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Dynamic Characteristics of Neuron Models and Active Areas in Potential Functions

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There have been many researches of various neuron models that typically take the form of ordinary nonlinear differential equations of several dimensions. Nonlinear systems are usually able to display different dynamic behaviors depending on system parameters and an external input. The pattern of neuronal spiking is of great importance because it is believed that it codifies the information transmitted by neurons. These dynamics have been thoroughly investigated in each individual model based on the bifurcation theory, where we can discuss the characteristics around a critical point and the perturbation in the vicinity of equilibrium points. However, many important aspects of the situation are poorly understood and lack the satisfying universality of the structural stability discussion and the global dynamic behavior of the models.

We have proposed a concept of potential with active areas to discuss a global landscape for dynamics of various models [1],[2]. For examples, let us consider the following Hodgkin-Huxley type equations with three variables,

$$\tau_x\dot{x} = u + z - g(x), \qquad \tau_z\dot{z} = z_\infty(x) - z + \theta, \qquad \tau_u\dot{u} = u_\infty(x) - u, \tag{1}$$

where τ_x, τ_z, τ_u, and θ are time constants of x, z, u, and an external input, respectively. Equations (1) can be transformed into the following two-variable (x, u) system,

$$\ddot{x} + \eta(x)\dot{x} = -\frac{\partial U_2(x,u,\theta)}{\partial x}, \qquad \tau_u\dot{u} = u_\infty(x) - u, \tag{2}$$

$$\eta(x) = \frac{g'(x)}{\tau_x} + \frac{1}{\tau_z}, \quad \frac{\partial U_2(x,u,\theta)}{\partial x} = \frac{1}{\tau_x\tau_z}\left\{g(x) - z_\infty(x) - \frac{\tau_z}{\tau_u}u_\infty(x) - \theta - \left(1 - \frac{\tau_z}{\tau_u}\right)u\right\}, \tag{3}$$

where $\eta(x)$ is a damping factor, and $U_2(x, u, \theta)$ can be read as a potential function for variable x, however $U_2(x, u, \theta)$ varies with time because of $u = u(t)$.

Equations (1) can be also transformed into the following one-variable (x) system,

$$\dddot{x} + b_2(x)\ddot{x} + b_1(\dot{x}, x)\dot{x} = -\frac{\partial U_3(x,\theta)}{\partial x}, \tag{4}$$

$$b_2(x) = \frac{1}{\tau_x}\left\{g'(x) + \frac{\tau_x}{\tau_z} + \frac{\tau_x}{\tau_u}\right\} = \eta(x) + \frac{1}{\tau_u}, \quad \frac{\partial U_3(x,\theta)}{\partial x} = \frac{1}{\tau_x\tau_z\tau_u}\{g(x) - z_\infty(x) - u_\infty(x) - \theta\}, \tag{5}$$

$$b_1(\dot{x}, x) = \frac{g''(x)}{\tau_x}\dot{x} + \frac{1}{\tau_x}\left\{\frac{g'(x) - z'_\infty(x)}{\tau_z} + \frac{g'(x) - u'_\infty(x)}{\tau_u} + \frac{\tau_x}{\tau_z\tau_u}\right\}, \quad \frac{\partial^2 U_3(x,\theta)}{\partial x^2} = b_0(x), \tag{6}$$

$$\frac{\partial^2 U_3(x,\theta)}{\partial x^2} = \frac{1}{\tau_x\tau_z\tau_u}\{g'(x) - z'_\infty(x) - u'_\infty(x)\}, \quad \frac{\partial U_2(x,u,\theta)}{\partial x} = \tau_u\frac{\partial U_3(x,\theta)}{\partial x} + \left(1 - \frac{\tau_z}{\tau_u}\right)\frac{u_\infty(x) - u}{\tau_x\tau_z}, \tag{7}$$

where $b_0(x)$ is the curvature of the potential $U_3(x, \theta)$. A equilibrium point x_0 is stable when $b_2(x_0) > 0$, $b_1(\dot{x} = 0, x_0) > 0$, $b_0(x_0) > 0$, and $B_1(x_0) = b_2(x_0)b_1(\dot{x} = 0, x_0) - b_0(x_0) > 0$ according to the Hurwitz's theorem. We can define three active areas b_2-active area, b_1-active area, and B_1-active area where $b_2(x) < 0$, $b_1(\dot{x} = 0, x) < 0$, and $B_1(x) < 0$, respectively. We can discuss the global dynamics of neuron models in terms of the shape of the potential and the disposition of active areas in the potential.

For simplicity, let us assume the potential $U_3(x, \theta)$ has the following form,

$$U_3(x,\theta) = \frac{x^4 - 2\gamma x^2 - \theta x}{\tau_x\tau_z\tau_u}, \quad \frac{\partial U_3(x,\theta)}{\partial x} = \frac{4x(x^2 - \gamma) - \theta}{\tau_x\tau_z\tau_u}, \quad \frac{\partial^2 U_3(x,\theta)}{\partial x^2} = b_0(x) = \frac{12x^2 - 4\gamma}{\tau_x\tau_z\tau_u}, \tag{8}$$

$$\frac{\partial U_2(x,u,\theta)}{\partial x} = \frac{4x^3 - 4\gamma x - \theta}{\tau_x\tau_z} - \frac{1}{\tau_x\tau_z}\left(1 - \frac{\tau_z}{\tau_u}\right)\{u_\infty(x) - u\}, \tag{9}$$

where the potential $U_3(x, \theta)$ has a double or single well shape depending on the sign of γ, and hence there are three equilibrium points $x = 0$ and $x = \pm\sqrt{\gamma}$ when $\gamma > 0$ and $\theta = 0$.

In order to set arbitrary values for a center and a width of an active area independently each other, let us assume for Eqs. (5) and (6) that

$$b_2(x) = \frac{1}{\tau_x}\left\{(x - c_2)^2 - \beta_2\right\}, \quad b_1(\dot{x} = 0, x) = \frac{1}{\tau_x}\left\{(x - c_1)^2 - \beta_1\right\}, \quad B_1(x) = b_2(x)b_1(x) - b_0(x). \tag{10}$$

The center and width of b_2-active area are c_2 and $2\sqrt{\beta_2}$, respectively, and those of b_1-active area are c_1 and $2\sqrt{\beta_1}$, respectively. Then we obtain the following equations,

$$g'(x) = (x - c_2)^2 - \alpha, \quad \alpha = \beta_2 + \tau_x \left(\frac{1}{\tau_z} + \frac{1}{\tau_u} \right), \; g(x) = \frac{x^3}{3} - c_2 x^2 + (c_2^2 - \alpha) x, \tag{11}$$

$$z'_\infty(x) = \frac{-\tau_x \tau_u}{\tau_z - \tau_u} \left[b_2(x) - \tau_z b_1(x) + \tau_z^2 b_0(x) - \frac{1}{\tau_z} \right], \; u'_\infty(x) = \frac{\tau_x \tau_z}{\tau_z - \tau_u} \left[b_2(x) - \tau_u b_1(x) + \tau_u^2 b_0(x) - \frac{1}{\tau_u} \right], \tag{12}$$

(13)

$$z_\infty(x) = \frac{\tau_z \tau_u}{\tau_u - \tau_z} \left[\left(\frac{4}{\tau_u} + \frac{1 - \tau_z}{3\tau_z} \right) x^3 + \frac{c_1 \tau_z - c_2}{\tau_z} x^2 - \left\{ \frac{4\gamma}{\tau_u} - \frac{c_2^2 - \beta_2}{\tau_z} + c_1^2 - \beta_1 + \frac{\tau_x}{\tau_z^2} \right\} x \right], \tag{14}$$

(15)

$$u_\infty(x) = \frac{\tau_z \tau_u}{\tau_z - \tau_u} \left[\left(\frac{4}{\tau_z} + \frac{1 - \tau_u}{3\tau_u} \right) x^3 + \frac{c_1 \tau_u - c_2}{\tau_u} x^2 - \left\{ \frac{4\gamma}{\tau_z} - \frac{c_2^2 - \beta_2}{\tau_u} + c_1^2 - \beta_1 + \frac{\tau_x}{\tau_u^2} \right\} x \right], \tag{16}$$

$$U_2(x, u, \theta) = \frac{1}{\tau_x \tau_u} \left[\left(2\frac{\tau_u}{\tau_z} + \frac{1 - \tau_u}{12} \right) x^4 + \frac{c_1 \tau_u - c_2}{3} x^3 \right. \tag{17}$$

$$\left. - \frac{1}{2} \left\{ 8\gamma \frac{\tau_u}{\tau_z} - (c_2^2 - \beta_2) + \tau_u (c_1^2 - \beta_1) + \frac{\tau_x}{\tau_u} \right\} x^2 + \frac{\tau_u}{\tau_z} \left\{ \left(1 - \frac{\tau_z}{\tau_u} \right) u - \theta \right\} x \right], \tag{18}$$

where integral constants are zero. The model shows a burst oscillation when $c_2 = 0.3, \alpha = 2.2 (\beta_2 = 1.195), c_1 = 0, \beta_1 = 0.5, \gamma = 0.5, \tau_x = \tau_z = 2.0, \tau_u = 200$, and $\theta = 0$ with the initial condition $x(t = 0) = 0.01$, and $u(t = 0) = z(t = 0) = 0$, on the other hand it shows a chaotic oscillation when $c_2 = 0.55, \alpha = 0.5825 (\beta_2 = 0.02), c_1 = -0.45, \beta_1 = 0.08, \gamma = 0.56, \tau_x = 0.5, \tau_z = 1.0, \tau_u = 8.0$, and $\theta = 0$ with the same initial condition.

Figure 1(a) shows the burst output $x(t)$ on the contour map of time series of the potential function Eq. (18) with the active area where $\eta(x) < 0$. Figures 1(b) and 1(c) show bird's eye views of the burst oscillation shown in Fig. 1(a) and the chaotic oscillation on the potential $U_3(x, \theta = 0)$ with the active areas. In Fig. 1(a) black line, red line, blue line, and gray area denote the output, trajectories of the equilibria with a positive curvature, a trajectory of the equilibria with a negative curvature, and the active area, respectively. In Figs. 1(b) and 1(c), red lines, purple and blue areas denote the outputs, the b_2-active areas, and the b_1-active areas, respectively.

The characteristics of the b_1-active area is not explained just as a part which injects energy to a quasi-particle with characteristics depending on the third differential term, but they are explained in relation to the particle motion on the time dependent potential $U_2(x, u)$. The b_1-active area dominates the slow dynamics of the burst oscillation because of the large τ_u. On the other hand, the b_2-active area dominates the fast dynamics of the burst oscillation. The dynamic characteristics of neuron models including chaotic behaviors are connected with a shape of a potential and a disposition of active areas.

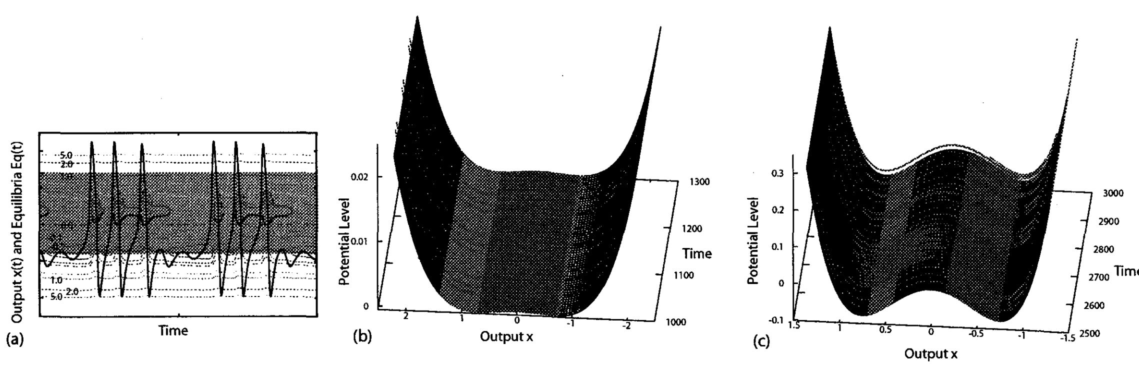

Figure 1: Burst and chaotic oscillations bound by the potential with the active areas

Conclusions

We have discussed the dynamic characteristics of a neuron model in terms of a concept based on a potential with active areas. Burst oscillations with undershoot occur depending on two active areas disposed in substantially overlapping each other on the potential. Chaotic properties appear depending on two active areas without overlapping. The global curvature of the potential ensures these oscillations not to diverge.

References

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