_{n_i}, t) \equiv 0_{n_i}, \quad t \in (-\infty, \infty) \\ \nabla w_i' \cdot g_i(x_i, x_i(t-\tau), t) \equiv c_{i4} d_{i1} |x_i(t-\tau)| \end{cases} \tag{3-9}$$

where d_{i1} is a real constant. The system which satisfies (3-7) to (3-9) is named ΣD in this chapter. For the system ΣD , we have the following inequalities in a similar way as (3-4) (by using the conditions (2-3), (3-8) and (3-9)).

$$\dot{W}(t) \leq AW(t) + DW(t) + RF(W(t))$$

where

$$D \triangleq \text{diag}(c_{i4} d_{i2}) \tag{3-10}$$

$$d_{i2} \triangleq \begin{cases} d_{i1}/c_{i1}, & d_{i1} > 0 \\ d_{i1}/c_{i2}, & d_{i1} < 0 \end{cases}$$

We consider the following differetial equation corresponding to the above inequalities

$$\dot{V}(t) = AV(t) + DV(t-\tau) + RF(V(t)) \tag{3-11}$$

The system described by (3-11) is also an imaginary system and we call it system ID . According to the comparison principles (Appendix(B)), the inequalitiy between the solutions of (3-10) and those of (3-11) of the form

$$V(t) \geq W(t) \quad , \quad t \geq 0$$

holds, so far as the relations $V(t) \geq W(t)$ are satisfied during the interval $-\tau \leq t \leq 0$. Therefore, the conditions for ASIL of the solutions of (3-11) give the conditions of $\sum D$ for ASIL. Now, when the function g_i are given, especially, by

$$g_i(x_i, x_i(t-\tau), t) \equiv D_i x_i(t-\tau) \tag{3-12}$$

where D_i is a $n_i \times n_i$ real constant matrix, we have the matrix D of (3-10) in the following form

$$D = \text{diag}\,(c_{i4} \|D_i\| / c_{i1}) \tag{3-13}$$

If the norm of the function $h_i(x_i, t)$ given by (3-7) is evaluated by the same way as (3-2), that is,

$$|h_i(x, t)| \leq \sum_{\substack{j=1 \\ \neq i}}^{m} \gamma_{ij} |x_j| \quad , \quad \gamma_{ij} \geq 0 \tag{3-14}$$

then the equation corresponding to (3-10) has a linear form as

$$\dot{V}(t) = AV(t) + DV(t-\tau) + BV(t) \tag{3-15}$$

We call again the system $\sum D$ whose interconnecting relations are restricted as in (3-14) $\sum D_L$ and we call the corresponding imaginary system, ID_L . Of course, system $\sum D_L$ is a specified class of $\sum D$.

Now, let us derive the stability conditions of the imaginary system described by (3-6), (3-11) and (3-15) in order to obtain the stability conditions of the overall system. Apparently, the stability conditions for (3-6) can easily be obtained in simple algebraic form. However, to get the stability conditions for the other two equations, is not so easy on account of the existence of time deviating arguments.

Section 3.3 Extended Popov-Type Theorems (by Li Xun-Jing)

In this section, we introduce frequency domain criteria to examine the properties of the imaginary systems I, ID and ID_L corresponding to Σ, ΣD and ΣD_L, respectively. For multi-input multi-output systems, several frquency domain theorems of stability have been reported before now, (25), (32), (51), (62). Here, we employ the extended Popov-type theorems established by Li Xun-Jing (32). The reason why these theorems are especially adopted is that they can give the stability conditions for wider class of systems that have dead time elements. In applying the theorems to the imaginary systams, some notices will be required. Because, in contrast to the imaginary systems that have closed loop equations, the system equations in the theorems are written in two forms ; open loop equation and feed back loop equation. As will be necessitated later, we show the outline of Li's theorems below.

$$\begin{cases} \dot{x}(t) = A x(t) + D x(t-\tau) + R f(z(t)) \\ z(t) = -C x(t) \quad , \tau > 0 \end{cases} \qquad (3\text{-}16)$$

where A, D, R and C $\in \mathbb{R}^{m\times m}$; $f(z(t))$ is a m vector whose j-th component $f_j(z)$ depends only on the j-th component z_j of the vector z , that is, $f_j(z) = f_j(z_j)$ and $f(0) = 0$.
We assume that the function $f(z(t))$ also satisfies the following "sector" conditions,

$$0 \leq z_i f_i(z_i) \leq h_i z_i^2 \tag{3-17}$$

where h_i is a positive constant. The transfer matrix of system (3-16) is

$$G(s) = -C(sI - A - e^{-s\tau}D)^{-1}R \tag{3-18}$$

For this system, the following theorems hold.

Theorem 3.1[32]

If the real parts of the roots of the characteristic equation

$$|sI - A - e^{-s\tau}D| = 0 \tag{3-19}$$

are negative, and there exist two square diagonal matrices P and Q of order m such that

1° the elements of the matrix P are all non-negative ;

2° for any real number ω $(0 \leq \omega \leq \infty)$, the matrix

$$W(\omega) = PH^{-1} + \frac{1}{2}\left\{(P+j\omega Q)G(j\omega) + \left[(P+j\omega Q)G(j\omega)\right]^*\right\} \tag{3-20}$$

where $H = \text{diag}\,(h_i)$

is positive definite,

then for any continuous vector function $f(z)$ satisfying the conditions of (3-17), the null solution of the system (3-16) is ASIL. That is, the system is absolutely stable in the large.

The next theorem treats the same system as above except less ristriction on $f(z)$.

Theorem 3.2 [32]

If all the roots of the equation (3-19) have negative real parts and there exist two diagonal matrices P and Q of order m with non-negative elements such that, for any arbitrary real number ω $(0 \leq \omega \leq \infty)$, the matrix

$$W(\omega) = (P+j\omega Q)G(j\omega) + \left[(P+j\omega Q)G(j\omega)\right]^{*} \qquad (3\text{-}21)$$

is positive definite, then for any continuous vector function satisfying the condition

$$0 \leq z_i f_i(z_i) \qquad (3\text{-}22)$$

the null solution of system (3-16) is ASIL.

In addition, we remark that circle criterion for multi-input multi-output systems given in (51) is a special case of the above theorems and is obtained by putting D = 0(Null Matrix), P = I and Q = 0 in Theorem 3.1. However, in the circle criterion, the function f can be assumed to be non-stationary.

Section 3.4 Main Theorems

By the arguments so far presented, the theorems of the previous section give the stability conditions of Σ and ΣD, when they are applied to the imaginary systems I and ID, respectively. In this section, we will give some theorems for some classes of large scale systems. First of all, consider the imaginary system ID_L, which corresponds to the large scale system ΣD_L. In Theorem 3.1, if we put $f(z(t)) = -Hz(t)$, where H is a $m \times m$ real constant matrix, we have

$$\dot{x}(t) = A\,x(t) + D\,x(t-\tau) + RHC\,x(t)$$

Now, if we choose R, H and C as follows

$$\begin{cases} R = I \text{ (Unit Matrix)} \\ H = \text{diag}\,(\ c_{i4}\ ) \\ C = \text{offdiag}\,(\ \gamma_{ij}/\ c_{j1}\ ) \end{cases} \qquad (3\text{-}23)$$

in order to correspond the third term of the right hand side of the aobve equation with that of (3-15), the transfer matrix of ID_L can be given from (3-18) as

$$G(s) = \text{offdiag}\,\left(-\frac{\gamma_{ij}/c_{j1}}{s + c_{i3}/c_{i2} - c_{i4}d_{i2}e^{-s\tau}}\right) \qquad (3\text{-}24)$$

We obtain the following stability theorems for $\sum D_L$ by applying Théorem 3.1 to the imaginary system ID .

Theorem 3.3

If the following two conditions are satisfied, $\sum D_L$ is ASIL.

1° All the roots of the equations

$$s + c_{i3}/c_{i2} - c_{i4}d_{i2}e^{-s\tau} = 0 \quad (i=1,2,\cdots,m)$$

have negative real parts.

2° There exist a square diagonal matrix P with non-negative elements of order m and also a square diagonal matrix Q of the same order such that the matrix

$$W(\omega) = PH^{-1} + \frac{1}{2}\left\{(P+j\omega Q)G(j\omega) + \left[(P+j\omega Q)G(j\omega)\right]^*\right\} \qquad (3\text{-}25)$$

is positive definite for any real ω $(0 \leq \omega \leq \infty)$.

Here, H and G(s) are given by (3-23) and (3-24), respectively.

Since (3-6) can be obtained by putting $D \equiv 0$ (Null Matrix) in (3-15), the system $\sum_L$ can be considered to be a special class of ΣD_L. Therefore, we can also obtain the stability condition of Σ_L by applying Theorem 3.1 to the imaginary system I_L. In this case, the transfer matrix of I_L is given by

$$G(s) = \text{offdiag}\left(- \frac{\gamma_{ij}/c_{j1}}{s + c_{i3}/c_{i2}} \right) \tag{3-26}$$

Theorem 3.4

If there exist two square diagonal matrices P and Q defined in the condition 2° of Theorem 3.3, such that the matrix defined by (3-25) is positive definite for any real ω $(0 \leq \omega \leq \infty)$, then Σ_L is ASIL. Here, $G(j\omega)$ in (3-25) is given by (3-26).

Next, we will deal with the large scale system whose interconnecting relations are not able to be evaluated linearly, that is, Σ and ΣD. In this case, we apply Theorem 3.2 to the corresponding imaginary systems. The transfer matrix of I and ID are given, respectively as

$$G(s) = \left\{ - \frac{r_{ij}}{s + c_{i3}/c_{i2} - c_{i4}d_{i2}e^{-s\tau}} \right\} \tag{3-27}$$

$$G(s) = \left\{ - \frac{r_{ij}}{s + c_{i3}/c_{i2}} \right\} \tag{3-28}$$

Theorem 3.5

If the following conditions are satisfied, ΣD is ASIL.

1° The same condition as 1° of Theorem 3.3

2° There exist two square diagonal matrices P and Q with non-negative elements of order m such that the matrix

$$W(\omega) = (P+j\omega Q)G(j\omega) + \left[(P+j\omega Q)G(j\omega)\right]^* \qquad (3\text{-}29)$$

is positive definite for any ω ($0 \leq \omega \leq \infty$). Here, $G(j\omega)$ of (3-29) is given by (3-27).

The stability conditions for the system Σ_1 can be obtained by equating all the coefficients of $e^{-s\tau}$ to zero. Thus the above theorem reduces to the following theorem for the system Σ_1 .

Theorem 3.6

If the same condition as 2° of the above theorem holds, when $G(j\omega)$ is given by (3-28), then Σ_1 is ASIL.

Section 3.5 Some Examples

Comparison the theorems in section 3.3 with other theorems obtained up to now will be made in another chapter (chapter 5). The discussions of this section are confined to giving some examples and remarks on the theorems.

Though the theorems in this chapter can be applicable to the various classes of large scale systems, the application of those theorems to actual systems meets with the following computational difficulties.

1° Every theorem reqires to choose 2m arbitrary parameters that are the diagonal elements of matrices P and Q.

2° Checking positive definiteness of the matrix-valued function for any real value of ω usually needs laborious work or troublesome manipulations.

A way to overcome the above difficulties is to utilize a computer with a graphic display terminal. By a computer with a display device, we will immediately be able to check positive definiteness of the matrix, and to seek for parameters in a cut-and-try manner. Before giving several applications of the theorems with a computer, we show a example where the stability condition can be obtained in an analytic form by the theorems of this chapter.

Example 1

Consider the system $\sum D_L$ where m = 2. We assume that the inequalities

$$c_{i3}/c_{i2} - c_{i4}d_{i2} > 0 \qquad (i=1,2) \tag{3-30}$$

are satisfied. Then, the condition 1° of Theorem 3.3 is also met(cf. Appendix(C)). Now, we choose P=I and Q=0(Null Matrix), respectively. The matrix $W(\omega)$ defined by (3-25) takes the form of

$$W(\omega) = \begin{bmatrix} h_1^{-1} & \frac{1}{2}\left\{g_{12}(j\omega)+g_{21}(-j\omega)\right\} \\ \frac{1}{2}\left\{g_{12}(-j\omega)+g_{21}(j\omega)\right\} & h_2^{-1} \end{bmatrix} \tag{3-31}$$

Here, $g_{ij}(j\omega)$ $(i,j=1,2\ ;\ i\neq j)$ are the offdiagonal elements of the transfer matrix of ID_L and given as

$$g_{ij}(j\omega) = -\frac{\gamma_{ij}/c_{j1}}{j\ \ + c_{i3}/c_{i2} - c_{i4}d_{i2}e^{-\tau\omega j}} \quad (i,j=1,2\ ;\ i\neq j) \qquad (3\text{-}32)$$

Since $h_i^{-1} > 0$ $(i=1,2)$, the condition that the matrix of (3-31) is positive definite for any ω $(0 \leq \omega \leq \infty)$ is equivalent to the condition that the following inequality holds for any ω $(0 \leq \omega \leq \infty)$

$$(h_1h_2)^{-1} - \frac{1}{4}\left|g_{12}(j\omega) + g_{21}(-j\omega)\right|^2 > 0 \qquad (3\text{-}33)$$

Considering the inequality

$$\left|j\omega + a - be^{-\tau\omega j}\right| \geq \left|j\omega + a\right| - \left|be^{-\tau\omega j}\right| \geq a-b$$

where $a,\ b>0$; $a-b>0$ and

$$\left|g_{12}(j\omega) + g_{21}(-j\omega)\right| \leq \left|g_{12}(j\omega)\right| + \left|g_{21}(-j\omega)\right|$$

we can easily verify that $\left|g_{12}(j\omega) + g_{21}(-j\omega)\right|$ takes its maximum value at $\omega = 0$. Consequently, if the inequality

$$(h_1h_2)^{-1} - \frac{1}{4}(g_{12}(0) + g_{21}(0))^2 > 0 \qquad (3\text{-}34)$$

holds, then (3-33) is satisfied. The converse is trivial. Therefore, we can conclude that the matrix given by (3-31) is positive definite for any ω $(0 \leq \omega \leq \infty)$, if and only if the inequality (3-34) holds. By (3-23) and (3-32), (3-34) can be written as

$$(c_{14}c_{24})^{-1} - \frac{1}{4}\left(\frac{\gamma_{12}c_{12}}{c_{21}c_{13}-c_{21}c_{14}c_{12}d_{12}} + \frac{\gamma_{21}c_{21}}{c_{11}c_{23}-c_{11}c_{24}c_{21}d_{22}}\right)^2 > 0 \qquad \cdots\cdots\ (3\text{-}35)$$

As an application of the result obtained above, we take the system whose system equations are written as

$$\begin{cases} \dot{x}_s = -\rho_s x_s + \sigma + \sum_{j=1}^{4} k_{sj}\, x_j(t-\tau) & , \quad s=1,2,3,4 \\ \dot{\sigma} = \sum_{s=1}^{4} \beta_s x_s + r\,p_2\sigma - f(\sigma) \end{cases} \tag{3-36}$$

where $\rho_4 \geq \rho_3 \geq \rho_2 \geq \rho_1 > 0$, $r>0$, $\tau>0$, $f(0)=0$, $\sigma f(\sigma)>0$. The case where there exists no dead time element, i.e. $K=\{k_{ij}\}$ $=0$ (null Matrx) in the above equation, was studied by Piontkovskii and Rutkovskaya as the exemplary application of the vector Lyapunov function method (52).

We constitute subsystems of the system (3-36) as follows

subsystem S_1 ; $\dot{x}_s = -\rho_s x_s$, $s=1,2,3,4$

subsystem S_2 ; $\dot{\sigma} = r\,p_2\sigma - f(\sigma)$

For the subsystem S_1, S_2 defined above, we choose the Lyapunov function that satisfies (2-3) as

$$c^*(\sum x_s^2)^{\frac{1}{2}} \quad \text{and} \quad c^{**}(\sigma^2)^{\frac{1}{2}}$$

respectively. Here, c^* and c^{**} are arbitrary positive constants. Then, we can calculate the values of the constant $c_{ij}(i=1,2\ ;\ j=1,2,3,4)$, $\gamma_{ij}(i,j=1,2\ ;\ i\neq j)$ and $d_{i2}(i=1,2)$. By the condition of (3-35), the stability condition of the system (3-36) is obtained such as

$$\begin{cases} \rho_1 - \|K\| > 0 \\ (\rho_1 - \|K\|)r\,p_2 + \max_s 2|\beta_s| < 0 \end{cases} \tag{3-37}$$

Particularly, when no dead time elements exist, i.e. $K=0$, (3-37) reduces to the form of

$$\rho_1 r\,p_2 + \max_s 2|\beta_s| < 0 \tag{3-38}$$

Thus, in a special case we could attain the stability condition in an analytic form wothout numerieal experiments. Moreover, it can be proved that under a certain condition some theorems in section 3.3 will be transformed to those in an algebraic form. However, the detailed discussion about them will be left for in chapter 5.

The following two examples show how the checking positive definiteness of the matrix is aided by a computer with a graphic display terminal.

Example 2

Let us consider the system described by

$$\begin{cases} \dot{x}_1 = A x_1 + b\sigma^3 + B x_1(t-\tau) + h(x_1) \\ \dot{\sigma} = -\rho\sigma - r\sigma^3 + q\left(\sum_{s=1}^{n} x_{1s}^2\right)^{\frac{3}{2}} + p\,\sigma(t-\tau) \end{cases} \tag{3-39}$$

where

$$x_1 \triangleq (x_{11}, x_{12}, \cdots, x_{1n})' \quad ;$$

$$h(x_1) \triangleq -h_1\left(\sum_{s=1}^{n} x_{1s}^2\right)(x_{11}, x_{12}, \cdots, x_{1n})'$$

$$= -h_1 x_1 |x_1|^2 \quad ;$$

h_1, ρ, r, p, q and τ are all positive constants ; $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

In this case we take two subsystems of the form

subsystem S_1 ; $\dot{x}_1 = A x_1$

subsystem S_2 ; $\dot{\sigma} = -\rho\sigma$

If we take the function of (2-3) for subsystem S_1 as $W_1 = (x_1' \cdot x_1)^{\frac{1}{2}}$ then we have $c_{11} = c_{12} = c_{14} = 1$. Similarly, for S_2 $W_2 = (\sigma^2)^{\frac{1}{2}}$ makes $c_{21} = c_{22} = c_{24} = 1$ and $c_{23} = \rho$. The constants and functions in Theorem 3.5 will be written in this case as

$$W_{e1}(\omega) = (P+j\omega Q)G_{e1}(j\omega) + \left[(P+j\omega Q)G_{e1}(j\omega)\right]^* \tag{3-40}$$

where $G_{e1}(j\omega)$ is given as

$$G_{e1}(j\omega) = \begin{bmatrix} \dfrac{h_1}{j\omega + c_{13} - \|B\| e^{-\tau\omega j}} & \dfrac{-|b|}{j\omega + c_{13} - \|B\| e^{-\tau\omega j}} \\ \\ \dfrac{-q}{j\omega + \rho - p e^{-\tau\omega j}} & \dfrac{r}{j\omega + \rho - p e^{-\tau\omega j}} \end{bmatrix} \qquad (3\text{-}41)$$

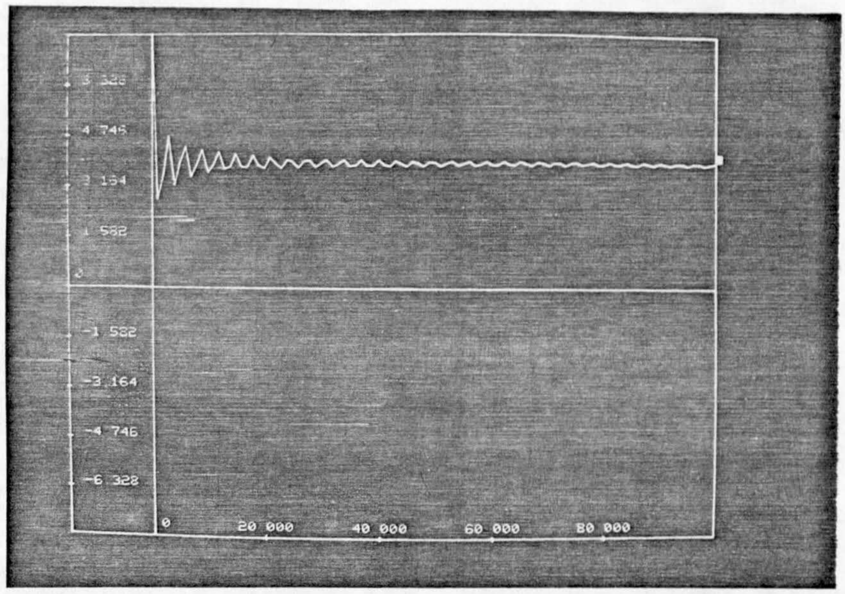
Theorem 3.5 reads that if the matrix given by (3-40) is positive definite for any ω $(0 \leq \omega \leq \infty)$, the system of (3-39) is ASIL. Here, we choose the matrices $P = \mathrm{diag}(p_i)$ and $Q = \mathrm{diag}(q_i)$ in (3-40) as

$$p_1 = c_{13}, \qquad p_2 = \rho, \qquad q_1 = q_2 = 1$$

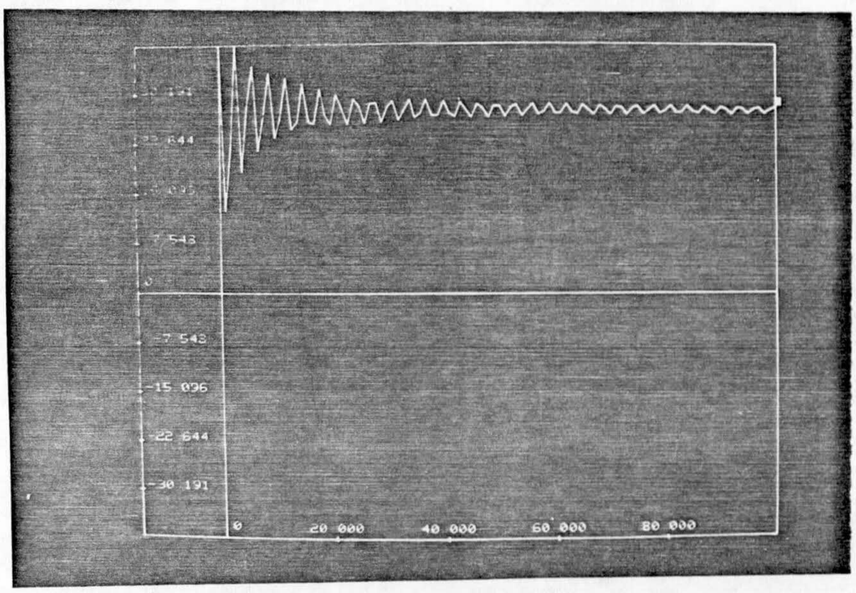
If the actual values of the constants of the system are given, we can check positive definiteness of the matrix by Sylvester's method using a computer with a display. Fig. 3.1 shows the output of the system (3-39), displayed on the graphic terminal, in which $h_1=1$, $|b|=0.5$, $q=3$, $r=10$, $c_{13}=2$, $\|B\|=1$, $\tau=2$, $\rho=1.15$, $p=1$. The abscissa of the figure denotes the frequency and the ordinate, the values of (1,1) element of $W_{e1}(\omega)$ and $\det(W_{e1}(\omega))$. Note that though checking positive definiteness is required for any real values of ω by Theorem 3.5, it is enough to inspect it in some frequency ranges decided by the relations among system parameters. For, the absolute value of each element of the matrix (3-40) usually converges to a finite constant value in the high frequency region. According to the result in Appendix(C), the condition 1° of Theorem 3.5 is satisfied, if the relation such that

$$c_{13} > \|B\| \quad \text{and} \quad \rho > p$$

hold.



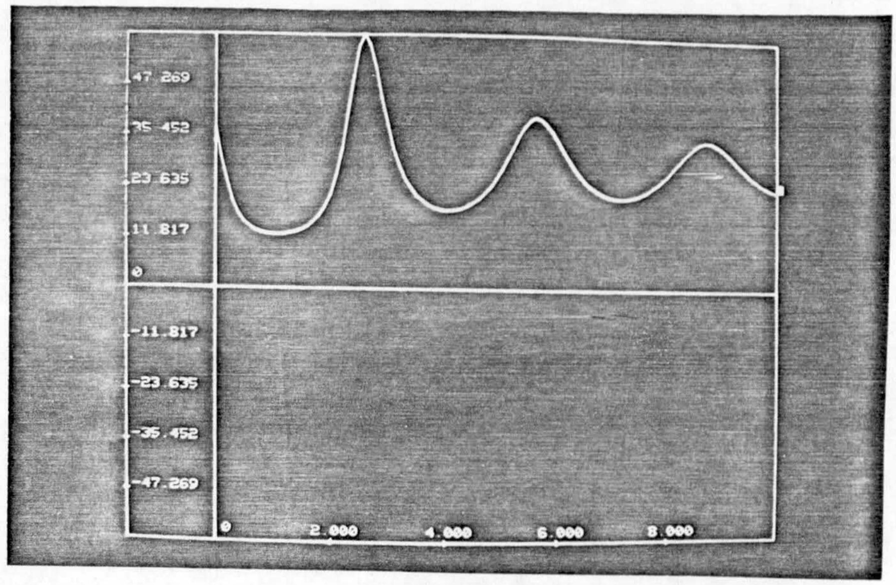
(a) (1,1)-element of the Matrix $W_{e1}(j\omega)$, the frequency runs from ω=0 to ω=100



(b) det ($W_{e1}(j\omega)$) , from ω=0 to ω=100

continued to next page.

Fig. 3.1 Graphical Display Output to Check Positive Definiteness of the Matrix (3-40)



(c) ditto , from ω=0 to ω=10

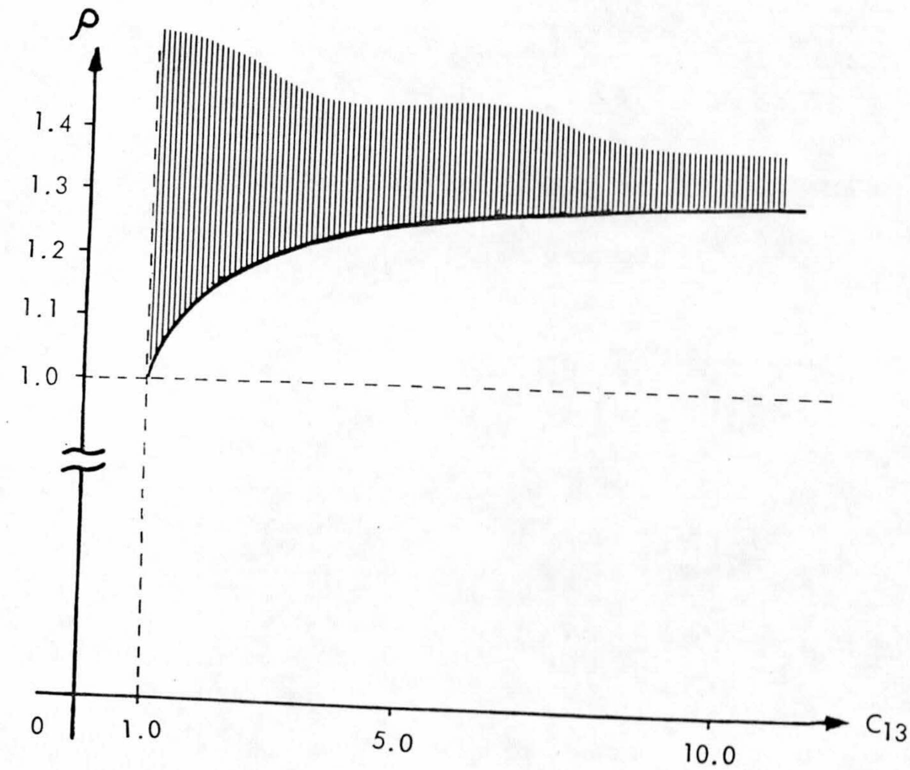


Fig. 3.2 Stability Regions for the System of Example 2 , in Which $h_1 = 1, \|B\| = 1, \tau = 2, |b| = 0.3, q = 3, p = 1, r = 10$

Of course, these relations are also satisfied in this example. Therefore, from these considerations, the system can be proved to be ASIL.

The stability regions obtained by this method in (c_{13} , ρ) plane of the system where the parameters, except c_{13} and ρ , are fixed, are depicted in Fig. 3.2. In this case, the constants are set as $h_1=1$, $|b|=0.3$, $q=3$, $r=10$, $\|B\|=1$, $\tau=2$ and $p=1$.

Now, the theorems in this chapter generally assume that all subsystems are ESIL. When some subsystems are exponentially unstable and others ESIL, the same arguments so far developed can be valid. We can replace $-c_{13}$ given in the assumption (2-3) by c_{13} throughout the discussion from the assumptions to the conclusion. Thus, we can obtain the theorems of large scale systems including the unstable subsystems. An application of them will be shown below.

Example 3

Consider the system whose equations are written as

$$\begin{cases} \dot{x}_1 = A_1 x_1 + b_1 f_1(\sigma_1) + g_1(x_1, x_1(t-\tau), t) , & \sigma_1 = c_1' x_2 \\ \dot{x}_2 = A_2 x_2 + b_2 f_2(\sigma_2) + g_2(x_2, x_2(t-\tau), t) , & \sigma_2 = c_2' x_1 \end{cases} \tag{3-42}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$; $b_1, c_2 \in \mathbb{R}^{n_1}$; $b_2, c_1 \in \mathbb{R}^{n_2}$; $f_i(\sigma_i)$ (i=1,2) are scalar functions satisfying

$$f_i(0) = 0 , \quad 0 \le \sigma_i f_i(\sigma_i) \le K_i \sigma_i , \quad K_i > 0 \tag{3-43}$$

The function $g_i(x_i, x_i(t-\tau), t)$ in (3-42) is given by

$$\mathcal{g}_i(\mathcal{X}_i,\ \mathcal{X}_i(t-\tau),\ t)=-(r_i^2+|\mathcal{X}_i|^2)\,|\mathcal{X}_i(t-\tau)|^2(\chi_{i1},\ \chi_{i2},\cdots,\ \chi_{in_i})'$$

$$,\ i=1,2 \qquad \cdots\cdots\cdots \quad (3\text{-}44)$$

where r_1 and r_2 are positive conststants. We take the equation of each subsystem as

subsystem S_1 ; $\dot{\mathcal{X}}_1 = A_1\mathcal{X}_1$

subsystem S_2 ; $\dot{\mathcal{X}}_2 = A_2\mathcal{X}_2$

We assume that instead of the second inequality of (2-3), the inequality

$$\dot{\mathcal{W}}_i\Big|_{S_i} \le c_{i5}\,|\mathcal{X}_i|^2 \quad , \quad i=1,2$$

holds for each subsystem and assume further that the function $\mathcal{W}_i$ in (2-3) is given by the form of $\mathcal{W}_i=\mathcal{X}_i'\cdot\mathcal{X}_i$ (i=1,2). Then, we have $c_{i1}=c_{i2}=c_{i4}=1$ (i=1,2) and $(\nabla\mathcal{W}_i)'\cdot\mathcal{g}_i \le -r_i\mathcal{W}_i(t-\tau)$ (i=1, 2). By choosing the matrices P and Q of Theorem 3.3 as P=I and Q=0 and applying the theorem to the system described by (3-42) to (3-44), we can conclude the system is ASIL, if the matrix of the form

$$2W_{e2}(\omega) = \mathrm{diag}(2) + \mathrm{offdiag}\left(-\frac{|b_i||c_i|K_i}{j\omega-c_{i5}+r_i e^{-\tau\omega j}} - \frac{|b_j||c_j|K_j}{-j\omega-c_{j5}+r_j e^{\tau\omega j}}\right)$$

is positive definite for all ω $(0\le\omega\le\infty)$ and if the condition 1° of Theorem 3.3 is satisfied for the equations

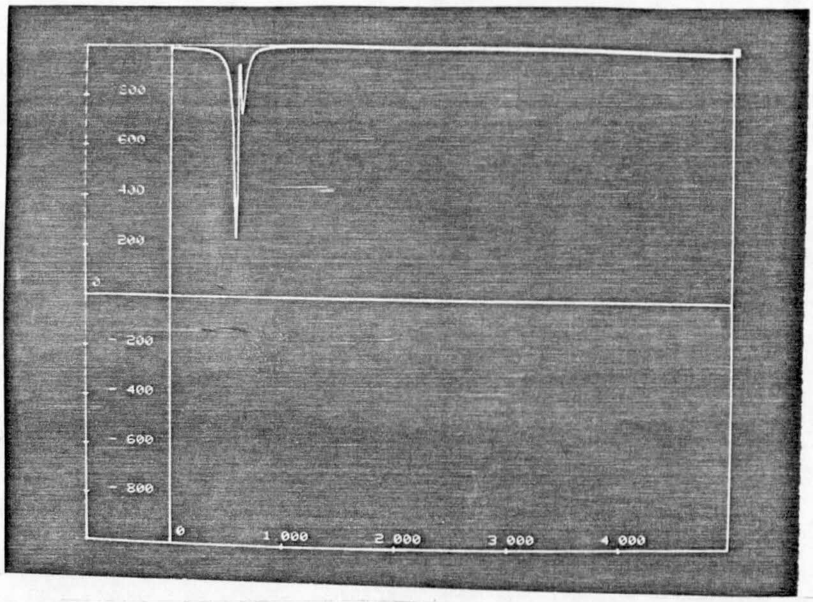
$$s - c_{i5} + r_i e^{-s\tau} = 0 \quad (i=1,2)$$

Figs. 3.3 (a) to (c) present the displayed computational result of $\det(W_{e2})$ vs. frequency ω for various ranges of ω, when the parameters of the system are given as

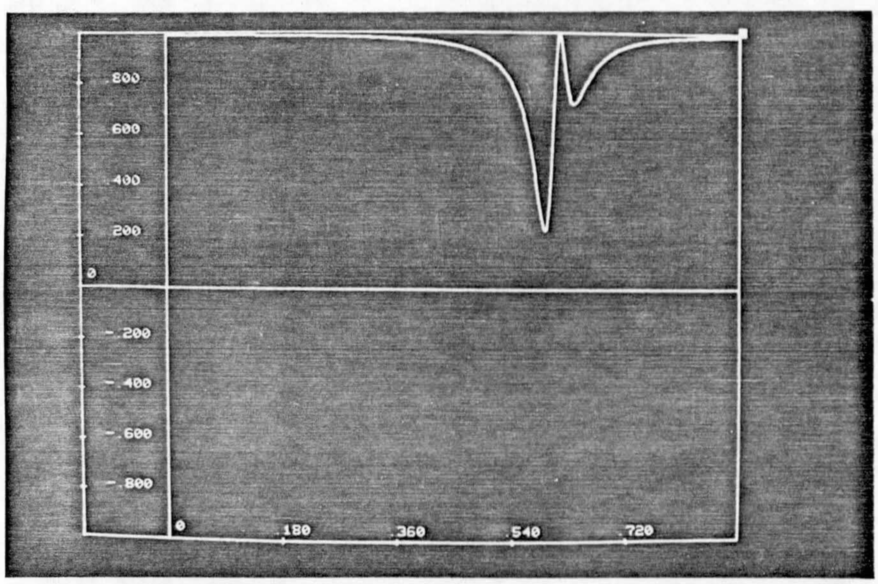
$b_1=1$, $|c_1|=1$, $K_1=0.02$, $c_{15}=0.08$, $r_1=1$, $\tau=1.047$

$b_2=1$, $|c_2|=1$, $K_2=0.02$, $c_{25}=0.8$, $r_2=1$.

For these values of the parameters, it can be shown from Appendix



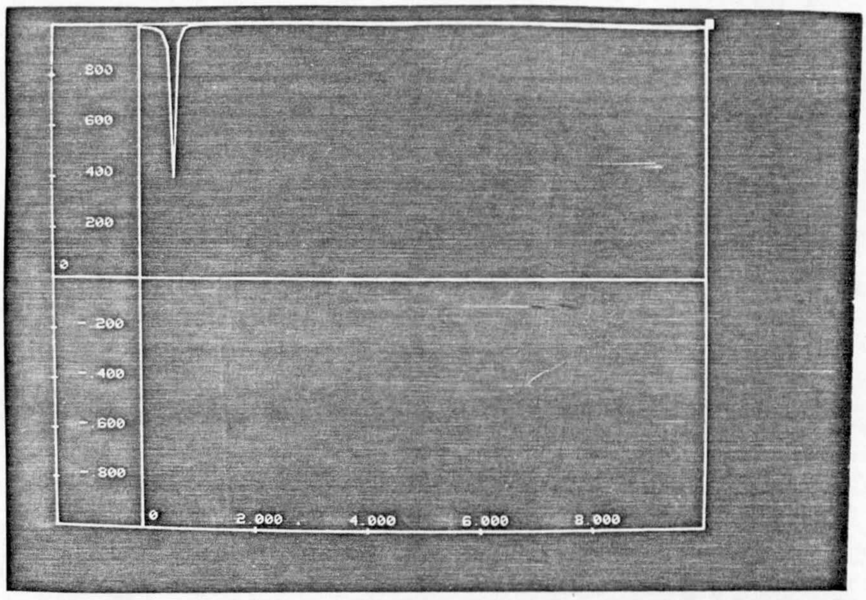
(a) The Frequency Range is $\omega = 0 \sim \omega = 5$



(b) $\omega = 0 \sim \omega = 0.9$

continued to next page.

Fig. 3.3 Display Output of det ($W_{e2}(j\omega)$) for Some Ranges of ω



(c) $\omega = 0 \sim \omega = 10$

(C) that the condition 1° of Theorem 3.3 is satisfied. Therefore, the system has proved to be ASIL.

Thus, if computers with graphic display devices are available, the theorems of this chapter will provide a valuable aid for examining the stability properties of various kinds of large scale systems.

Chapter 4 Conditions for Positive Definiteness of Rational Matrices

Section 4.1 Introduction

In this chapter we deal with conditions for positive definiteness of rational matrices in order to compare the frequency domain stability theorems of the preceding chapter with other theorems obtained previously and to reduce the frequency domain conditions more tractable forms. The problem discussed in this chapter is formulated more precisely as follows : Under what conditions a square matrix $G(s) = \left\{ g_{ij}(s) \right\}$, where $g_{ij}(s)$ is a real rational function, is positive definite for any pure imaginary value of s. In general, the condition for positive definiteness of a matrix can be checked by Strum's test, if the actual values of the elements are given. However, this test requires troublesome manipulations as the increase of the dimension of the matrix. Besides, to get the condition for positive definiteness in an analytic form is generally not so easy by any means. So, we shall be able to have an answer to this problem only in a restricted circumstances.

In this chapter, we discuss the conditions for positive definiteness or positive semidefiniteness of the matrix as the following procedures, which will be shown in section 4.3.

First, a sufficient condition for positive definiteness of the matrix is given in a matrix from. Next, under certain constraints to diagonal elements and pairs of offdiagonal elements of the matrix, a necessary condition for positive definiteness is given. Finally, it is shown that the above two conditions are equivalent to each other and give a necessary and sufficient condition for positive definiteness in an algebraic form by imposing some other constraints to the elements of the matrix. Whether the elements of the matrix satisfy these constraints or not can be examined by a simple calculation or by drawing the vector loci of them. Discussions for positive semidefiniteness of the matrix are given in parallel with those for positve definiteness. In the next section, some preliminary lemmas concerning the properties of some classes of matrices are presented. In the last section of this chapter, several examples are given to show the usefulness of the results of this chapter.

Section 4.2 Some Preliminary Propositions

Let us consider a square matrix $A(s) = \{a_{ij}(s)\}$ of order m whose elements are real rational functions of complex argument s. In this chapter, we write the matrix A(s) as follows:

$$A(s) = \mathrm{diag}(\, a_{ii}(s)\,) + \mathrm{offdiag}(\, a_{ij}(s)\,)$$

Now, let us show the definitions of the class of positive definite matrices and positive semidefinte matrices.

<u>Definition 4.1</u>

A matrix A ($\in \mathbb{C}^{m\times m}$) is positive definite, if the inequality $\overset{*}{Z}(A+\overset{*}{A})Z > 0$ holds for any non-zero vector Z ($\in \mathbb{C}^m$). The class of positive definte matrices will be written simply as p.d.

<u>Definition 4.2</u>

A matrix A ($\in \mathbb{C}^{m\times m}$) is positive semidefinte, if for any vector Z ($\in \mathbb{C}^m$), the inequality $\overset{*}{Z}(A+\overset{*}{A})Z \geq 0$ holds. The class of positive semidefinte matrices will be written simply as p.s.d.

Similar definitions are given for matrices in real number field.

Definition 4.3

A matrix A ($\in \mathbb{C}^{m\times m}$) is in the class Q ($Q_0$), if for any non-zero vector x , there exist a subscript k such that

$$|x_k| \neq 0 \quad , \qquad \mathrm{Re}\, \bar{x}_k (Ax)_k > 0 \ (\geq 0)$$

where x_k and $(Ax)_k$ is the k-th element of x and Ax , respectively.

The properties of the matrices in Q_0 have already examined in (38). It is supposed that the class Q should also have analogous properties with Q_0 . And this is indeed the truth and is easily verified. We have the following proposition.

Proposition 4.1

The following properties of a matrix A are equivalent :

1° For any non-zero vector x ($\in \mathbb{C}^m$), there exist a natural number k such that

$$|x_k| \neq 0 \quad , \qquad \mathrm{Re}\, \bar{x}_k (Ax)_k > 0 \qquad ;$$

2° For any non-zero vector x ($\in \mathbb{C}^m$), there exists a diagonal matrix D_x with non-negative elements satisfying

$$x^* D_x x > 0 \qquad , \qquad \mathrm{Re}\left[x^* D_x A x\right] > 0 \qquad ;$$

3° $\det(A + D) = 0 \quad , \quad \forall D \in \mathcal{D}_0$;

4° $A + D \in Q \quad , \quad \forall D \in \mathcal{D}_0$;

5° $\mathrm{Re}\left[\lambda_i (A+D)(N)\right] > 0 \quad , \quad \forall D \in \mathcal{D}_0 \quad , \quad \forall N \subseteq M$

Here, $(A+D)(N)$ is a principal submatrix of $(A+D)$ corresponding to the index set N and $\lambda_i (A+D)(N)$ is characteristic root of $(A+D)(N)$;

6 $\mathrm{Re}[\lambda_i(A+D)] > 0$, $\forall D \in \mathcal{D}_0$.

The properties of the matrices in Q_0 shown in (38) are the same as above, but where the second inequality of the property 2° is replaced by $\mathrm{Re}[x^* D_x A x] > 0$ and $\mathcal{D}_0$ by $\mathcal{D}$ below the property 2°.

<u>Definition 4.4</u>

A matrix A ($\in \mathbb{C}^{m\times m}$) belongs to the class $\mathcal{L}$, if there exist m constant positive numbers d_i ($i \in M$) satisfying

$$d_i \,\mathrm{Re}\, a_{ii} > \sum_{j\neq i}^{m} d_j\,|a_{ij}| \quad , \quad i \in M$$

The class $\mathcal{L}_0$ is defined analogously so as to include the case where equalities hold in the above relations.

The following proposition indicates the relations between $\mathcal{L}$ and Q and between $\mathcal{L}_0$ and Q .

<u>Proposition 4.2</u>

$$A \in \mathcal{L} \Rightarrow A \in Q \quad , \quad A \in \mathcal{L}_0 \Rightarrow A \in Q_0$$

The latter half of the above proposition was proved in (38). The first half can be shown analogously, or can be shown easily by Gershgorin's theorem (29).

<u>Definition 4.5</u>

If a matrix A ($\in \mathbb{R}^{m\times m}$) satisfying

$$a_{ij} \leq 0 \ , \ i \neq j \quad , \quad \forall i,j \in M$$

have positive (non-negative) principal minors of any order, it is said that A belongs to the class $\mathbb{K}$ ($\mathbb{K}_0$).

The matrices defined by Definition 4.5 are known as M-matrices (semi M-matrices). We will show some of their properties for subsequent arguments.

Proposition 4.3

Let A ($\in \mathbb{R}^{m \times m}$) be the matrix such that

$$a_{ij} \leq 0 \quad , \quad i \neq j \quad , \quad \forall i,j \in M$$

Then the following conditions are equivalent :

1° All principal minors of A are positive ;

2° The real part of each characteristic root of A is positive ;

3° There exists a vector $x > 0$ such that $Ax > 0$;

4° Each real characteristic root of A is positive ;

Proposition 4.4

Let A ($\in \mathbb{R}^{m \times m}$) be the matrix such that

$$a_{ij} \leq 0 \quad , \quad i \neq j \quad , \forall i,j \in M$$

Then the following conditions are equivalent :

1° Each principal minor of A is non-negative ;

2° Each real characteristic root of A as well as of each principal minor of A is non-negative ;

3° $A + \varepsilon I \in \mathbb{K}$, $\forall \varepsilon > 0$ where I is a unit matrix.

Proposition 4.5

Assume that the elements of the matrix A satisfy

$$a_{ij} \leq 0 \quad , \quad i \neq j \quad , \quad \forall i,j \in M \quad .$$

If there exists a vector $x > 0$ such that $Ax > 0$, then A is in $\bar{\mathbb{K}}_o$. Conversely, let A be in $\mathbb{K}_o$. If A is irreducible[†], then there exists a vector $x > 0$ such that $Ax > 0$.

Proposition 4.6

Let A and B ($\in \mathbb{R}^{m\times m}$) be matrices satisfying the following properties : $b_{ij} \le 0$, $i \neq j$, $\forall i,j \in M$; $A \in \mathbb{K}$ ($\mathbb{K}_0$) ; $A \le B$, then B is in $\mathbb{K}$ ($\mathbb{K}_0$).

If a matrix A is symmetric with respect to its principal diagonal, the following property of it is trivial.

Proposition 4.7

If a matrix A ($\in \mathbb{C}^{m\times m}$) is such that $A = A^*$ or for A ($\in \mathbb{R}^{m\times m}$) $A = A'$, then

$A \in Q$ (Q_0) $\Longleftrightarrow$ $A \in$ p.d (p.s.d)

Section 4.3 Condition for Positive Definiteness and Positive Semidefiniteness

In this section, we consider the conditions that a real-rational matrix A of the form of

$$A(s) = \mathrm{diag}(\ a_{ii}(s)\ ) + \mathrm{offdiag}(\ a_{ij}(s)\ )$$

belongs to p.d or p.s.d. First, we try to get a sufficient

---

† A matrix A is said to be reducible, if there exist a nonvoid $N \subset M$ ($N \neq M$), such that $a_{ij} = 0$ for $i \in N$ and $j \in M-N$. A matrix is irreducible if it is not reducible.

condition for positive definiteness of the matrix. If we put the matrix $G(j\omega)$ as $G(j\omega) = A(j\omega)+A(-j\omega)$, $G(j\omega)$ can be written as

$$G(j\omega) = \mathrm{diag}(\ g_{ii}(\omega)\ ) + \mathrm{offdiag}(\ g_{ij}(j\omega)\ )$$

where

$$g_{ii}(\omega) = 2\ \mathrm{Re}\ a_{ii}(j\omega) \quad , \quad \forall i \in M$$

$$g_{ij}(j\omega) = a_{ij}(j\omega) + a_{ji}(-j\omega) \quad , \quad i \neq j \quad , \forall i,j \in M$$

By this matrix, we define a real symmetric matrix $\overset{\circ}{G}$ in the form of

$$\overset{\circ}{G} = \mathrm{diag}(\ \overset{\circ}{g}_{ii}\ ) + \mathrm{offdiag}(\ \overset{\circ}{g}_{ij}\ )$$

where

$$\overset{\circ}{g}_{ii} = \inf_{\omega \geq 0} g_{ii}(\omega) \quad , \quad \forall i \in M$$

$$\overset{\circ}{g}_{ij} = -\sup_{\omega \geq 0} |g_{ij}(j\omega)| \quad , \quad i \neq j \quad , \quad \forall i,j \in M$$

In this case, we have the following theorem.

<u>Theorem 4.1</u>

$$\overset{\circ}{G} \in \mathrm{p.d} \quad \Longrightarrow \quad A(j\omega) \in \mathrm{p.d} \quad , \quad \forall \omega \in \mathbb{R}$$

This theorem gives a sufficient condition for positive definiteness of the matrix A. Of course, for $\overset{\circ}{G}$ to be in p.d it should be that

$$\inf_{\omega \geq 0} \mathrm{Re}\ a_{ii}(j\omega) > 0 \quad , \quad \forall i \in M$$

$$\sup_{\omega \geq 0} |a_{ij}(j\omega)| < \infty \quad , \quad i \neq j \quad , \quad \forall i,j \in M$$

Notice that when $a_{ij}(s)$ expresses the input output relation of a physical system, such as the transfer function, the element $\overset{\circ}{g}_{ij}$ of the matrix $\overset{\circ}{G}$ can be obtained by a simple calculation or by drawing the vector loci of $a_{ij}(s)$ and $a^{*}_{ji}(s)$.

Proof of Theorem 4.1

Assume $\overset{\circ}{G} \in$ p.d. Since $\overset{\circ}{g}_{ij} = \overset{\circ}{g}_{ji} \leq 0$, $i \neq j$, $\forall i, j \in M$ by the definition of $\overset{\circ}{G}$, $\overset{\circ}{G}$ is also in $\mathbb{K}$. From the property 3° of Proposition 4.3, there exist m positive constants d_i $(i \in M)$ satisfying

$$d_i \overset{\circ}{g}_{ii} + \sum_{j \neq i}^{m} d_j g_{ij} > 0 \quad , \qquad \forall i \in M$$

This shows $\overset{\circ}{G}$ is in $\mathcal{L}$. Furthermore, the following inequality are satisfied ;

$$d_i \mathrm{Re} g_{ii}(\omega) \geq d_i g_{ii} \geq -\sum_{j \neq i}^{m} d_j \overset{\circ}{g}_{ij} \geq \sum_{j \neq i}^{m} d_j \, g_{ij}(j\omega) \quad ,$$
$$\forall \omega \geq 0 \quad , \ \forall i \in M$$

That is, $G(j\omega)$ is also in $\mathcal{L}$ for any value of ω $(0 \leq \omega \leq \infty)$. By Proposition 4.2, $G(j\omega)$ belongs to Q for any value of ω. Considering that $G(j\omega)$ is a Hermitian matrix, it follows that by Proposition 4.7

$$G(j\omega) \in \text{p.d} \quad \text{for } \omega \in \mathbb{R}$$

Thus by Definition 4.1, $A(j\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$ is proved.

Q.E.D.

A sufficient condition for positive semidefiniteness can be given as well in the following theorem.

Theorem 4.2

If G^{o} is irreducible and belongs to p.s.d, then $A(j\omega)$ belongs to p.s.d for any ω $(0 \leq \omega \leq \infty)$.

The proof of the above theorem is similar to that of Theorem

4.1, but where the property 2° of Proposition 4.4 and the latter half of Proposition 4.5 should be used instead of the property 4° of Proposition 4.3 and the property 3° of Proposition 4.3, respectively.

Now, we have considered so far a sufficient condition for positive definiteness or positive semidefinteness of a matrix A. Next, we will consider a necessary one. We write the matrix

$$G(j\omega) = A(j\omega) + A(-j\omega)$$

in the form of

$$G(j\omega) = \mathrm{Re}G(j\omega) + j\mathrm{Im}G(j\omega) = U(\omega) + jV(\omega).$$

Then the matrices $U(\omega) \triangleq \mathrm{Re}G(j\omega)$ and $V(\omega) \triangleq \mathrm{Im}G(j\omega)$ are given respectively by

$$U(\omega) = \mathrm{diag}(\ u_{ii}(\omega)\ ) + \mathrm{offdiag}(\ u_{ij}(\omega)\ )$$

where

$$u_{ii}(\omega) = g_{ii}(\omega) = a_{ii}(j\omega) + a_{ii}(-j\omega) \quad , \ \forall i \in M$$

$$u_{ij}(\omega) = \mathrm{Re}g_{ij}(j\omega) = \mathrm{Re}\left\{a_{ij}(j\omega)+a_{ji}(-j\omega)\right\} \ , \ i \neq j \ , \ \forall i,j \in M$$

and

$$V(\omega) = \mathrm{offdiag}(\ v_{ij}(\omega)\ )$$

where

$$v_{ij}(\omega) = \mathrm{Im}g_{ij}(j\omega) = \mathrm{Im}\left\{a_{ij}(j\omega)+a_{ji}(-j\omega)\right\} \ , \ i \neq j \ , \ \forall i,j \in M \ .$$

Using the above notations, we have the following theorem.

<u>Theorem 4.3</u>

$$A(j\omega) \in \mathrm{p.d}, \ \forall \omega \in \mathbb{R} \implies U(\omega) \in \mathrm{p.d}, \ \forall \omega \in \mathbb{R}$$

<u>Proof</u>

assume that $A(j\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$. By Definition 4.1, for any non-zero vector z ; $z^* G(j\omega) z > 0$, $\forall \omega \in \mathbb{R}$ is satisfied. Now, when the vector z is written as $z = x + j \cdot y$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $z^* G(j\omega) z$ is expressed as

$$z^* G(j\omega) z = (x', y') \begin{bmatrix} U(\omega) & V(\omega) \\ V'(\omega) & U(\omega) \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore, the condition that the Hermitian matrix $G(j\omega)$ of order m is positive definite is equivalent to the real symmetric matrix of order 2m such that

$$X(\omega) = \begin{bmatrix} U(\omega) & V(\omega) \\ V'(\omega) & U(\omega) \end{bmatrix}$$

is positive definite. The conclusion, that is, $U(\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$ comes from a necessary condition for the matrix $X(\omega)$ to be positive definite.

Q.E.D.

Suppose that there exists a real number a $(0 \leq a \leq \infty)$ such that

$$\left.\begin{array}{l} g_{ii}(\omega) \geq g_{ii}(a) > 0 \quad , \forall \omega \geq 0 \quad , \quad \forall i \in M \\ -\infty < \mathrm{Re}\, g_{ij}(ja) \leq \mathrm{Re}\, g_{ij}(j\omega) \leq 0 \quad , \forall \omega \geq 0 \ , \ i \neq j \ , \forall i,j \in M \end{array}\right\} \quad (1)$$

Under these assumptions, we have the following theorem.

<u>Theorem 4.4</u>

$$A(j\omega) \in \text{p.d} \ , \forall \omega \in \mathbb{R} \implies U(a) \in \text{p.d} \iff U(\omega) \in \text{p.d} \ , \forall \omega \geq 0$$

Proof

The first half is obvious from Theorem 4.3. We will prove ($\Longrightarrow$) of the latter half. By assumption (1), $U(\omega)$ has non-zero offdiagonal elements and is symmetric for any real ω . Therefore, it is verified by the property 4° of Proposition 4.3 that $U(a) \in$ p.d and $U(\omega) \in$ p.d, $\forall \omega \geq 0$ are equivalent to $U(a) \in \mathbb{K}$ and $U(\omega) \in \mathbb{K}$, $\forall \omega \geq 0$, respectively. The conclusion can be immediately obtained from Proposition 4.6.

Q.E.D.

As was understood by the above proof, under the assumption (1) the existence of the matrix $V(\omega)$ makes the difference between the necessary and sufficient condition for positive definiteness of the matrix $A(j\omega)$ for any real ω and a necessary one given in the above theorem. As to the condition for positive semidefiniteness, the following results can be obtained corresponding to Theorem 4.3 and Theorem 4.4.

Theorem 4.5

$$A(j\omega) \in \text{p.s.d} \ , \forall \omega \in \mathbb{R} \ \Longrightarrow \ U(\omega) \in \text{p.s.d} \ , \forall \omega \geq 0$$

Theorem 4.6

If the assumption (1) is satisfied, then we have a proposition such that

$$A(j\omega) \in \text{p.s.d} \ , \forall \omega \in \mathbb{R} \Longrightarrow U(a) \in \text{p.s.d} \iff U(\omega) \in \text{p.s.d} \ , \forall \omega \geq 0$$

Now, we consider the case where the following conditions are met instead of the condition (1) ;

$$\left.\begin{array}{l} g_{ii}(\omega) \geq g_{ii}(a) > 0 \;, \;\; \forall \omega \geq 0 \;, \;\; \forall i \in M \\ -\infty < \mathrm{Re}\, g_{ij}(ja) = -\sup_{\omega \geq 0} |g_{ij}(j\omega)| \leq 0 \;, \;\; i \neq j \;, \;\; \forall i,j \in M \end{array}\right\} \quad (2)$$

Then from the definition of the matrix $\overset{\circ}{G}$, we have $U(a) = \overset{\circ}{G}$. Moreover, by Theorem 4.1 and the first half of Theorem 4.4 , the following theorem can be attained.

Theorem 4.7

Under the condition (2), we have a proposition such that
$A(j\omega) \in$ p.d , $\forall \omega \in \mathbb{R} \iff U(a) = \overset{\circ}{G} \in$ p.d

Similar theorem on positive semidefinteness can be obtained as follows.

Theorem 4.8

Under the condition (2), we have a proposition such that
$A(j\omega) \in$ p.s.d , $\forall \omega \in \mathbb{R} \iff U(a) = \overset{\circ}{G} \in$ p.s.d and $U(a) = \overset{\circ}{G}$; irreducible

Whether the elements of the matrix A(s) satisfy the condition (2) or not can be checked as well as the condition (1), Theorem 4.1 and Theorem 4.2 by a simple calculation or by drawing the vector loci of them. Fig. 4.1 shows the regions for the condition (2) on the complex plane. The first inequality of the condition (2) is depicted in Fig. 4.1 (a) and the other in Fig. 4.1 (b). Note that the first inequality of the condition (2) is the same as that of the condition (1) and the second one of both conditions is not included by each other.

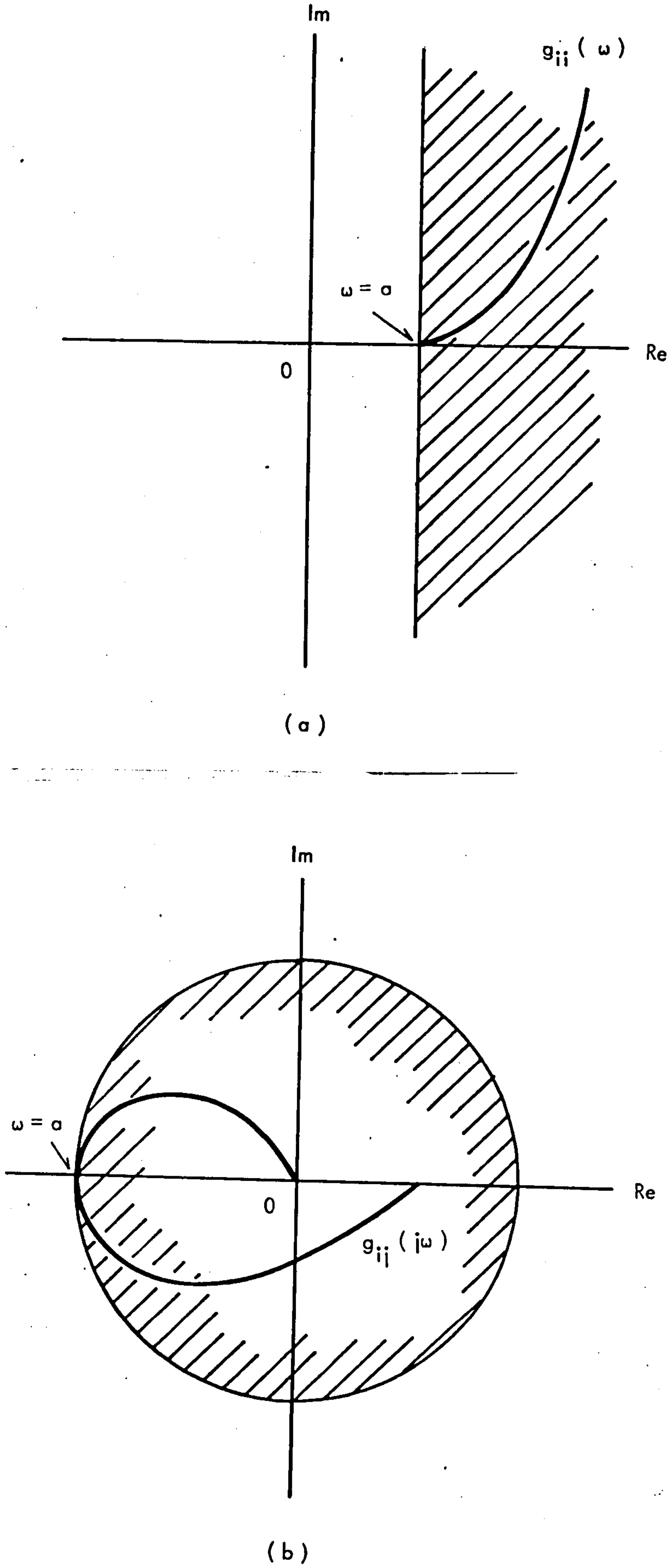


Fig. 4.1 Regions for Condition (2)

Theorem 4.7 and Theorem 4.8 show that under appropriate conditions we can obtain the necessary and sufficient condition for positive definiteness of the matrix $A(j\omega)$ for any real value of ω in a matrix form.

Section 4.4 Some Illustrative Examples

In this section, we give some examples of rational matrices satisfying the assumptions of the theorems in the previous section. As already mentioned, with respect to positive definiteness Theorem 4.1 gives a sufficient condition and Theorem 4.3 gives a necessary one. If we assume the condition (2), then the necessary and suffcent condition can be derived as in Theorem 4.7. Checking whether a matrix A(s) meets the condition (2) or not can be carried out by a observation of the A(s) or by a simple calculation if the element of A(s) has a simple form or by drawing the vector loci of the elements, otherwise.

We consider first the case where the elements of A(s) have comparatively simple forms.

<u>Example 1</u>

Let A(s) be the form of

$$A(s) = \mathrm{diag}(\ K_i\ ) + \mathrm{offdiag}(\ -\frac{N_{ij}}{s + d_{ij}})$$

whrere

$$K_i > 0, \qquad d_{ij} > 0\ (i \neq j), \qquad \forall\, i, j \in M$$

Apparently, the elements of this matrix satisfy the condition (2) and $U(a)$ equals to G° where $a = 0$. Fig.4.2 shows the vector locus of the offdiagonal (i,j) element, that of conjugate (j,i) element and the sum of the both loci. In this figure, a_{ij} denotes the (i,j) offdiagonal element of $-A(s)$. By virtue of Theorem 4.7, the necessary and sufficient condition for positive definiteness of this matrix can be obtained immediately as follows

$$A(0) \in \text{p.d}$$

In addition, it might be noticed that this matrix has the same offdiagonal elements as appeared in (3-24), which was given as the transfer matrix of the imaginary system I_L .

Example 2

Consider the matrix

$$A(s) = \left\{ a_{ij}(s) \right\}$$

$$= \begin{bmatrix} \dfrac{s}{s^2+s+1} + K_1 & \dfrac{1}{s^2+s+1} \\ \dfrac{-1}{s^2+s+1} & \dfrac{s+1}{s^2+s+1} + K_2 \end{bmatrix}$$

As known well, by hitherto-obtained frequency domain stability criteria extended to large scale systems, the stability condition reads that a matrix, whose elements are real rational functions of s of comparatively high order, belongs to the class p.d. However, the work required for checking this condition is not a easy one as aforesaid. Hirai and Kurematsu proposed a graphical method of checking positive definiteness of the real rational matrix, taking the above matrix as an example, but the method is applicable only when m is less than or equal to 2.[24]

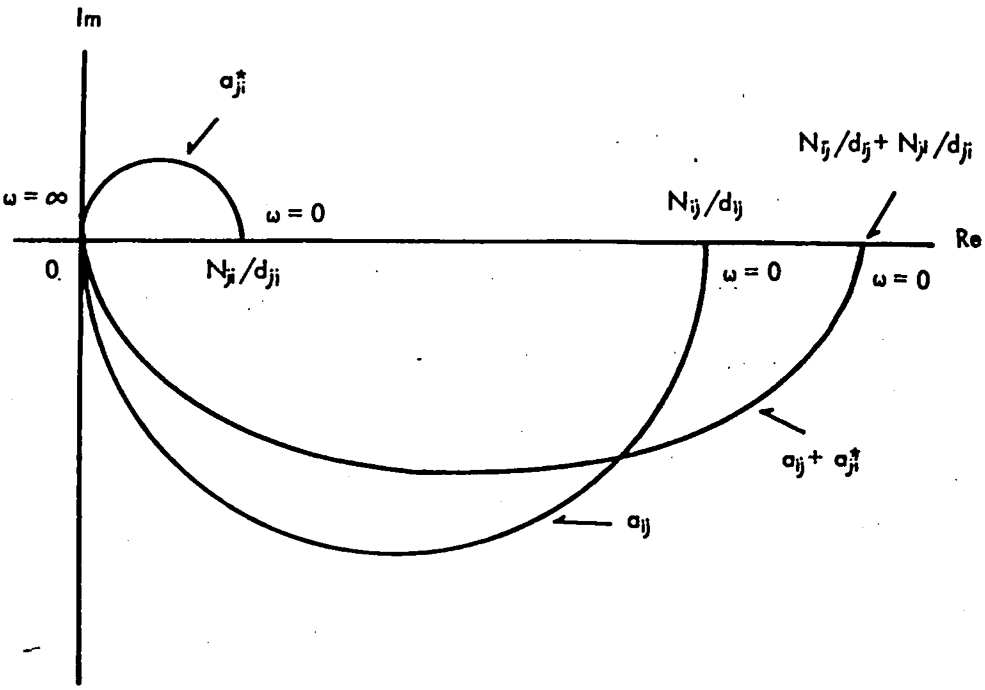


Fig. 4.2 Vector Loci of the Elements of the Matrix in Example 1

The vector loci of $a_{11}(s)$, $a_{22}(s)$ and $a_{12}(s)+a_{21}^*(s)$ of this matrix are depicted in Figs. 4.3 (a), (b) and (c), respectively. From these figures, it can easily be shown that the matrix satisfies the condition (1) where $a = \infty$, but not the condition (2). Therefore, we can not obtain the necessary and sufficient condition for positive definiteness by Theorem 4.7. But we can obtain a sufficent condition and a necessary one separately by Theorem 4.1 and Theorem 4.4, respectively. The matrix G° in Theorem 4.1 is calculated as

$$\frac{1}{2} G^\circ = \begin{bmatrix} K_1 & \tilde{a} \\ \tilde{a} & K_2 \end{bmatrix}$$

where

$$\tilde{a} = \frac{\sqrt{6+6\sqrt{23}}}{72} = 0.073\cdots$$

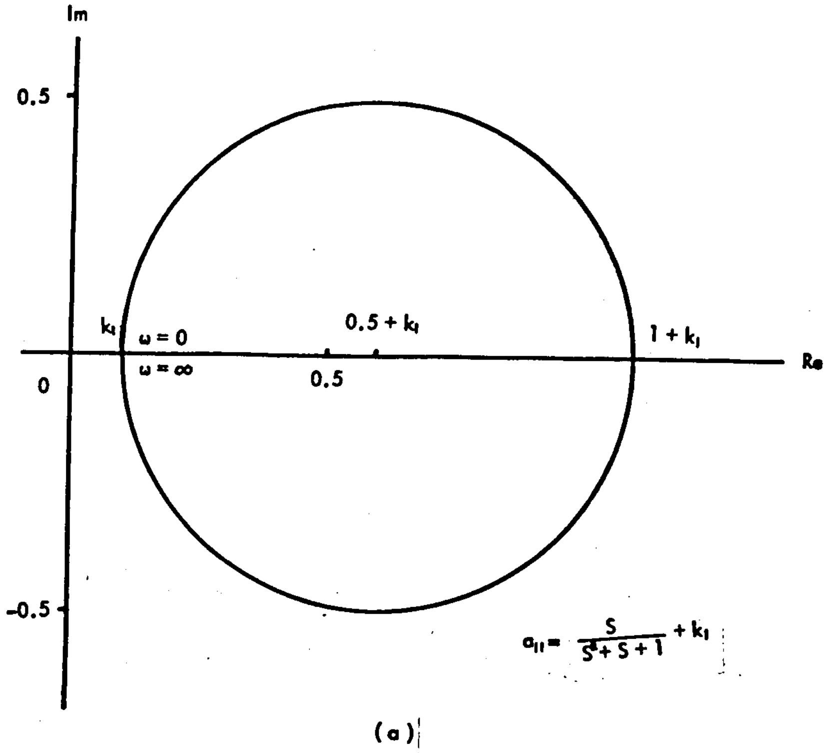
Whence, a sufficent condition for the matrix to be in p.d is given by Theorem 4.1 as

$$K_1 K_2 > \tilde{a}^2 = 0.0053\cdots$$

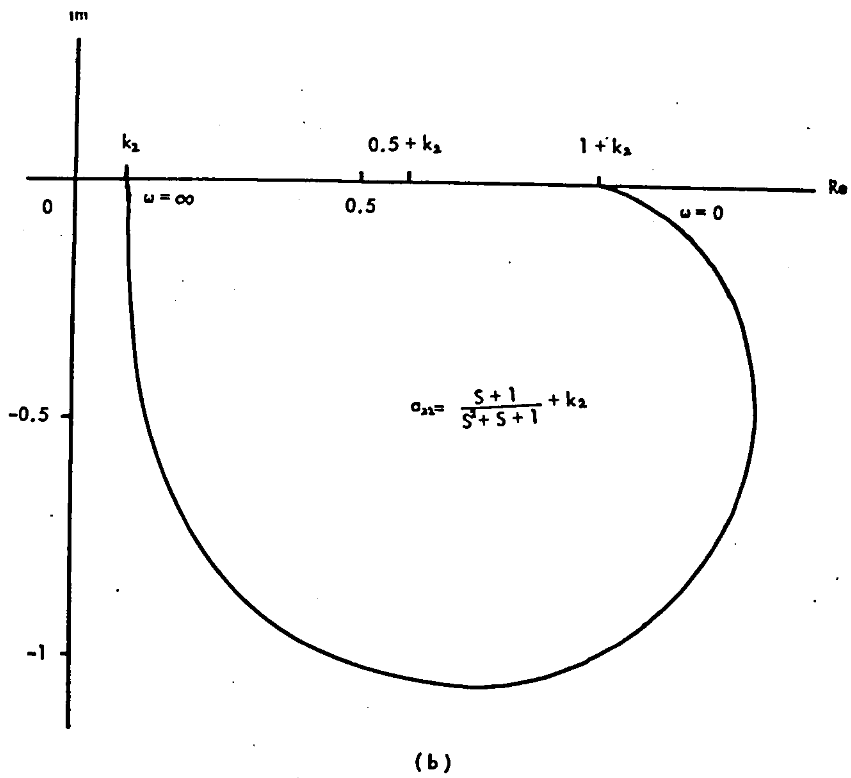
On the other hand, a necessary condition is given by Theorem 4.4 as

$$A(j\infty) = \mathrm{diag}(\ K_i\ ) \in \mathrm{p.d}$$

That is, $K_1 > 0$ and $K_2 > 0$. Hirai and Kurematsu obtained the necessary and sufficient condition for positive definiteness of this matrix by their method as $K_1 > 0$, $K_2 > 0$.[24] Consequently, in this case Theorem 4.4 gives a necessary and sufficient condition.



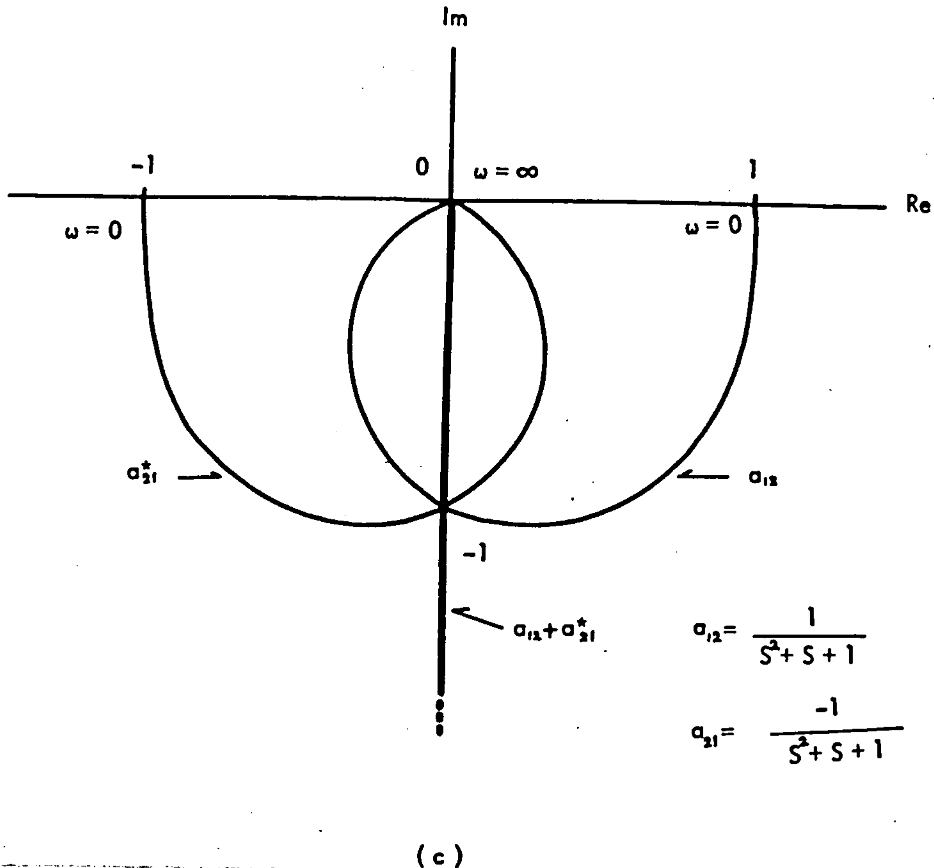
(a) Vector Locus of $a_{11}(s)$



(b) Vector Locus of $a_{22}(s)$

continued to next page.

Fig. 4.3 Vector Loci of the Elements of the Matrix in Example 2



(c) Vector Loci of $a_{12}(s)$ and $a^*_{21}(s)$

Example 3

We consider the following transfer matrix of order 3.

$$A(s) = \left\{ a_{ij}(s) \right\}$$

$$= \begin{bmatrix} K_1 & \dfrac{-1}{s+1} & \dfrac{-2}{s+1} \\ \dfrac{-1}{(s+1)(2s+1)(3s+1)} & K_2 & \dfrac{-2}{s+3} \\ -\dfrac{s^2+2s+3}{2s^2+4s+2} & \dfrac{1}{s+6} & \dfrac{s}{s+1} + K_3 \end{bmatrix}$$

Figs. 4.4 (a) to (c) are the vector loci of the offdiagonal elements, those of conjugate elements in symmetric position and the sum of them. Fig. 4.4 (d) is the vector locus of the third diagonal element. We can confirm from these figures that the matrix satisfies the condition (2) and $a_{-} = 0$. Whence, by Theorem 4.7 the necessary and sufficient condition for positive definiteness can be obtained as

$$A(0) \in \text{p.d}$$

Example 4

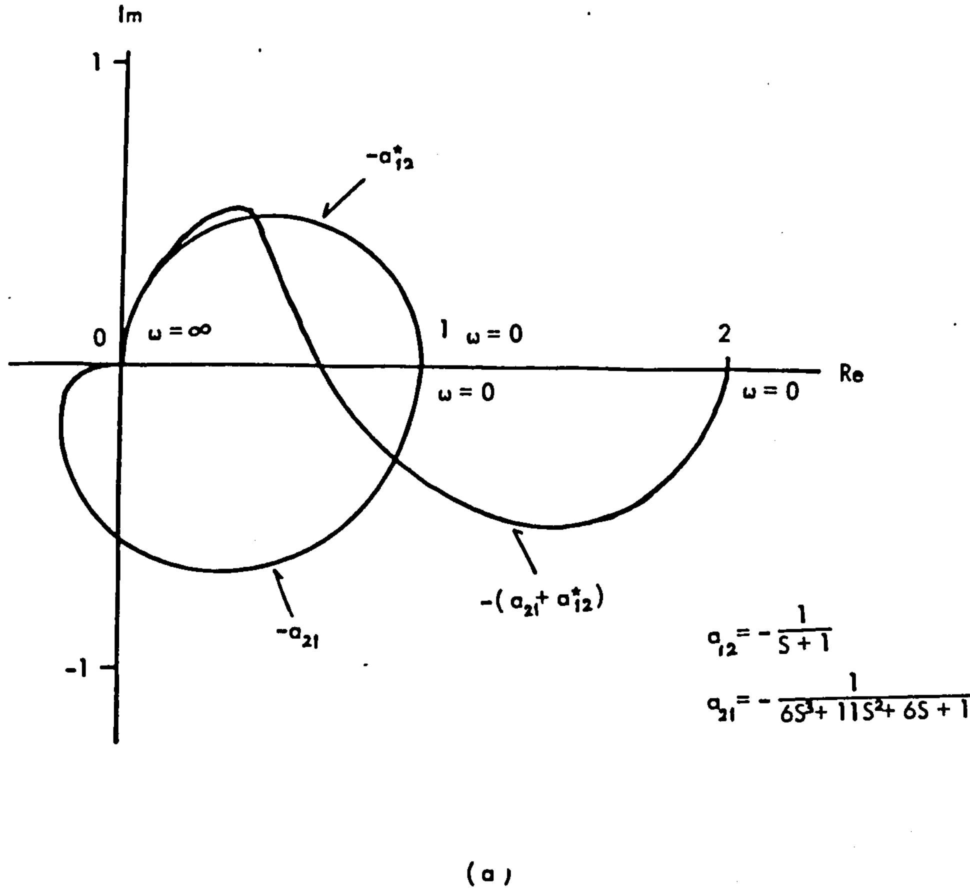
Let A(s) be such that

$$A(s) = \left\{ a_{ij}(s) \right\}$$

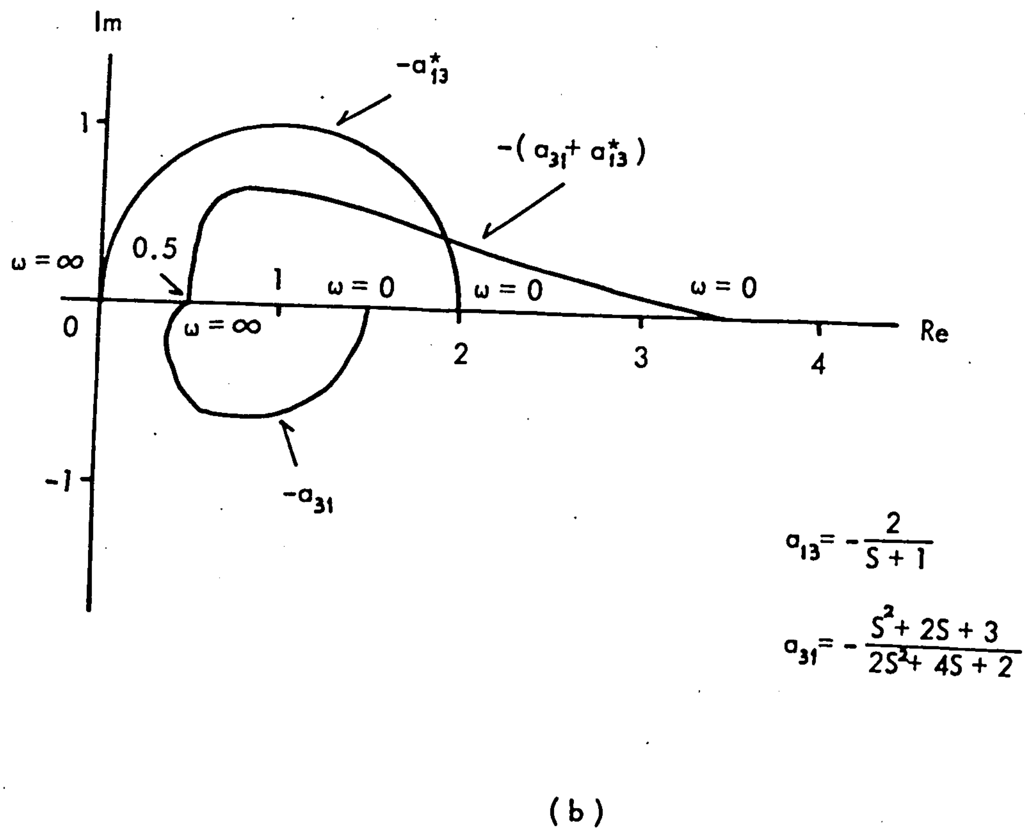
$$= \begin{bmatrix} K_1 & \dfrac{-1}{s^3+2s+1} \\ \dfrac{-1}{s^3+2s+2} & K_2 \end{bmatrix} .$$

where

$$K_1 > 0, \quad K_2 > 0.$$



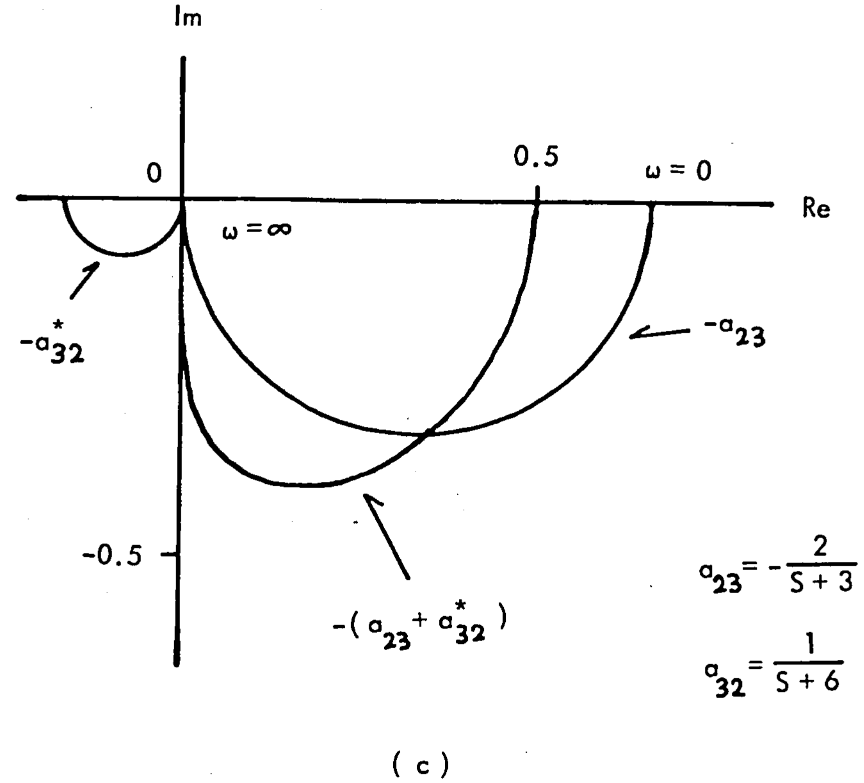
(a) Vector Loci of $-a_{21}(s)$ and $-a_{12}{}^*(s)$



(b) Vector Loci of $-a_{31}(s)$ and $-a_{13}{}^*(s)$

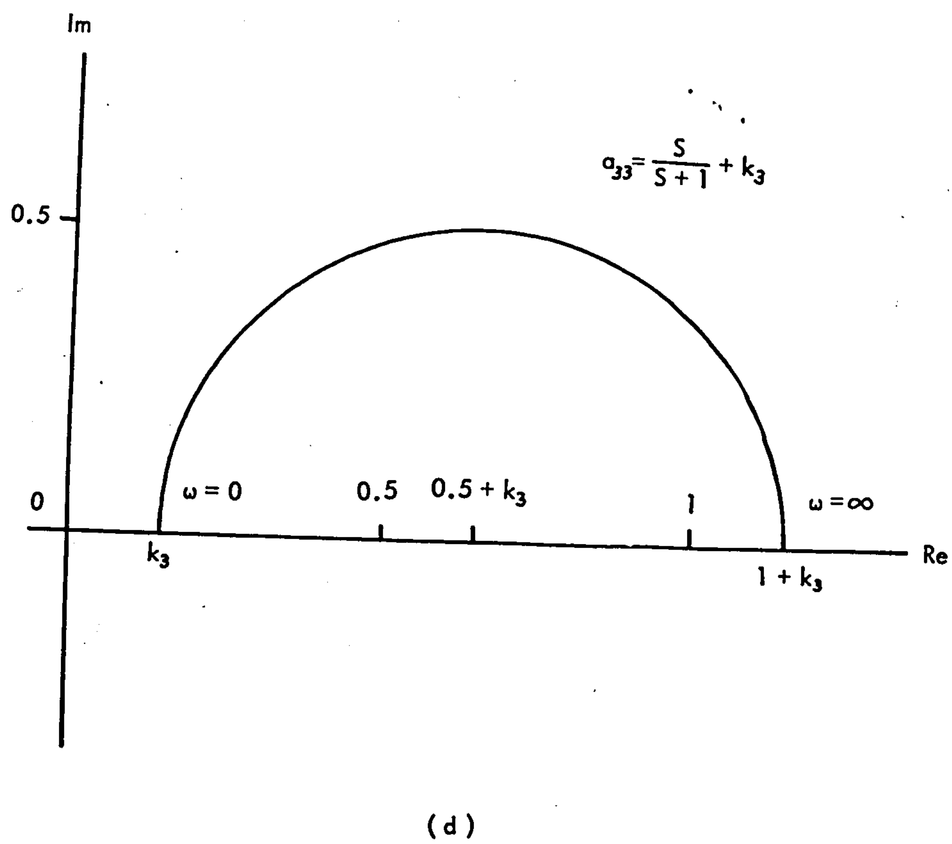
continued to next page.

Fig. 4.4 Vector Loci of the Elements of the Matrix in Example 3



(c)

(c) Vector Loci of $-a_{23}(s)$ and $-a_{32}{}^*(s)$



(d)

(d) Vector Locus of $a_{33}(s)$

The vector locus of the offdiagonal elements of this matrix is shown in Fig. 4.5, where a_{21}^{*} denotes the conjugate of a_{21}. From this figure, we can see that the matrix satisfies the condition (2), where $a = 0$ or $a = \sqrt{2}$. Therefore, the necessary and sufficient condition for positive definiteness is obtained by Theorem 4.7 as

$$A(0) = A(+j\sqrt{2}) \in \text{p.d}$$

Whence, we have

$$K_1K_2 > 1/2, \quad K_1 > 0, \quad K_2 > 0$$

Thus, it has been shown that, under certain conditions for the elements of a real rational matrix, the necessary and sufficient condition for positive definiteness or positive semidefinteness can be transformed into a simple algebraic condition. The conditions for the elements of the matrix are checked by relatively easy procedures. These results will give fundamental means to transform the frequency domain method into algebraic one equivalently. Details of an approach and results of the transformation will be developed in the next chapter. Even when the condition (2) is not met for a matrix, we can obtain a sufficient condition by Theorem 4.1 and a necessary one by Theorem 4.3, independently. Especially, Theorem 4.3 provides a tractable step for checking positive definiteness or semidefiniteness of the matrix. Finally, it should be noted that the results of this chapter are considered to have a close relation to the applications of multivariable circle criteria obtained by Rosenbrock (s-1) or Cook (s-2). However the detailed discussion about it will not be given.

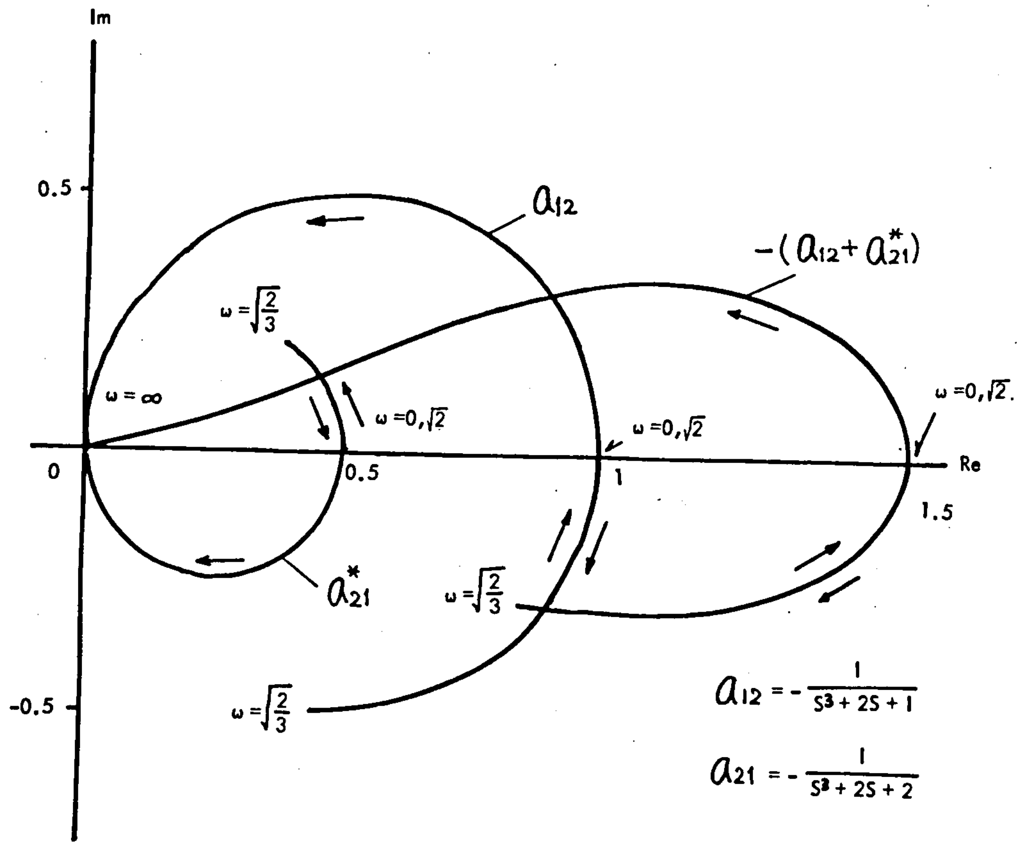


Fig. 4.5 Vector Loci of the Offdiagonal Elements of the Matrix in Example 4

Chapter 5 Transformation of Frequency Domain Stability Theorems into Algebraic Forms

Section 5.1 Introduction

In the preceding chapter, we derived the conditions for positive definiteness and positive semidefiniteness of a real rational matrix in a matrix form under some restricted circumstances. We show, in this chapter, the stability conditions in frequency domain of chapter 3 can be transformed into algebraic conditions. By this transformation the stability conditions of some of the theorems in chapter 3 become comparable with those of the theorems which have already been obtained by other authors in an algebraic form. We will give some further considerations about the theorems in chapter 3 by comparing them with other theorems in an algebraic form. Furthermore, we show that by the results of the preceding chapter it is also possible in some cases to transform frequency domain criteria of large scale systems established by other authors into an algebraic form equivalently. And we also show that even if this equivalent transformation is impossible, we can obtain a sufficient condition for the criteria in a matrix form. Whence, in the latter case, we have a sufficient condition for stability, though of course a restricted one, in an algebraic form. In the same way, a necessary condition for the criteria is presented and it is shown by an example that the condition provides a simple test for checking the conditions of the criteria.

Section 5.2 Considerations on Theorems in Chapter 3

The procedures required for assuring stability of large scale systems by theorems in chapter 3 are to check positive definiteness of the real rational matrix constituted mainly by the system transfer matrix. So, by utilizing the results of the preceding chapter, we will be able to compare some of the result in chapter 3 woth other theorems previously obtained. Since we have also to make use of some other properties of matrices belonging to the class $\mathbb{K}$, let us collect them in the following lemmas.

<u>Lemma 5.1</u>[17]

Let $A \in \mathbb{K}$. If D is a diagonal matrix with positive diagonal elements, then DA and DA belong to $\mathbb{K}$ as well.

<u>Lemma 5.2</u>[2]

Let A ($\in \mathbb{R}^{m \times m}$) be a matrix such that

$$a_{ij} \leq 0 \quad , \quad i \neq j \quad , \quad \forall i,j \in M .$$

The necessary and sufficient condition for A to be in the class $\mathbb{K}$ is that there exists a diagonal matrix W ($\triangleq \mathrm{diag}(w_i)$) with positive diagonal elements such that the matrix given by

$$B = \frac{1}{2} (WA + A'W)$$

is positive definite.

We, first, consider Theorem 3.4. This theorem treats the system Σ_L. Many stability theorems for the system Σ_L have been developed up to now, because its imaginary system I_L

are linear and can be examined easily. Now, if we put the matrices P and Q in Theorem 3.4 as

$$P = I(\text{unit matrix}), \quad Q = \Theta(\text{null matrix})$$

then the matrix $W(\omega)$ given by (3-25) and (3-26) becomes

$$W(\omega) = \mathrm{diag}(c_{i4}^{-1}) + \mathrm{offdiag}\left[-\frac{1}{2}\left(\frac{\gamma_{ij}/c_{j1}}{j\omega + c_{i3}/c_{i2}} + \frac{\gamma_{ji}/c_{i1}}{-j\omega + c_{j3}/c_{j2}}\right)\right] \quad (5\text{-}1)$$

By Theorem 3.4 the stability condition of Σ_L is obtained as the condition for positive definiteness of $W(\omega)$ for any real value of ω. Recalling example 1 of chapter 4, this condition is equivalent to positive definiteness of the matrix

$$W(0) = \mathrm{diag}(c_{i4}^{-1}) + \mathrm{offdiag}\left[-\frac{1}{2}\left(\frac{c_{i2}\gamma_{ij}}{c_{i3}c_{j1}} + \frac{c_{j2}\gamma_{ji}}{c_{j3}c_{i1}}\right)\right] \quad (5\text{-}2)$$

Now, we put matrices L_1 and X_1 as follows

$$\left.\begin{aligned} L_1 &= \mathrm{diag}(c_{i3}/c_{i2}) + \mathrm{offdiag}(-c_{i4}\gamma_{ij}/c_{j1}) \\ X_1 &= \mathrm{diag}(c_{i2}/c_{i3}c_{i4}) \end{aligned}\right\} \quad (5\text{-}3)$$

Then $W(0)$ can be written as

$$W(0) = \frac{1}{2}(X_1L_1 + L_1'X_1)$$

Noting that the offdiagonal elements of L_1 are non-positive, it is shown by Lemma 5.2 that $W(0) \in$ p.d is a necessary condition to $L_1 \in \mathbb{K}$. The condition $L_1 \in \mathbb{K}$ is no other than the codition of the theorem given by Šiljak (59). Namely, the condition of Theorem 3.4, where the matrices P and Q are chosen as I and Q, respectively, implies that of Šiljak's theorem. However, a comparison of Theorem 3.4 in itself with Šiljak's is not known now. A comparison between Šiljak's theorem and the theorems given by other authors is made by Araki(3). He states in (3) that his theorems include Šiljak's, but he also states there the converse has not yet been proven.

Next, we consider Theorem 3.3 which gives the stability condition of the system ΣD_L . To begin with, remind that we could have the same discussions as those in chpter 4 about non-rational matrices whose elements include $e^{-\tau s}$, and that the results of chapter 4 are also true for these matrices. Let us assume that for the system ΣD_L the following inequalities hold.

$$c_{i3}/c_{i2} > c_{i4}d_{i2} , \qquad i \in M \tag{5-4}$$

As mentioned in example 1 of chapter 3, the condition 1° of Theorem 3.3 is met in this case. Therefore, to examine stability of the system ΣD_L , it is left to check positive definiteness of the matrix $W(\omega)$ given by (3-25) and (3-24). We will choose the matrices P and Q in Theorem 3.3 as $P = I$ and $Q = \Theta$ as well as in Theorem 3.4. Then, the offdiagonal elements of $W(\omega) \triangleq \{w_{ij}(\omega)\}$ become

$$w_{ij}(\omega) = -\frac{1}{2}\left(\frac{\gamma_{ij}/c_{j1}}{j\omega + c_{i3}/c_{i2} - c_{i4}d_{i2}e^{-j\omega\tau}} + \frac{\gamma_{ji}/c_{i1}}{-j\omega + c_{j3}/c_{j2} - c_{j4}d_{j2}e^{j\omega\tau}}\right) \tag{5-5}$$

$$\text{for } i \neq j$$

Let us manipulate $\sup|w_{ij}(\omega)|$ for $\omega \geq 0$, in the next place. If a and b are positive constants such that $a > b$, we have

$$0 < a - b \leq |j\omega + a| - b \leq |j\omega + a - be^{\pm j\omega\tau}| \tag{5-6}$$

for any value of ω . The last term of the above inequality takes the value a-b only when $\omega = 0$. Considering that the inequalities (5-4) are assumed, we can obtain from

(5-6) the supremum of $|w_{ij}(\omega)|$ as follows

$$\sup|w_{ij}(\omega)| = |w_{ij}(0)| = \frac{1}{2}\left(\frac{\gamma_{ij}/c_{j1}}{c_{i3}/c_{i2}-c_{i4}d_{i2}} + \frac{\gamma_{ji}/c_{i1}}{c_{j3}/c_{j2}-c_{j4}d_{j2}}\right),$$

$$\text{for } i \neq j \qquad \cdots\cdots\cdots \quad (5\text{-}6)$$

Consequently, we have

$$-\sup|w_{ij}(\omega)| = w_{ij}(0), \qquad i \neq j , \quad \forall i,j \in M \qquad (5\text{-}7)$$

Since the diagonal elements of $W(\omega)$ are positive constants, the matrix W°, which corresponds to G° in Theorem 4.1 of chapter 4 is given by

$$W^{\circ} = W(0)$$

By Theorem 4.1, $W^{\circ} \in$ p.d is a sufficient condition for $W(\omega)$ to be in p.d. And it is evident that this condition is also a necessary one. Therefore, $W^{\circ} = W(0) \in$ p.d is the necessary and sufficient condition for $W(\omega) \in$ p.d, $\forall\, \omega\, (0 \leq \omega \leq \infty)$.

From (3-24) and (3-25), it follows that

$$W(0) = \text{diag}(c_{i4}^{-1}) + \text{offdiag}\left\{-\frac{1}{2}\left(\frac{\gamma_{ij}/c_{j1}}{c_{i3}/c_{i2}-c_{i4}d_{i2}} + \frac{\gamma_{ji}/c_{i1}}{c_{j3}/c_{j2}-c_{j4}d_{j2}}\right)\right\}$$

$$\cdots\cdots\cdots \quad (5\text{-}8)$$

As the offdiagonal elements of the above matrix are non-positive, the condition $W(0) \in$ p.d is equivalent to the condition that $W(0) \in \mathbb{K}$ from the properties of the matrices in $\mathbb{K}$.

If we put square matrices L_2 and X_2 as

$$\left.\begin{aligned} L_2 &= \text{diag}\left(\frac{c_{i3}-c_{i2}d_{i2}c_{i4}}{c_{i2}}\right) + \text{offdiag}\left(-\frac{c_{i4}\gamma_{ij}}{c_{j1}}\right) \\ X_2 &= \text{diag}\left(\frac{c_{i2}}{c_{i3}c_{i4}-c_{i2}d_{i2}c_{i4}}\right) , \end{aligned}\right\} \qquad (5\text{-}9)$$

it follows that

$$W(0) = \frac{1}{2} (X_2L_2 + L_2'X_2)$$

According to analogous discussions to those in the case of Theorem 3.4, it can also be shown that $W(0) \in \mathbb{K}$ only if $L_2 \in \mathbb{K}$. All this can be put in the form of a theorem thus;

Theorem 5.1

Assume the relation (5-4) holds. The system $\sum D_L$ is ASIL, if the matrix W(0) given by (5-8) belongs to the class p.d.

This theorem shows that under appropriate conditions Theorem 3.4 can partly be transformed into the theorem in an algebraic form equivalently. A comparison of Theorem 3.4 with other theorems will be omitted here, because there are few adequate theorems to compare with so far as our research. As well, as to Theorem 3.5 and Theorem 3.6, where large scale systems with non-finite "sector conditions" were dealt, we will not treat here because of the same reason mentioned above. Note that, in the case of Theorem 3.5 and Theorem 3.6, the transformation into an algebraic form is not an easy work on account of the existence of non-constant diagonal elements in the matrix $W(\omega)$ given in (3-27) to (3-29). However, the examples in the next section will give some refferences to the applications of these theorems.

Section 5.3 Applications to Checking the Conditions of Frequency Domain Theorems by Other Authors

The results of chapter 4 are useful not only for rewriting the theorems of chapter 3 but also for applying frequency domain theorems of large scale systems obtained by other researchers up to now. We will give some examples to illustrate usefulness of the results of chapter 4. Most of frequency domain theorems requires checking positive definiteness of a real-rational matrix with complex argument s for its stability condition. We take here Partovi and Nahi's theorem (51) as a representative of them. The outline of their theorem is shown below for subsequent arguments.

Partovi and Nahi's Theorem

Let a system be formulated by the following equations

$$\dot{x} = Ax + Bf(\sigma, t) \quad , \quad \sigma = Cx \tag{5-10}$$

where

$$x \in \mathbb{R}^m \; ; \; A,B,C \in \mathbb{R}^{m\times m} \; ; \; f(\sigma, t) \in \mathbb{R}^m \; ,$$

and

$$f'(\sigma, t) = \left[f_1(\sigma_1, \sigma_2, \cdots, t), \ f_2(\sigma_1, \sigma_2, \cdots, t), \cdots, f_m(\sigma_1, \sigma_2, \cdots, t) \right]$$

Here, each component $f_i(\sigma, t)$ is piecewise continuous in σ and t such that (5-10) has a unique solution and satisfies the following condition

$$\left.\begin{aligned} 0 < \varepsilon \le \frac{f_i(\sigma_1, \sigma_2, \cdots, \sigma_m, t)}{\sigma_i} \le K_i - \varepsilon \\ f_i(0, 0, \cdots, 0, t) = 0 \ , \quad t \ge 0 \end{aligned}\right\} \quad i \in M \tag{5-11}$$

Then, we have the following stability criterion.†

† The expressions used in their theorem are not faithfully followed in detail for convenience's sake.

For the system of (5-10), if A has all its characteristic roots with negative real parts, $f(\sigma, t)$ satisfies the condition (5-11) and $W(\omega)$ given by

$$W(\omega) = \left[G(j\omega) + K\right] + \left[G(j\omega) + K\right]^{*} \tag{5-12}$$

where

$$\begin{aligned} G(j\omega) &= -C(j\omega I - A)^{-1}B \\ K &= \mathrm{diag}(K_i^{-1}) \end{aligned} \tag{5-13}$$

is positive definite for all ω , then the system is absolutely stable (that is, the system is ASIL for any functions satisfying (5-11)).

We consider the application of this theorem in the following.

<u>Example 1</u>

Let a system be written by the equation

$$\left.\begin{aligned} \dot{X}_1 &= -4X_1 + 2X_2 - \frac{X_1-X_2}{(1+X_2^2)^2}K_1 \\ \dot{X}_2 &= X_1 - X_2 - \frac{2X_1+X_2}{(1+X_1^2)^2}K_2 \end{aligned}\right\} \tag{5-14}$$

where $K_1 > 0$ and $K_2 > 0$. Taking $\sigma_1 = X_1 - X_2$, $\sigma_2 = 2X_1 + X_2$, we have

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & 2 \\ 1 & -1 \end{bmatrix}$$

and the transfer matrix

$$G(s) = -C(sI - A)^{-1}B = \begin{bmatrix} \dfrac{s}{s^2 + 5s + 2} & -\dfrac{s + 2}{s^2 + 5s + 2} \\ -\dfrac{2s + 1}{s^2 + 5s + 2} & \dfrac{s}{s^2 + 5s + 2} \end{bmatrix} \tag{5-15}$$

The matrix $W(\omega)$ of (5-12) can be written as

$$W(\omega) = \left[G(j\omega) + K\right] + \left[\overset{*}{G}(j\omega) + K\right] \qquad (5\text{-}16)$$

where $G(s)$ is given by (5-14) where $s=j\omega$ and

$$K = \text{diag}(\ K_i^{-1}\ ) \quad , \quad i=1,2.$$

Here, $f_1(\sigma_1)$ and $f_2(\sigma_2)$ are taken as

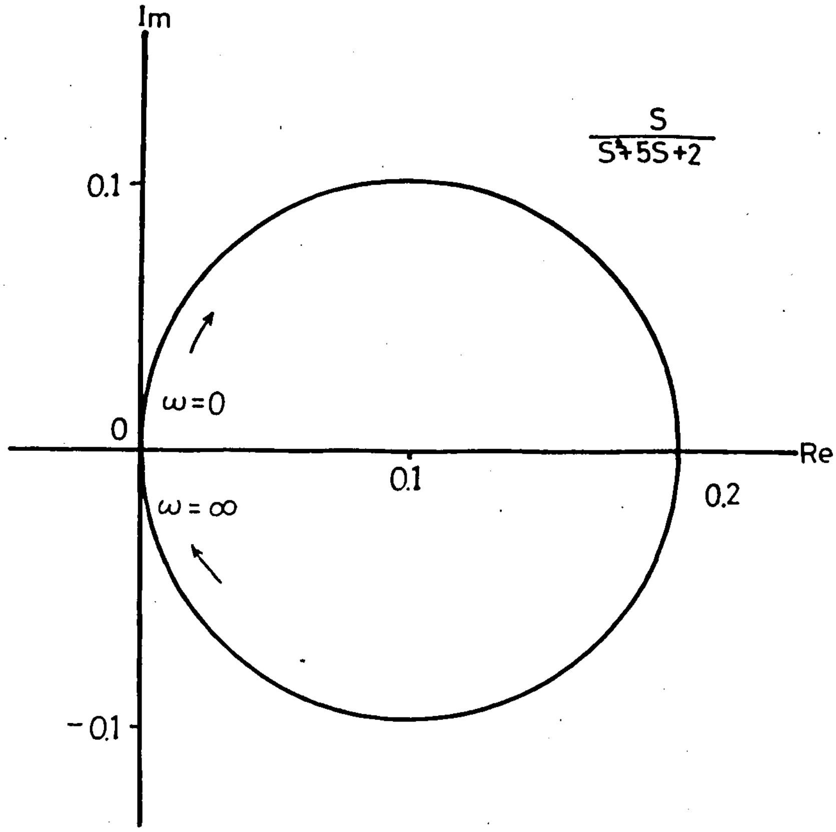
$$f_1(\sigma_1) = \frac{X_1 - X_2}{(1 + X_2^2)^2} K_1 \quad , \quad f_2(\sigma_2) = \frac{2X_1 + X_2}{(1 + X_1^2)^2} K_2 \qquad (5\text{-}17)$$

and

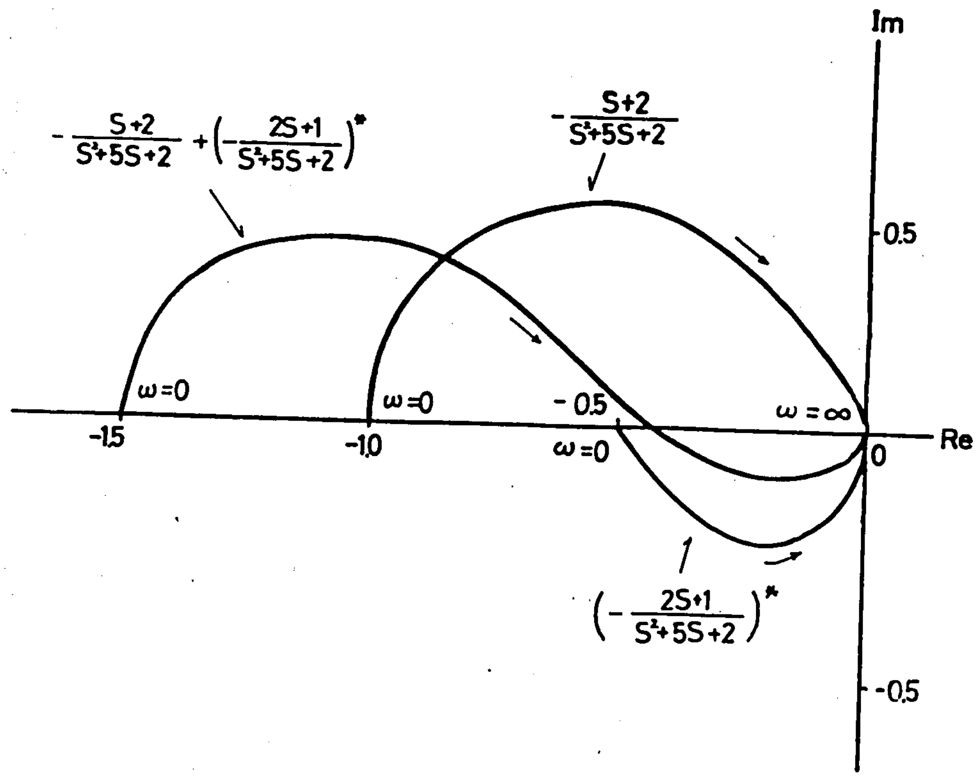
$$0 \le \frac{f_1(\sigma_1)}{\sigma_1} \le K_1 \quad , \quad 0 \le \frac{f_2(\sigma_2)}{\sigma_2} \le K_2 .$$

The condition for the matrix $W(\omega)$ of (5-16) to be positive definite for all ω gives the stability condition of the system (5-14). For checking positive definiteness of a matrix, Sylvester's method and Strum's test are usually employed. And these methods generally requires laborious manipulations as the increase of the order of the matrix. However, if the elements of a matrix satisfy a certain constraint, such as the condition (2) in chapter 4, we can obtain the condition for positive definiteness directly. The matrix $W(\omega)$ given by (5-15) and (5-16) is one of such matrices that satisfy the condition (2). To see this, we show the vector the vector locus of the diagonal elements of $G(s)$ in Fig. 5.1 (a), and the loci of the offdiagonal elements $g_{12}(s)$ and $g_{21}(s)$ and the sum of them are in Fig. 5.1 (b). In this case, a real number "a" in the condition (2) is found to be 0. And we have the condition for positive definiteness from Theorem 4.7 as

$$\left[G(0) + K\right] + \left[G(0) + K\right]' \in \text{p.d} \qquad (5\text{-}18)$$



(a) Diagonal Element



(b) Offdiagonal Elements

Fig. 5.1 Vector Loci of the Elements of the Matrix (5-15) and the stability condition

$$K_1K_2 < \frac{16}{9} \quad , \quad K_1 > 0 \ , \quad K_2 > 0 \tag{5-19}$$

Even if it is proved that the condition (2) is not satisfied for $W(\omega)$, we can mostly get a sufficient condition for positive definiteness in the process of checking the condition (2) by Theorem 4.1. See the following example ;

Example 2

Let the system equations be

$$\left.\begin{aligned} \dot{X}_1 &= -X_1 + X_2 - f_1(\sigma_1) \quad , \quad \sigma_1 = X_1 - X_2 \\ \dot{X}_2 &= -X_1 - f_2(\sigma_2) \quad , \quad \sigma_2 = -X_1 + X_2 \end{aligned}\right\} \tag{5-20}$$

where $f_i(\sigma_i)$ (i=1,2) satisfies the "sector condition" shown in Fig. 5.2 and is written as

$$0 \leq \sigma_i f_i(\sigma_i) \leq K_i \sigma_i^2 \quad , \quad K_i > 0 \quad (i=1,2) \tag{5-21}$$

We have

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} , \quad C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} , \quad A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$G(s) = -C(sI - A)^{-1}B$$

$$= \begin{bmatrix} \dfrac{s+1}{s^2+s+1} & -\dfrac{s}{s^2+s+1} \\ -\dfrac{s+1}{s^2+s+1} & \dfrac{s}{s^2+s+1} \end{bmatrix} \tag{5-22}$$

$$K = \mathrm{diag}(K_i^{-1}) \quad , \quad i=1,2 .$$

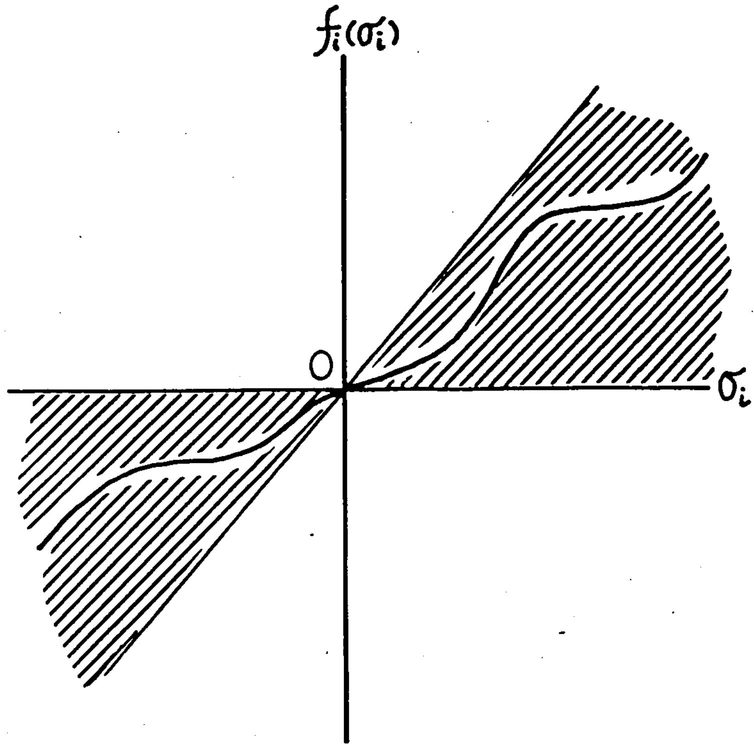


Fig. 5.2 Sector Condition

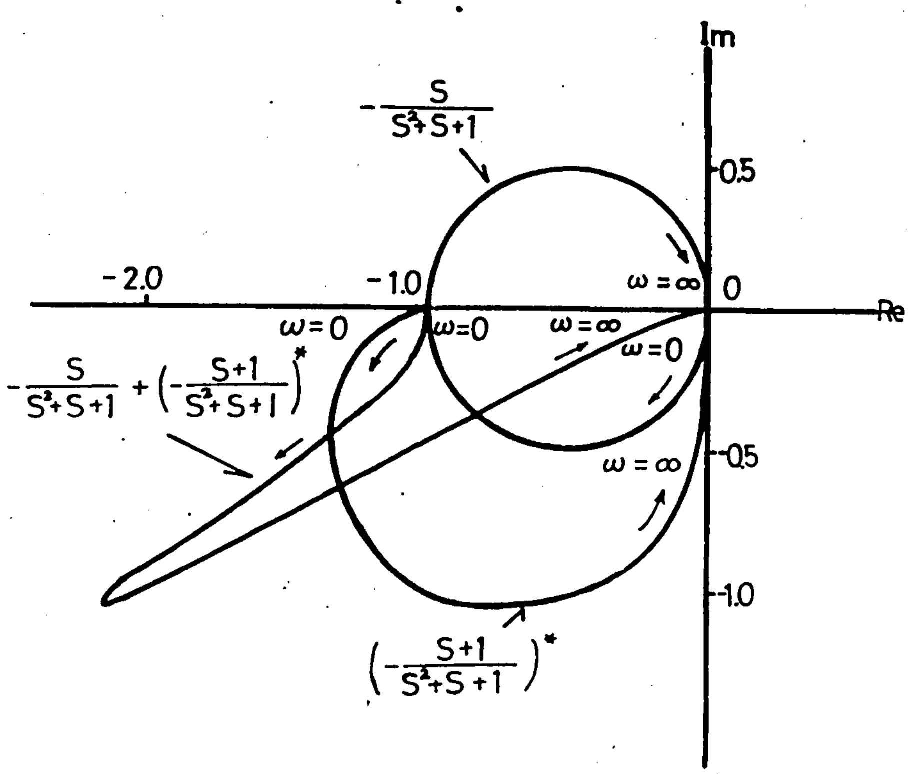


Fig. 5.3 Vector Loci of the Offdiagonal Elements of the Matrix (5-22)

Fig. 5.3 shows the vector loci of the two offdiagonal elements of G(s) and the locus of the sum of them. The vector locus of the diagonal element is not shown, because it is nothing but the sign-changed one of the diagonal element. As readily been found by these pictures, $W(\omega) \triangleq \{w_{ij}(\omega)\} = [G(j\omega) + K] + [G(j\omega) + K]^*$ does not satisfy the condition (2).
But the matrix W^o corresponding to G^o in Theorem 4.1 is given by

$$W^o = \mathrm{diag}(\, w_{ii}^o \,) + \mathrm{offdiag}(\, w_{ij}^o \,)$$

where

$$w_{ii}^o = \inf_{\omega \geq 0} w_{ii}(\omega) \;, \quad i=1,2$$

$$w_{ij}^o = -\sup_{\omega \geq 0} \left| w_{ij}(j\omega) \right| \quad ; \quad i \neq j \quad ; \; i,j=1,2$$

and this matrix is obtainale from these pictures. We have in this case

$$w_{11}^o = 2K_1 \quad , \qquad w_{22}^o = 2K_2$$

$$w_{12}^o = w_{21}^o = -\frac{5}{2} \quad .$$

Therefore, by Theorem 4.1 we get the stability condition such that

$$K_1 K_2 > \frac{25}{16} \tag{5-23}$$

Lastly, we show Theorem 4.3 could be used for checking positive-definiteness in the preliminary stages.

Example 3

Let us consider the system described by

$$\dot{X}_1 = -X_1 + X_2 - \frac{0.6X_1 - 0.4X_1 \sin t}{(1 + X_1^2)^2} K_1 \tag{5-24}$$

$$\dot{X}_2 = -X_1 - \frac{X_1K_2}{(1 + X_1^2)^2} - \frac{X_2K_2}{(1 + X_2^2)^2} \quad , K_1, K_2 > 0 \qquad (5\text{-}24)$$

when K_1 and K_2 equal to 1, this system is the same that was given by Partovi and Nahi as an example of their theorem. Taking $\sigma_1 = X_1$ and $\sigma_2 = X_1 + X_2$, we have

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^{\dagger}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$G(s) = -C(sI - A)^{-1}B$$

$$= \begin{bmatrix} \frac{s}{s^2 + s + 1} & \frac{1}{s^2 + s + 1} \\ \frac{s - 1}{s^2 + s + 1} & \frac{s + 2}{s^2 + s + 1} \end{bmatrix}^{\dagger\dagger} \qquad (5\text{-}25)$$

and

$$0 \leq \frac{f_1(\sigma_1, t)}{\sigma_1} \leq K_1 \quad , \quad 0 \leq \frac{f_2(\sigma_2, t)}{\sigma_2} \leq K_2$$

where

$$f_1(\sigma_1, t) = \frac{0.6X_1 - 0.4X_2 \sin t}{(1 + X_1^2)^2} K_1$$

$$f_2(\sigma_2, t) = \frac{X_1 + X_2}{(1 + X_1^2)^2} K_2 \qquad (5\text{-}26)$$

By Partovi and Nahi's theorem, the system (5-24) is ASIL, if the matrix

$$W(\omega) = \left[G(j\omega) + K\right] + \left[G(j\omega) + K\right]^* \qquad (5\text{-}27)$$

is positive definite for all ω. Here, $G(j\omega)$ is given

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† In the Partovi and Nahi's paper (51), the matrix B was written as B = I. But this is not valid.

†† There, the second diagonal element of G(s) was also miscalculated as $\frac{s + 1}{s^2 + s + 1}$.

by (5-25) and K is by

$$K = \mathrm{diag}(\ K_i^{-1}\ ) \quad , \quad i=1,2$$

Now, let us put K_1 and K_2 as $K_1=1$ and $K_2=7$, respectively. According to Theorem 4.3, $\mathrm{Re}W(\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$ is a necessary condition for $W(\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$. We remark the second diagonal element of G(s). The real part of it is given by

$$\mathrm{Re}\ g_{22}(j\omega) = \frac{2 - \omega^2}{\omega^4 - \omega^2 + 1}$$

By simple manipulations, we obtain

$$\mathrm{Re}\ g_{22}(j\omega) \geq 1 - \frac{2}{3}\sqrt{3} = -\ 0.1546\cdots$$

Then, we have

$$\mathrm{Re}\ W_{22}(\omega) = 2(\mathrm{Re}\ g_{22}(j\omega) + 1/7) \geq -\ 0.0237\cdots$$

and the condition $\mathrm{Re}W(\omega) \in$ p.d, $\forall \omega \in \mathbb{R}$ is not satisfied. Thus, it has turned out that the system given by (5-24) where $K_1=1$ and $K_2=7$ cannot be proved to be ASIL by Partovi and Nahi's theorem.

Chapter 6 Stability of Large Scale Systems Containing Unstable Subsystems

Section 6.1 Introduction

Frequency domain stability criteria of large scale systems so far obtained have such a significant feature that they can be applied on the basis only of the output responses of the systems, not necessary of the explicit expressions of the system equations. The theorems in chapter 3 have also an analogous feature, if the parameters of the imaginay systems could be obtained. To obtain these parameters seems to be possible, if, for example, subsystems are described by simple linear equations. On the other hand, algebraic methods of analyzing large scale systems assume that the knowledges of the mathematical constructions of the systems should previously be given. Therefore, for algebraic approaches attention has been focused on relaxing the assumptions on the properties of subsystems and the interconnecting relations between them and on obtaining the stability conditions for systems with a variety of interconnecting relations. With respect to the assumptions on the properties of subsystems, exponential stability, asymptotic stability and exponential instability have been assumed in turn.

In this chapter, we deduce the stability conditions of large scale systems containing unstable subsystems. A new class of assumptions for the interconnecting relations are put in the theorems of this chapter. Grujić and Šiljak (20) also derived the stability conditions of large scale systems with stable and unstable subsystems under analogous assumptions. In the last section of this chapter, a comparison between the obtained results and Grujić and Šiljak's theorems will be made by giving some illustrative examples.

Section 6.2 System Equations and Assumptions

Let us consider large scale systems described by (1-6). We write here again the equations (1-6) where inputs to the systems are put 0 as follows

$$\dot{x}_i = f_i(x_i, t) + g_i(x, t), \quad i \in M \tag{6-1}$$

Both the functions f_i and g_i satisfy

$$f_i(0_{n_i}, t) \equiv g_i(0_n, t) \equiv 0_{n_i}, \quad t \in R \tag{6-2}$$

The i-th isolated subsystem is described by

$$\dot{x}_i = f_i(x_i, t) \tag{6-3}$$

For each subsystem (6-3), we assume that there exist non-negative fuctions $v_i(x_i, t)$, $\phi_i(|x_i|)$, and positive constants c_{i1} and c_{i2} satisfying the inequalities

$$c_{i1}\phi_i(|x_i|) \le v_i(x_i, t) \le c_{i2}\phi_i(|x_i|) \tag{6-4}$$

where $\phi_i(r)$ is a scalar monotonously increasing function satisfying $\phi_i(0) = 0$, $\phi_i(r) \to \infty (r \to \infty)$. We assume further that the subsystems are classified into two groups ; one is composed of stable (exponentially stable) subsystems and is written as S, and the other is of unstable (exponentially unstable) subsystems and written as U. The number of subsystems belonging to S and U are assumed to be ℓ ($\ell \le m$) and $m-\ell$, respectively. The derivatives with respect to time t of the function v_i along the solutions of the equation (6-3) are assumed to be evaluated as

$$\left.\begin{aligned} -c_{i3}\phi_i(|x_i|) \le \dot{v}_i(x_i, t) \le -c_{i4}\phi_i(|x_i|), \quad \forall i \in S \\ c_{i4}\phi_i(|x_i|) \le \dot{v}_i(x_i, t) \le c_{i3}\phi_i(|x_i|), \quad \forall i \in U \end{aligned}\right\} \qquad (6\text{-}5)$$

where $0 < c_{i4} \le c_{i3}$ and $S \cup U = M = \{1, 2, \cdots, m\}$.

The assumptions for the interconnecting relations are as follows ; there exist real-valued functions $P_{ij}(x, t)$ and $Q_{ij}(x, t)$ such that

$$Q_{ii}(x, t)\phi_i(|x_i|) + \sum_{j \ne i} Q_{ij}(x, t)\phi_j(|x_j|) \le (\nabla v_i)' \cdot g_i(x, t) \le P_{ii}(x, t)\phi_i(|x_i|) + \sum_{j \ne i} P_{ij}(x, t)\phi_j(|x_j|),$$

$$\forall i \in S, U \qquad \cdots\cdots\cdots\cdots \qquad (6\text{-}6)$$

where

$$Q_{ij}(x, t) \le P_{ij}(x, t), \quad x \in R^m, \quad t \in R, \quad \forall i, j \in M \qquad (6\text{-}7)$$

and v_i satisfies (6-4) and (6-5). We put the lower bound of P_{ij} as p_{ij} and the upper bound of Q_{ij} as q_{ij}. That is,

$$\inf_{\substack{x \in R^m \\ t \in R}} P_{ij}(x, t) = p_{ij}, \quad \sup_{\substack{x \in R^m \\ t \in R}} Q_{ij}(x, t) = q_{ij} \qquad (6\text{-}8)$$

$$\forall i, j \in M$$

From (6-7), it follws that

$$q_{ij} \le p_{ij}, \quad \forall i, j \in M \qquad (6\text{-}9)$$

From (6-6) we have the inequalities

$$\sum_{j=1}^{m} q_{ij}\phi_j(|x_j|) \le (\nabla v_i)' \cdot g_i(x, t) \le \sum_{j=1}^{m} p_{ij}(|x_j|), \qquad (6\text{-}10)$$

$$\forall i \in S \cup U = M$$

The above mentioned are the all assumptions put here to the systems. In this chapter, we also call the systems which satisfy these assumptions Σ . For the sequel discussions we will give some inequalities with respect to v_i.

Let us differentiate v_i with respect to t for some i such that $i \in S$ along the solutions of (6-1). We obtain from (6-4), (6-5) and (6-10),

$$\dot{v}_i \leq -c_{i4}\phi_i(|x_i|) + p_{ii}\phi_i(|x_i|) + \sum_{\substack{j=1 \\ \neq i}}^{m} p_{ij}\phi_j(|x_j|)$$

$$\leq \frac{-c_{i4} + p_{ii}}{c_{ik}} v_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{p_{ij}}{c_{jk}} v_j \quad , \quad i \in S \tag{6-11}$$

where the subscript k is given by

$$k \triangleq \begin{cases} 1 \quad , \quad \text{for } -c_{i4}+p_{ii} > 0 \text{ or } p_{ij} > 0 \ (i \neq j) \\ 2 \quad , \quad \text{for } -c_{i4}+p_{ii} < 0 \text{ or } p_{ij} < 0 \ (i \neq j) \end{cases} \tag{6-12}$$

In the same way, for v_i, $i \in U$ we have

$$\dot{v}_i \geq \frac{c_{i4} + q_{ii}}{c_{ik}} v_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{q_{ij}}{c_{jk}} v_j \tag{6-13}$$

where

$$k \triangleq \begin{cases} 2 \quad , \quad \text{for } c_{i4}+q_{ii} > 0 \text{ or } q_{ij} > 0 \ (i \neq j) \\ 1 \quad , \quad \text{for } c_{i4}+q_{ii} < 0 \text{ or } q_{ij} < 0 \ (i \neq j) \end{cases} \tag{6-14}$$

Moreover, using again (6-4), (6-5) and (6-10), we have the inequalities with the reversed inequality signs in (6-11) and (6-13) such as

$$\dot{v}_i \geq \frac{-c_{i3} + q_{ii}}{c_{ik}} v_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{q_{ij}}{c_{jk}} v_j \quad , \quad i \in S \tag{6-15}$$

$$\dot{\mathcal{V}}_i \leq \frac{c_{i3} + p_{ii}}{c_{ik}} \mathcal{V}_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{p_{ij}}{c_{jk}} \mathcal{V}_j \quad , \quad i \in U \qquad (6\text{-}16)$$

Here, the denominator c_{ik} of the fractional expressions of the type a/c_{ik} takes the value as follows ;

$$c_{ik} \triangleq \begin{cases} c_{i2} & , \quad a > 0 \quad , \quad \text{for } i \in S \\ c_{i1} & , \quad a < 0 \quad , \quad \text{for } i \in S \end{cases}$$
$$c_{ik} \triangleq \begin{cases} c_{i1} & , \quad a > 0 \quad , \quad \text{for } i \in U \\ c_{i2} & , \quad a < 0 \quad , \quad \text{for } i \in U \end{cases} \qquad (6\text{-}17)$$

Multiplying by a positive number k_i the both sides of (6-11), we have

$$k_i \dot{\mathcal{V}}_i \leq \frac{-c_{i4} + p_{ii}}{c_{ik}} k_i \mathcal{V}_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{p_{ij}}{c_{jk}} k_i \mathcal{V}_j \quad , \quad i \in S \qquad (6\text{-}18)$$

and (6-13) by a negative number $-k_i$ $(k_i > 0)$

$$-k_i \dot{\mathcal{V}}_i \leq - \frac{c_{i4} + q_{ii}}{c_{ik}} k_i \mathcal{V}_i - \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{q_{ij}}{c_{jk}} k_i \mathcal{V}_j \quad , \quad i \in U \qquad (6\text{-}19)$$

In the same way, multiplying the both sides of (6-15) by a negative $-h_i$ $(h_i > 0)$ and (6-16) by a positive h_i yield

$$-h_i \dot{\mathcal{V}}_i \leq - \frac{-c_{i3} + q_{ii}}{c_{ik}} h_i \mathcal{V}_i - \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{q_{ij}}{c_{jk}} h_j \mathcal{V}_j \quad , \quad i \in S \qquad (6\text{-}20)$$

$$h_i \dot{\mathcal{V}}_i \leq \frac{c_{i3} + p_{ii}}{c_{ik}} h_i \mathcal{V}_i + \sum_{\substack{j=1 \\ \neq i}}^{m} \frac{p_{ij}}{c_{jk}} h_i \mathcal{V}_j \quad , \quad i \in U \qquad (6\text{-}21)$$

Section 6.3 Some Preliminary Lemmas

This section deals with some preliminary lemmas required for deriving the main results. The first lemma is a special case of Lemma 2.1.

Lemma 6.1

The solutions of scalar differentail inequality $\dot{X} \leq aX$, $X(0)=X_0$ satisfies $X \leq e^{at} X_0$, $t \geq 0$.

The proof of this lemma is omitted here because of its simplicity.

Lemma 6.2

Let $v^{(1)}$ and $v^{(2)}$ be the vectors with the elements $v_i(t)$ of order ℓ and $m-\ell$, respectively, and $v_i(t)$ is a non-negative scalar function of t. Moreover, let $h^{(1)}$ and $k^{(1)}$; $h^{(2)}$ and $k^{(2)}$ be the vectors of the same order as $v^{(1)}$ and $v^{(2)}$ with positive elements. We assume for the values defined by

$$\begin{aligned} h_m^{(2)} &= \min (h_{\ell+1}^{(2)} , h_{\ell+2}^{(2)} , \cdots , h_m^{(2)}) \\ h_M^{(2)} &= \max (h_1^{(2)}, h_2^{(2)}, \cdots , h_\ell^{(2)}) \\ k_m^{(1)} &= \min (k_1^{(1)} , k_2^{(1)}, \cdots , k_\ell^{(1)}) \\ k_M^{(1)} &= \max (k_{\ell+1}^{(1)} , k_{\ell+2}^{(1)} , \cdots , k_m^{(1)}) \end{aligned} \qquad (6\text{-}22)$$

that the following inequality

$$k_m^{(1)} h_m^{(2)} > k_M^{(2)} h_M^{(1)} \qquad (6\text{-}23)$$

is satisfied. Then if the relations

$$\begin{aligned} h^{(2)\prime} \cdot v^{(2)} - h^{(1)\prime} \cdot v^{(1)} &\leq N_1 e^{\mu t} , \quad \text{for } t \geq 0 \\ k^{(1)\prime} \cdot v^{(1)} - k^{(2)\prime} \cdot v^{(2)} &\leq N_2 e , \quad \text{for } t \geq 0 \end{aligned} \qquad (6\text{-}24)$$

where μ, λ, N_1 and N_2 are real constants and $\lambda, \mu > 0$, then there exist positive constants L and α such that

$$\upsilon_i \leq L\, e^{-\alpha t}, \qquad t \geq 0, \quad i \in M \tag{6-25}$$

Proof

We first show that under the condition (6-23) and (6-24) both N_1 and N_2 in (6-24) can not be non-positive. Let us assume the contrary ; that is, $N_1 \leq 0$, $N_2 \leq 0$ or $N_1 = N_2 = 0$. From (6-24) we have

$$h^{(2)\prime} \cdot \upsilon^{(2)} - h^{(1)\prime} \cdot \upsilon^{(1)} \leq 0$$

$$k^{(1)\prime} \cdot \upsilon^{(1)} - k^{(2)\prime} \cdot \upsilon^{(2)} \leq 0$$

From these relations and (6-22), it follows that

$$h_m^{(2)} \sum^{(2)} \upsilon_i - h_M^{(1)} \sum^{(1)} \upsilon_i \leq 0$$

$$k_m^{(1)} \sum^{(1)} \upsilon_i - k_M^{(2)} \sum^{(2)} \upsilon_i \leq 0$$

where $\sum^{(1)}$ and $\sum^{(2)}$ denote the summmations over the all elements of $\upsilon^{(1)}$ and $\upsilon^{(2)}$. Dividing the both sides of the first inequality by $h_m^{(2)}$ and of the second by $k_M^{(2)}$ and adding them, we get

$$\left(\frac{k_m^{(1)}}{k_M^{(2)}} - \frac{h_M^{(1)}}{h_m^{(2)}} \right) \sum^{(1)} \upsilon_i \leq 0 \tag{6-26}$$

Since $\sum^{(1)} \upsilon_i$ is non-positive, we obtain the following inequality.

$$k_m^{(1)} h_m^{(2)} \leq h_M^{(1)} k_M^{(2)}$$

This contradicts the assumption (6-23). Therefore, we assume $N_1 > 0$, $N_2 > 0$ in the sequel. If either N_1 or N_2 is negative, we have the same discussions as the following by putting it to 0.

By analogous arguments to those in deriving (6-26), the following inequality can be obtained.

$$\left(\frac{k_m^{(1)}}{k_M^{(2)}} - \frac{h_M^{(1)}}{h_M^{(2)}}\right)\sum^{(1)} \mathcal{V}_i \le \frac{N_1}{h_m^{(2)}} e^{\mu t} + \frac{N_2}{k_M^{(2)}} e^{\lambda t} \quad , \quad t \ge 0 \qquad (6\text{-}27)$$

As the parenthesized part of the above inequality is positive by (6-23), we can assert the existence of positive constants $\tilde{L}$ and $\tilde{\alpha}$ such that

$$\sum^{(1)} \mathcal{V}_i \le \tilde{L}\, e^{-\tilde{\alpha} t}$$

Since $\sum^{(2)} \mathcal{V}_i$ is evaluated as

$$\sum^{(2)} \mathcal{V}_i \le \frac{h_M^{(1)}}{h_m^{(2)}} \sum^{(1)} \mathcal{V}_i + \frac{N_1}{h_m^{(2)}} e^{\mu t} ,$$

for some positive $\tilde{\tilde{L}}$ and $\tilde{\tilde{\alpha}}$ the following inequality also holds.

$$\sum^{(2)} \mathcal{V}_i \le \tilde{\tilde{L}}\, e^{-\tilde{\tilde{\alpha}} t}$$

The function $\mathcal{V}_i$ being non-negative, thus the existence of L and α satisfying (6-25) has been proved.

Q.E.D.

Lemma 6.3

Let us assume the same condition as in Lemma 6.2 and assume that the following inequality holds in place of (6-23)

$$k_m^{(1)}\, h_m^{(2)} < k_M^{(2)}\, h_M^{(1)} \qquad (6\text{-}28)$$

and that the constants λ , μ in (6-24) are positive.
If N_1 and N_2 in (6-24) can be chosen negative numbers, there exist positive numbers L and α satisfying

$$\sum^{(j)} v_i \geq L\ e^{at} \quad , \quad j=1,2 \tag{6-29}$$

Proof

As in the case of lemma 6.2, we have

$$\begin{aligned} h_m^{(2)} \sum^{(2)} v_i - h_M^{(1)} \sum^{(1)} v_i &\leq N_1\ e^{\mu t} \\ k_m^{(1)} \sum^{(1)} v_i - k_M^{(2)} \sum^{(2)} v_i &\leq N_2\ e^{\lambda t} \end{aligned} \tag{6-30}$$

From these inequalities, (6-27) can be deduced. As the parenthesized part of (6-27) is, in this case, negative by (6-28), the evaluation of (6-29) for j=1 is obtained by dividing the both sides of (6-27) by the value of this part. As to $\sum^{(2)} v_i$, from the second inequality of (6-30)

$$\sum^{(2)} v_i \geq \frac{k_m^{(1)}}{k_M^{(2)}} \sum^{(1)} v_i - \frac{N_2}{k_M^{(2)}} e^{\lambda t}$$

holds. And from this, the conclusion for j=2 is evident.

Q.E.D.

Section 6.4 Stability Theorems and Instability Theorem

We can assume without loss of generality as $S = \{1,2,\cdots,\ell\}$, $U = \{\ell+1, \ell+2, \cdots, m\}$ by rearranging the order of subsystems in the sequel discussions. Let us define a matrix A composed of the coefficients of (6-11) and (6-13) as follows

$$A = \begin{bmatrix} A_1 & A_3 \\ A_4 & A_2 \end{bmatrix} \tag{6-31}$$

where the order of A_1, A_2, A_3 and A_4 are such that

A_1 ; $\ell \times \ell$, A_2 ; $(m-\ell)\times(m-\ell)$, A_3 ; $\ell\times(m-\ell)$, A_4 ; $(m-\ell)\times\ell$,

Here, A_i(i=1,2,3,4) are given by

$$\left.\begin{aligned}
A_1 &\triangleq \mathrm{diag}\left(\frac{-c_{i4}+p_{ii}}{c_{ik}}\right) + \mathrm{offdiag}\left(\frac{p_{ij}}{c_{jk}}\right) \\
A_2 &\triangleq \mathrm{diag}\left(\frac{c_{\ell+i,4}+q_{\ell+i,\ell+i}}{c}\right) + \mathrm{offdiag}\left(\frac{q_{\ell+i,\ell+j}}{c_{\ell+j,k}}\right) \\
A_3 &\triangleq \left\{\frac{p_{i,\ell+j}}{c_{\ell+j,k}}\right\} \\
A_4 &\triangleq \left\{\frac{q_{\ell+i,j}}{c_{jk}}\right\}
\end{aligned}\right\} \qquad (6\text{-}32)$$

where the subscript k of the denominator c_{ik} of the elements of matrices A_1 and A_3 is defined by (6-12) and of A_2, A_4 by (6-14). Analogously, for the inequalities (6-15) and (6-16) we define a coefficient matrix B as follows.

$$B = \begin{bmatrix} B_1 & B_3 \\ B_4 & B_2 \end{bmatrix} \qquad (6\text{-}33)$$

where B_1 ; $\ell\times\ell$, B_2 ; $(m-\ell)\times(m-\ell)$, B_3 ; $\ell\times(m-\ell)$, B_4 ; $(m-\ell)\times\ell$. Each matrix $B_i (i=1,2,3,4)$ is defined as

$$\left.\begin{aligned}
B_1 &\triangleq \mathrm{diag}\left(\frac{-c_{i3}+q_{ii}}{c_{ik}}\right) + \mathrm{offdiag}\left(\frac{q_{ij}}{c_{jk}}\right) \\
B_2 &\triangleq \mathrm{diag}\left(\frac{c_{\ell+i,3}+p_{\ell+i,\ell+j}}{c_{\ell+i,k}}\right) + \mathrm{offdiag}\left(\frac{p_{\ell+i,\ell+j}}{c_{\ell+j,k}}\right) \\
B_3 &\triangleq \left\{\frac{q_{i,\ell+j}}{c_{\ell+j,k}}\right\} \\
B_4 &\triangleq \left\{\frac{p_{\ell+i,j}}{c_{jk}}\right\}
\end{aligned}\right\} \qquad (6\text{-}34)$$

where the subscript of c_{ik} in B_1 and B_3 is defined by the first half of (6-17) and B_2 and B_4 by the latter half.

We also define a square matrix C of the same order as A and B in the following way.

$$C = \begin{bmatrix} A_1 & A_3 \\ B_4 & B_2 \end{bmatrix} \qquad (6\text{-}35)$$

where A_1, A_3, B_2 and B_4 are defined by (6-32) and (6-34). Then we have the following theorem for stability of the system Σ .

Theorem 6.1

If the matrix C', the transposed matrix of C, has at least one real characteristic root with negative sign and the elements of the corresponding characteristic vector have the same sign, the system Σ is ESIL.

Proof

Let a vector V be such that

$$V \triangleq (v^{(1)'}, v^{(2)'})' = (v_1, v_2, \cdots, v_m)' .$$

Then from (6-11), (6-16), (6-32), (6-34) and (6-35), we have

$$\dot{V}\Big|_{\Sigma} \leq CV \tag{6-36}$$

where C is given by (6-35) and $|_{\Sigma}$ denotes that the derivatives are taken along the solutions of Σ . By the assumption of the theorem, C' has a characteristic root λ (< 0) and a corresponding characteristic vector $k = (k_1, k_2, \cdots, k_m)'$, $k_i > 0$, $i \in M$ such that

$$C' k = \lambda k$$

that is,

$$\lambda k' = k' C \tag{6-37}$$

For a scalar non-negative function $\nu = k' \cdot V$, (6-36) and (6-37) yields

$$\dot{\nu}\Big|_{\Sigma} \leq \lambda \nu \tag{6-38}$$

By Lemma 6.1 we obtain

$$\nu \leq \nu_0 e^{\lambda t}, \quad t \geq 0, \quad \nu_0 = \nu(0) \tag{6-39}$$

As ν is the sum of the positively-multiplied Lyapunov function for each subsystem, it can be considered to be a Lyapunov function of the overall system. Therefore, (6-39) shows asymptotic stability in the large of the system. The proof for ESIL can be given in the similar way as in Theorem 2,1 and is omitted.

Q.E.D.

As a special case of Theorem 6.1, let us consider the case where all the offdiagonal elements of the matrix C are non-negative. In this case the upper bound p_{ij} of $P_{ij}(x, t)$ satisfies

$$p_{ij} \geq 0, \quad \forall i \in S, U, \quad i \neq j$$

We have the following theorem.

Theorem 6.2

Assume that all the offdiagonal elements of the matrix C given by (6-35) are non-negative. If the principal minors of -C of any order are positive, the system Σ is ESIL.

Proof

Note first by the property 1° of Proposition 4.3 the assumption of the theorem is equivalent to the condition that the matrix -C is in the class $\mathbb{K}$, that is, the class of M-matrices. We start from the inequalities (6-36). By the definition of the class $\mathbb{K}$, if -C is in $\mathbb{K}$ $-C'$ is also in $\mathbb{K}$. Therefore, by the property 3° of Proposition 4.3 there exists a vector $k > 0$ such that $-C' \cdot k > 0$. Hence, $k' \cdot C < 0$.

Then from (6-36) we have

$$\nu\Big|_{\Sigma} = k'\cdot V\Big|_{\Sigma} \leq k'\cdot CV < 0$$

Thus, ASIL of Σ is proved as in the case of Theorem 6.1. The proof for ESIL is also omitted.

Q.E.D.

Remark

Theorem 6.2 is already reported in more general form in (20). There, instead of (6-36) the following inequality was derived.

$$V\Big|_{\Sigma} \leq C\Phi \quad , \quad \Phi = (\phi_1, \phi_2, \cdots, \phi_m)'$$

where ϕ_i ($i \in M$) is a appropriate comparison function. But in this theorem the conclusion should be ASIL, not ESIL.

Under the condition that the offdiagonal elements are all non-negative, Theorem 6.1 is included by Theorem 6.2. Because from property 3° of Proposition 4.3 and the assumption of Theorem 6.1, $-C'\cdot k = -\lambda k > 0$ leads to $-C \in K$. It is evident that the converse is not true. Grujić and Šiljak pointed out in (20) the supposition that the offdiagonal elements might be negative should be improper so that it turns out to make the value of the Lyapunov function for some subsystem negative. This can be simply explained by the example such that the solution $v_1(t)$ of the inequality

$$\dot{v}_1 \leq av_1 + bv_2 \quad , \quad b < 0$$

may be negative when started from $v_1(0)=0$, $v_2(0) > 0$. But if $v_1=0$ implies $v_2=0$ invariably, it might not be the case just mentioned. We show an example of it in the later section.

Now, let us discuss the case where $\dot{v}_i$ can be evaluated below and above such as in (6-11) to (6-17). Then, we have the following theorem.

Theorem 6.3

Assume that both A' and B', the transposed matrices of A and B given respectively by (6-31) and (6-32), (6-33) and (6-34), have at least one negative characteristic root and a corresponding characteristic vector $\mathbf{k}$ and $\mathbf{h}$ written in the form of

$$\begin{aligned} \mathbf{k} &= (\mathbf{k}^{(1)\prime}, \mathbf{k}^{(2)\prime})' \\ \mathbf{h} &= (\mathbf{h}^{(1)\prime}, \mathbf{h}^{(2)\prime})' \end{aligned} \qquad (6\text{-}40)$$

where

$$\mathbf{k}^{(1)} = (k_1^{(1)}, k_2^{(1)}, \cdots, k^{(1)})', \quad \mathbf{k}^{(2)} = (k_{\ell+1}^{(2)}, k_{\ell+2}^{(2)}, \cdots, k_m^{(2)})'$$

$$\mathbf{h}^{(1)} = (h_1^{(1)}, h_2^{(1)}, \cdots, h_\ell^{(1)})', \quad \mathbf{h}^{(2)} = (h_{\ell+1}^{(2)}, h_{\ell+2}^{(2)}, \cdots, h_m^{(2)})'$$

Here, the signs of the elements of the vector $\mathbf{k}$ and $\mathbf{h}$ are specified as

$$\left.\begin{aligned} \text{sgn } k_1^{(1)} &= \text{sgn } k_i^{(1)} \neq 0 \\ \text{sgn } h_1^{(1)} &= \text{sgn } h_i^{(1)} = 0 \end{aligned}\right\} \quad i=2,3,\cdots,\ell$$

$$\left.\begin{aligned} \text{sgn } k_{\ell+1}^{(2)} &= \text{sgn } k_1^{(1)} = \text{sgn } k_i^{(2)} \\ \text{sgn } h_{\ell+1}^{(2)} &= \text{sgn } h_1^{(1)} = \text{sgn } h_i^{(2)} \end{aligned}\right\} \quad i=\ell+1, \ell+2, \cdots, m \qquad (6\text{-}41)$$

If the inequality (6-23) is satisfied for the values defined by (6-22), then the system Σ_1 is ESIL.

Proof -

If we could get the inequalities (6-24) under the assumptions of the theorem, the conclusion of the theorem can be obtained by Lemma 6.2. Without loss of generality we can take the sign of the elements of $\mathbf{k}^{(1)}$ and $\mathbf{h}^{(1)}$ positive and put

$$-\mathbf{k}^{(2)} = (-k^{(2)}_{\ell+1}, -k^{(2)}_{\ell+2}, \cdots, -k^{(2)}_{m})'$$

$$-\mathbf{h}^{(2)} = (-h^{(2)}_{\ell+1}, -h^{(2)}_{\ell+2}, \cdots, -h^{(2)}_{m})'$$

where $h_i > 0$ and $k_i > 0$, $i=\ell+1, \ell+2, \cdots, m$.

Now we choose the elements of the characteristic vector $\mathbf{k}$ and $\mathbf{h}$ as positive number k_i's and h_i's in (6-18) to (6-21). Summing up the inequalities (6-18) over $i \in S$, we have

$$\mathbf{k}^{(1)'} \cdot \dot{\mathbf{v}}^{(1)} \leq \mathbf{k}^{(1)'} \cdot [A_1 \ , \ A_3] \begin{pmatrix} \mathbf{v}^{(1)} \\ \mathbf{v}^{(2)} \end{pmatrix} \tag{6-42}$$

Also for the inequalities (6-19), we obtain

$$-\mathbf{k}^{(2)'} \cdot \dot{\mathbf{v}}^{(2)} \leq -\mathbf{k}^{(2)'} \cdot [A_4 \ , \ A_2] \begin{pmatrix} \mathbf{v}^{(1)} \\ \mathbf{v}^{(2)} \end{pmatrix} \tag{6-43}$$

Adding (6-42) and (6-43) leads to

$$\mathbf{k}' \cdot \dot{V}\Big|_{\Sigma} \leq \mathbf{k}' \cdot AV \tag{6-44}$$

Where A is given by (6-31). By the assumption of the Theorem 6.3 the matrix A' has a characteristic root λ (< 0) and a corresponding characteristic vector $\mathbf{k}$ such that

$$A'\mathbf{k} = \lambda \mathbf{k}$$

that is,

$$\mathbf{k}'A = \lambda \mathbf{k}'$$

Substituting this into (6-44), we have

$$\boldsymbol{k}' \cdot \dot{V} \Big|_{\Sigma} \leq \lambda \boldsymbol{k}' \cdot V$$

Therefore, by Lemma 6.1 it follows that

$$\boldsymbol{k}' \cdot V \leq \boldsymbol{k}' \cdot V_0 \, e^{\lambda(t-t_0)} \quad , \qquad V_0 = V(t_0)$$

Thus we obtain the latter half of (6-24). The first half is obtained in the same manner.

Q.E.D.

Now, we will give instability theorem for $\sum$ as follows.

Theorem 6.4

Let us assume that both the transposed matrix A' of A given by (6-31) and B' given by (6-33) have a positive characteristic root and the corresponding characteristic vector $\boldsymbol{k}$, $\boldsymbol{h}$ given by (6-40) and (6-41). If by taking some initial values such that

$$\boldsymbol{k}' \cdot V_0 < 0 \quad , \qquad \boldsymbol{h}' \cdot V_0 < 0 \quad , \qquad V_0 = V(t_0) \qquad (6\text{-}45)$$

the inequality (6-28) is satisfied, then the equilibrium state of the system $\sum$ is unstable.

Proof

As seen in the proof of Theorem 6.3, N_1 and N_2 in Lemma 6.3 correspond to $\boldsymbol{k}' \cdot V_0$ and $\boldsymbol{h}' \cdot V_0$, respectively and are negative by (6-45). Therefore, by Lemma 6.3 there exist positive constants L and α such that

$$\sum^{(j)} \mathcal{V}_i \geq L \, e^{\alpha t} \quad , \qquad j = 1,2$$

Since $\mathcal{V}_i$ is a non-negative function, this inequality means that at least one v_i tends to ∞ as t to ∞ . Then, by (6-4) we have $\phi_i(|x_i|) \to \infty$, that is, $|x_i| \to \infty$.

Q.E.D.

Section 6.5 Examples

In this section, we give an example for each theorem in the previous section and make some remarks relating to the results of Grujić and Šiljak. In the first place, the application of Theorem 6.1 will be given below.

Example 1

Let the system equations be such that

$$\left.\begin{aligned} \dot{x}_1 &= A_1 x_1 + g_1(x, t) \\ \dot{x}_2 &= A_2 x_2 + g_2(x, t) \end{aligned}\right\} \qquad (6\text{-}46)$$

where $x_1 = (x_{11}, x_{12})'$, $x_2 = (x_{21}, x_{22})'$, $x = (x_1', x_2')'$

$$A_1 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix}$$

and

$$g_1(x, t) = \begin{pmatrix} -(|x_{21}| + |x_{22}|) \operatorname{sgn} x_{11} \\ -(|x_{21}| + |x_{22}|) \operatorname{sgn} x_{12} \end{pmatrix}$$

$$g_2(x, t) = \begin{pmatrix} (2 + \sin t) \operatorname{sat} x_{11} \\ (2 + \sin t) \operatorname{sat} x_{22} \end{pmatrix}$$

Here, the function sat x is defined as

$$\text{sat } x \triangleq \begin{cases} x , & |x| < 1 \\ 1 , & |x| \geq 1 \end{cases}$$

We take subsystem S_i (i=1,2) as

$$S_1 \;;\quad \dot{x}_1 = A_1 x_1$$

$$S_2 \;;\quad \dot{x}_2 = A_2 x_2$$

and choose the functions v_i (i=1,2) as $v_i = (x_{i1}^2 + x_{i2}^2)^{1/2}$ (i=1,2). Then, we have

$$\nabla v_i = v_i^{-1} x_i \quad , \quad i=1,2$$

and

$$\dot{v}_1 \leq -4\, v_1 \quad , \quad \text{for } S_1$$

$$\dot{v}_2 \leq -2\, v_2 \quad , \quad \text{for } S_2$$

and

$$\begin{aligned}(\nabla v_1)' \cdot g_1 &= - v_1^{-1} (|x_{11}| + |x_{12}|)(|x_{21}| + |x_{22}|) \\ &\leq - v_1^{-1} (x_{11}^2 + x_{12}^2)^{1/2} (x_{21}^2 + x_{22}^2)^{1/2} \\ &= - v_2\end{aligned}$$

$$(\nabla v_2)' \cdot g_2 \leq v_2^{-1} (x_{11}x_{21} + x_{12}x_{22}) \leq v_1$$

From the above relations, the following inequalities can be obtained.

$$\dot{v} \leq A v$$

where

$$A = \begin{bmatrix} -4 & -1 \\ 1 & -2 \end{bmatrix}$$

Taking the matrix C in Theorem 6.1 as the aobve A and noting that A has characteristic root -3 and the corresponding characteristic vector (t,t)' where t is any real number, A satisfies the assumption of Theorem 6.1 and the sysyem is proved to be

ESIL. Since the matrix C has negative offdiagonal element, Theorem 6.2 can not be applied to this system.

The following example is an application of Theorem 6.3.

Example 2

Consider the system composed of two subsystems S_1 and S_2 that are described by

$$\begin{aligned} S_1 \; &; \quad \dot{x}_1 = -6\,x_1 \\ S_2 \; &; \quad \dot{x}_2 = x_2 \end{aligned} \tag{6-47}$$

The interconnecting functions between them are written as

$$g_1(x,\ t) = \begin{pmatrix} g_{11}(x,\ t) \\ g_{22}(x,\ t) \end{pmatrix} \triangleq \frac{1}{2}\left\{3+2\sqrt{2}+(2\sqrt{2}-3)\sin\ t\right\}\sqrt{x_{21}^2+x_{22}^2}\begin{pmatrix} \mathrm{sgn}\ x_{11} \\ \mathrm{sgn}\ x_{22} \end{pmatrix}$$

$$g_2(x,\ t) = \begin{pmatrix} g_{21}(x,\ t) \\ g_{22}(x,\ t) \end{pmatrix} \tag{6-48}$$

where

$$g_{21}(x,\ t) \triangleq -\frac{9+3\sin\ t}{2}\,x_{21} + \frac{4t^2+3}{t^2+1}\left(|x_{11}|+|x_{12}|\right)\ \mathrm{sgn}\ x_{21}$$

$$g_{22}(x,\ t) \triangleq -\frac{9+3\sin\ t}{2}\,x_{22} + \frac{4t^2+3}{t^2+1}\left(|x_{11}|+|x_{12}|\right)\ \mathrm{sgn}\ x_{22}$$

Putting $v_i = (x_{i1}^2 + x_{i2}^2)^{1/2}$, i=1,2, we have $\nabla v_i = v_i^{-1} x_i$, i=1,2 and

$$(\nabla v_1)' \cdot g_1 = \frac{1}{2}\{3+2\sqrt{2}+(2\sqrt{2}-3)\sin\ t\}(|x_{11}|+|x_{12}|)$$

$$(\nabla v_2)' \cdot g_2 = -\left(\frac{9+3\sin\ t}{2}\right)(x_{21}^2 + x_{22}^2)^{1/2}$$

$$+\frac{4t^2+3}{t^2+1}(x_{21}^2 + x_{22}^2)^{1/2}(|x_{11}|+|x_{12}|)(|x_{21}|+|x_{22}|).$$

From these relations it follows that

$$3 v_2 \leq (\nabla v_1)' \cdot g_1 \leq 4 v_2$$

$$3 v_1 - 6 v_2 \leq (\nabla v_2)' \cdot g_2 \leq 4 v_1 - 3 v_2 .$$

Since $\dot{v}_1 = -6 v_1$, $v_2 = v_2$, the following inequalities hold.

$$-6 v_1 + 3 v_2 \leq \dot{v}_1 \leq -6 v_1 + 4 v_2$$

$$3 v_1 - 5 v_2 \leq \dot{v}_2 \leq 4 v_1 - 2 v_2 .$$

This system can not be proved to be stable by Theorem 6.2, though the matrix C in the theorem is given by

$$C = \begin{bmatrix} -6 & 4 \\ 4 & -2 \end{bmatrix}$$

and the offdiagonal elements of this matrix are positive. For, as readily be checked, det(-C) is negative. Therefore, we will apply Theorem 6.3 to this system. The matrix A and B in this theorem are

$$A = \begin{bmatrix} -6 & 4 \\ 3 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

The matrix A' has a characteristic root -9 and a corresponding characteristic vector $(t_1, -t_1)'$ and B' has a characteristic root -8 and a vector $(2t_2, -t_2)'$. Here, t_1 and t_2 are any non-zero real numbers. Apparently, the elements of these vectors satisfy the condition (6-23) and all the assumptions of Theorem 6.3 are met. Thus, the system is proved to be ESIL by Theorem 6.3.

Finally, we give an example of Theorem 6.4.

Example 3

Let the matrices A and B of (6-31), (6-33) have the form of

$$A = \begin{bmatrix} 11 & -6 \\ 6 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 12 & -6 \\ 5 & -1 \end{bmatrix} \tag{6-49}$$

Then, A' has a characteristic root 5 and a corresponding characteristic vector $(t_1, -t_1)'$ and B' has a characteristic root 2 and a vector $(t_2, -2t_2)'$. Therefore, if we could choose the initial states of the system such that (6-45) is satisfied, this system can be proved to be unstable by Theorem 6.4.

For a comparison with Grujić and Šiljak's instability theorem, we show the outline of their results below.

In (20) two sufficient conditions for instability were given. Namely, a comparison inequality is derived as

$$\dot{v}\Big|_{\Sigma} \geq -AW \quad , \qquad W = (\phi_1, \phi_2, \cdots, \phi_m)' \tag{6-50}$$

where ϕ_i $(i \in M)$ is some comparison function, and instability condition is as either of the following ;

1° $-A \in \mathbb{K}$

2° For the matrix $A = \{a_{ij}\}$, there exists a row i such that

$$a_{ij} \begin{cases} < 0 & , \ i=j \\ \leq 0 & , \ i \neq j \ , \quad j = 1,2,\cdots,\ell-1,\ell+1,\cdots,m \end{cases}$$

In case of example 3, from (6-47), the inequalities corresponding to (6-48) become

$$\dot{v} \geq Dv$$

where

$$D = \begin{bmatrix} 12 & -6 \\ 6 & -1 \end{bmatrix}$$

This matrix satisfies neither the above condition.

It can be thus shown that there exist systems which can be proved to be stable or unstable by the theorems in this chapter and meanwhile are not proved by previously obtained theorem. It should be noted that Theorems 6.1 to 6.4 were derived without employing the comparison principle. Consequently, the existence of non-positive offdiagonal elements of the coefficient matrix of the comparison inequalities are allowed. Especially Theorem 6.3 was obtained in the way of utilizing more information on systems, that is, the lower bound of the derivative of v_i. All the theorems except Theorem 6.2 requires to get signs of a characteristic root and of the elements of the corresponding characteristic vector of a matrix for checking stability or instability. The manipulation for this is not seem to be easy one when the order of the matrix increases. However, we could rewrite the conditions of the theorem in more tractable form, if more restrictions were put to the systems as in case of Theorem 6.2.

Chapter 7 Concluding Remarks

Main results are summarized and some further works are summarized, in this chapter.

Stability theorems for various kinds of large scale systems have been studied under the assumption that the system were decomposed into stable or unstable subsystems. Studies were carried out by two representative ways, that is, frequency domain method and time domain method. The stability conditions in time domain method usually have an algebraic form.

In chapter 2 systems with time-varying interconnecting relations between subsystems were treated by the modified vector Lyapünov function method. In section 2.5 the obtained theorems were compared with the results previously obtained by Bailey and others. The theorems could give no better condition than those established before, when the "absolute value" of the interconnecting functions were evaluated linearly. However, the theorems could be applied even to systems with interconnections, the "absolute value" of which were not bounded linearly by the "absolute value" of the system states. Improvement of the condition of these theorems by adding more assumptions was shown in section 2.7. This could be performed by making use of much more information on systems ; namely periodicity of interconnection relations. The method employed throughout chapter 2 might well be said to be time domain method. On the other hand, frequency domain stability criteria of large scale systems were developed in chapter 3, where we derived stability criteria by the following procedures. First we constituted the comparison equations and the imaginary systems, the characteristics of which were subject to the comparison equations, from the properties of subsystems and the interconnections among them. Then, the extended Popov-type theorems established previously were applied to the imaginary systems. Thus, several frequency domain theorems were established according as the properties of subsystems and interconnectiong relations, including the case where each subsystem had dead time elements in its own feedback loop. In section 3.5 it was shown that the theorems obtained in this way became more useful by the aid of computers with a graphic display terminal.

In chapter 4, some conditions for positive definiteness and positive semidefiniteness of a matrix-valued function with argument $s=j\omega$ for any value of ω were considered. Under a certain condition we got the necessary and sufficient condition for positive definiteness or positive semidefiniteness in a very simple form. Whether it is the case where such a simple condition can be obtained or not can be checked mainly by drawing the vector loci of the elements of the matrix. In addition to the necessary and sufficient condition, a sufficient condition or a necessary one for positive definiteness or semidefiniteness was obtained separately in this chapter.

In chapter 5, the results of chapter 4 were used for comparing the theorems of chapter 3 with other theorems. For systems without the dead time elements, one of the theorems was comparable with those obtained by other authors and was shown to include some of them. In section 5.3 it was exhibited by some illustrative examples that the condition was also useful for applying frequency domain criteria of large scale systems by some other researchers.

In chapter 6, stability theorems of large scale systems including stable and unstable subsystems were established without the comparison principle. Instability theorem for these systems was also derived there. The obtained results were compared with Grujič and Šiljak's theorems. It was shown that there exist systems that were proved to be stable or unstable by the theorem in this chapter, but not by Grujič and Šiljak's.

Thus, we have obtained the means of investigating many kinds of large scale systéms. However, the following works seem to remain unsolved and to be investigated in the future.

1° The approaches to the problems of theis thesis were generally taken not so much from practical view points as from theoretical interest. So, to consolidate more tractable ways for applying the theorems to the actual systems is a future work to be done. In that case, considering large dimensions of the system, to make the most of computers with appropriate input-output devices should be necessary.

2° By taking advantages of the feature of systems, that is, employing much more information on systems, improvement of the stability condition could be anticipated. Stability theorems, therefore, might be established extensively according as the properties of the systems.

3° It may be interesting to deal with large scale systems where subsystems are coupled by interconnecting functions with retarded arguments, considering the phase-shifting effect of the interconnections.

4° To make more use of nonlinear characteristics included in the systems without simple linear evaluations is expected, though it may be very difficult.

Appendix

Appendix A. Stability Theorems of Large Scale Systems Previously Obtained by Other Authors

Some stability theorems of large scale systems previously obtained by other researchers are introduced for the purpose of comparison and reference. The theorems listed below are chosen from among those that are considered to be typical and and comparable with the theorems derived in this thesis, Symbols and notations used in these theorems are unified for the sake of convenience.

Let systems be written by (1-1) and (1-3), where vector function f_i and g_i satisfy the assumptions (1-2) and (1-3). The i-th isolated subsystem is expressed by (1-5). We call these systems Σ and discuss the stability of the equilibrium sate 0_n of Σ .

<u>Bailey's Theorem</u>[7]

Assume that for each subsystem described by (1-5) there exist a non-negative function V_i and positive constants c_{ij} (j=1,2,3, 4) satisfying (1-8). Namely, each subsystem is assured to be ESIL. Moreover, the interconnecting function $g_i(x , t)$ is expressed in a linear form of the states of subsystems as

$$g_i(x , t) = \sum_{i \neq j}^{m} C_{ij} x_j \tag{A-1}$$

where $C_{ij} \in \mathbb{R}^{n_i \times n_j}$ is a constant matrix. Then if a matrix $\bar{A}$ given by

$$\bar{A} = \text{diag}\left(-\frac{c_{i3}}{2c_{i2}}\right) + \text{offdiag}\left(\frac{c_{i4}^2 \sum_{j\neq i}^{m} \|C_j\|^2}{c_{i3}\, c_{j1}}\right) \tag{A-2}$$

is a Hurwitz matrix, the system $\sum$ is ESIL.

Araki's Theorem[2]

Assume that for each subsystem there exist a non-negative function $\mathcal{V}_i(\mathcal{X}_i, t)$ and positive constants c_{i3} and c_{i4} such that

$$\dot{\mathcal{V}}_i \Big|_{(1-5)} \leq -c_{i3}|\mathcal{X}_i|^2 \tag{A-3}$$

$$|\nabla \mathcal{V}_i| \leq c_{i4}|\mathcal{X}_i| \tag{A-4}$$

$$\mathcal{V}_i(\mathcal{X}_i, t) \rightarrow \infty \quad (|\mathcal{X}_i| \rightarrow \infty) \tag{A-5}$$

For the interconnecting gunction $\mathcal{G}_i$, the following inequality is assumed.

$$|\mathcal{G}_i(\mathcal{X}, t)| \leq \sum_{j\neq i}^{m} \varepsilon_{ij}|\mathcal{X}_j| \tag{A-6}$$

If the principal minors of any order of a matrix A given by

$$A = \text{diag}\left(\frac{c_{i3}}{c_{i4}}\right) + \text{offdiag}(-\varepsilon_{ij}) \tag{A-7}$$

are positive, then $\sum$ is ESIL. Moreover, when each subsystem is ESIL, that is, the inequaities (1-8) hold instead of (A-2) to (A-4), $\sum$ is ESIL.

Remark The above assumption with respect to the matrix A may be said in other words that A belongs to the class , i.e., the class of M-matrices. Some properties of these matrices are presented in Proposition 4.4 and 4.5 in chapter4 and Lemma 5.1 and 5.2 in chapter 5.

Šiljak's Theorem [59]

For each subsystem there exist a non-negative function $v_i(x_i.t)$ and positive constants $\eta_{ij}(j=1,2,3,4)$ such that

$$\eta_{i1}|x_i| \leq v_i(x_i,t) \leq \eta_{i2}|x_i| \qquad \text{(A-8)}$$

$$\dot{v}_i\Big|_{(1-5)} \leq -\eta_{i3}|x_i| \qquad \text{(A-8)}$$

$$|\nabla v_i| \leq \eta_{i4} \qquad \text{(A-10)}$$

The assumptions for interconnecting function $g_i(x,t)$ is the same as (A-5). If the principal minors of any order of a matrix B defined by

$$B = \text{diag}\left(\frac{\eta_{i3}}{\eta_{i2}}\right) + \text{offdiag}\left(-\frac{\eta_{i4}}{\eta_{j1}}\varepsilon_{ij}\right) \qquad \text{(A-11)}$$

is positive, the system Σ is ESIL.

Apparently, the assumption for the matrix B is also equivalent to the condition that B is in $\mathbb{K}$.

Appendix B Comparison Principle

Comparison principle is an important technique in the theorey of differential equations used for estimating a function satisfying a differential inequality by the external solutions, of the corresponding differential equation. In this appendix, some comparison theorems for differential equations and functional differential equations are introduced. We present first the theorem for differential inequalities.

Definition B.1

A vector function $f = (f_1, f_2, \cdots , f_m)'$ of a vector variable $x = (x_1, x_2, \cdots , x_m)'$ will be said to be of type K_1 with respect to x in a set S, if for each subscript i=1, 2,$\cdots$,m we have $f_i(a) \leq f_i(b)$ for any two points $a = (a_1, a_2, \cdots, a_m)'$ and $b = (b_1, b_2, \cdots, b_m)'$ in S with $a_i = b_i$ and $a_k \leq b_k$ ($k=1,2,\cdots,m$; $k \neq i$).

Theorem B.1

Let $f(x, t)$ be smooth and continuous for any arguments so that on an interval (a , b) the solution $x(t)$ of the differential eqation

$$\dot{x} = f(x , t) \tag{B-1}$$

is assured to be unique and continuous, and of type K_1 with respect to x for each fixed values of t. If $z(t)$ is continuous on (a , b), satisfies the differential inequality

$$\dot{z} \leq f(z , t) \tag{B-2}$$

and $z(a) \leq x(a)$, then $z(t) \leq x(t)$ for $a \leq t \leq b$.

Proof of this theorem is shown in (15).

Definition B.2

Let $f(x, y, t)$ be a vector function of vector variable $x \in R^m$, $y \in R^m$ and $t \in R$, continuous for any arguments. If the following conditions are satisfied, f is said to be type K_2.

1° For any y_1 and y_2 such that $y_1 \le y_2$, the inequality

$$f(x, y_1, t) \le f(x, y_2, t)$$

hold.

2° The function f is of type K_1 with respect to x.

Theorem B.2

Condsider the functional differential inequalities

$$\dot{z}(t) \le f(z(t), z(t-\tau), t), \quad t \ge 0, \quad \tau > 0 \tag{B-3}$$

and

$$\dot{y}(t) \ge f(y(t), y(t-\tau), t), \quad t \ge 0 \tag{B-4}$$

where f is of type K_2. If $z(t) \le y(t)$, $-\tau \le t \le 0$ holds, then $z(t) \le y(t)$, $t \ge 0$.

Theorem B.3

Let us assume the solution $x(t)$ of the functional differential equation

$$\dot{x}(t) = f(x(t), x(t-\tau), t) \tag{B-5}$$

exists and unique and the function f is of type K_2. If for the solution $z(t)$ of the inequality (B-3) $z(t) \le x(t)$ $-\tau \le t \le 0$ holds, then $z(t) \le x(t)$, $t \ge 0$.

For the case of $m = 1$, the proofs of the above theorems are given in (28). In this case the condition 1° of the definition B.2 is trivially satisfied. For $m \ge 2$, the proofs of the theorems can be given, for example, in an analogous manner to those given by Tokumaru et al. (67) and is omitted.

Appendix C. The Number of the Root with Positive Real Parts of the Equation $S + a + b\, e^{-\tau S} = 0$ (16)

In this appendix, we present the region of asymptotic stability in the space of the coefficients a and b for the trivial solution of the equation

$$x(t) + ax(t) + bx(t-\tau) = 0 \qquad \text{(C-1)}$$

where a, b, and τ are constants and $\tau > 0$. In this case the characteristic equation has the form

$$S + a + b\, e^{-\tau S} = 0 \qquad \text{(C-2)}$$

The stability condition is obtained as the condition for the absence of roots with positive real parts of the characteristic quasi-polynomial $\varphi(S) = S + a + b\, e^{-\tau S}$. Fig. C.1. shows the regions in the space of coefficients (a, b), having constant number of roots with positive real parts. In this figure, n denotes the number of roots with positive real parts. Therefore, the region corresponding to n=0 gives the stability ranges in this space. The equations for the line (1) and the straight line (2) are given respectively as follows

$$b = \frac{y}{\sin \tau y}, \qquad a = -\frac{y \cos \tau y}{\sin \tau y}.$$

where y is a parameter,

and $$a + b = 0 .$$

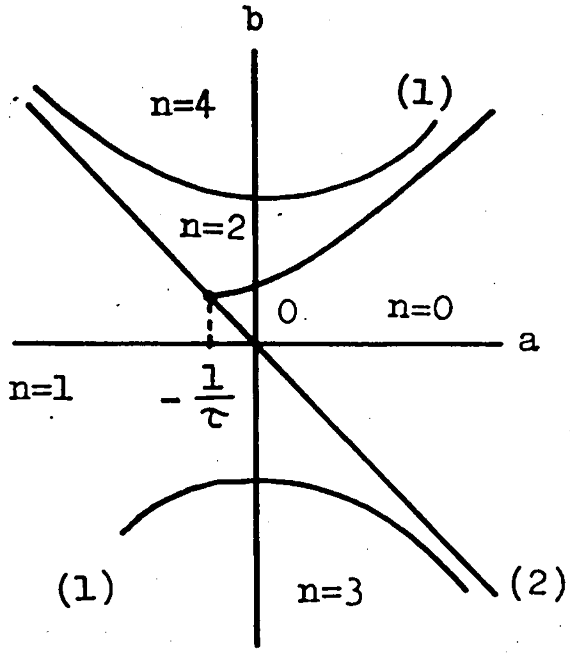

Fig. C.1 Parameter Plane Diagram

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