THEORY OF G-CATEGORIES TOWARD EQUIVARIANT ALGEBRAIC K-THEORY

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The notion of a G-category — a category with an action of a group G — was needed to make algebraic K-theory equivariant one. Though various notions have been used so far, the relations with them have not been explained explicitly yet. Beginning by introducing the notion of a G-category from point of view of Galois descent in linear categories, I deal comprehensively with various notions of G-categories and establish the comparison in the complete form. It is important for us to study simultaneously the limit categories together with G-categories and G-functors. The objects to appear in text are as follows.

G-category	G-functor	limit category
a category C with a G-descent datum	a morphism of Galois descent data	descended category $\Delta_H C$
a pseudo functor $\alpha : G \longrightarrow Cat$	a pseudo nat. transf. $G \overset{\longrightarrow}{\underset{\longrightarrow}{\downarrow}} Cat$	
a fibered category over G; $\gamma : D \to G$	a cartesian functor over G; $D \longrightarrow D'$, $\searrow\ \swarrow$ G	representation categ. $Cart_G(H,D)$ or $Cart_G(\underline{G/H},D)$
a lax functor	a lax nat. transf.	lax limit over G
a (strict) functor $\alpha : G \longrightarrow Cat$	a nat. transf. $G \overset{\longrightarrow}{\underset{\longrightarrow}{\downarrow}} Cat$	$\Delta_H \alpha(\cdot)$ or $\alpha(\cdot)^H$
an $O_G{}^{op}$-category $\beta : O_G{}^{op} \longrightarrow Cat$	a nat. transf. $O_G{}^{op} \overset{\longrightarrow}{\underset{\longrightarrow}{\downarrow}} Cat$	$\beta(G/H)$

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§1. Introduction: The notion of G-categories

In order to introduce the notion of G-categories *i.e.* categories on which the group G acts, I think, we are asked to fit it to the following problems. One of them is the problem of Galois descent. Let B/A be a Galois extension of rings (or a Galois covering $B \longrightarrow A$ of schemes) of Galois group G. I shall consider the notion of a linear category with a Galois descent datum of Galois group G originated in A.Grothendieck (see N.S.Rivano[18]). Let L be an A-linear category and L_B denotes the B-linear category deduced from L by extension of scalars from A to B. In the Galois case the usual datum of descent on L_B relative to B/A reduces to the following datum by using the isomorphism $B \otimes_A B \cong \prod_G B$.

For each $s \in G$ there is an equivalence of categories

$$\alpha_s : L_B \longrightarrow L_B$$

and for each pair $s, t \in G$ there is a natural isomorphism

$$a_{s,t} : \alpha_{st} \longrightarrow \alpha_s \circ \alpha_t$$

satisfying coherence conditions

$$(\alpha_s * a_{t,u}) \circ a_{s,tu} = (a_{s,t} * \alpha_u) \circ a_{st,u}$$

for any $s, t, u \in G$. Further the usual descended category can be rewritten in the following form by the Galois descent datum $(\alpha_s, a_{s,t})$ on L_B. The descended category $\Delta_G L_B$ has as objects the pairs $(X, (\lambda_s)_{s \in G})$ where X is an object of L_B and $\lambda_s : X \longrightarrow \alpha_s X$ is an isomorphism of L_B for each $s \in G$ such that

$$\lambda_e = id_X \quad (e = \text{the identity element of } G)$$
$$\alpha_s(\lambda_t)\bullet\lambda_s = (a_{s,t})_X \bullet \lambda_{st}$$

The morphisms of $\Delta_G L_B$ are defined to be morphisms of L_B commuting all λ_s. It is shown that the descended category $\Delta_G L_B$ is an A-linear category which is equivalent to the original L.([18])

Watching these data we find the fact that the descent situation may be formulated in the form independent of the ring extension B/A and of the linearity of the categories. So leaving theory of linear categories we interpret abstractly the descent data as the data toncerning any categories and any groups, and then we reach the notion of a category with a G-descent datum (or a G-category). This is the starting point of our theory of G-categories. However when we give the definition we had better normalize the data so that the equivalences and the natural isomorphisms corresponding to the identity element e of G are the identities.

Definition 1.1. Let G be a fixed group whose identity element is denoted by e and let C be a category. A datum $(\alpha_s, a_{s,t})$ is called a G-descent datum on C if for each element s of G α_s is an equivalence of categories

$$\alpha_s : C \longrightarrow C$$

and for each pair s,t of elements of G $a_{s,t}$ is a natural isomorphism

$$a_{s,t} : \alpha_{st} \longrightarrow \alpha_s \circ \alpha_t$$

satisfying

$$\alpha_e = Id_C \quad (= \text{the identity functor of } C)$$
$$a_{e,s} = id_{\alpha_s} = a_{s,e} \qquad (s \in G)$$
$$(\alpha_s * a_{t,u}) \circ a_{s,tu} = (a_{s,t} * \alpha_u) \circ a_{st,u} \qquad (s,\ t,\ u \in G)$$

where id_{α_s} denotes the identity natural transformation of the functor α_s.

Then we call $(C;\ \alpha_s, a_{s,t})$ a category with a G-descent datum or a G-category.

The notion of G-functors is obtained by applying the notion of morphisms of data of descent to the Galois descent case. Together with the notion of G-natural transformations between them as 2-arrows we have a 2-category denoted by $Des(G)$.

Definition 1.2. A G-functor of G-categories

$$(F, n_s)\ :\ (C;\alpha_s, a_{s,t}) \longrightarrow (C';\alpha'_s, a'_{s,t})$$

consists of a functor $F\ :\ C \longrightarrow C'$ of the underlying categories and a natural isomorphism for every $s \in G$

$$n_s\ :\ F \circ \alpha_s \longrightarrow \alpha'_s \circ F$$

such that

$$n_e = id_F$$

$$(a'_{s,t} * F) \circ n_{st} = (\alpha'_s * n_t) \circ (n_s * \alpha_t) \circ (F * a_{s,t}) \qquad (s, t \in G)$$

where id_F denots the identity morphism of the functor F.

A G-natural transformation of G-functors

$$t\ :\ (F, n_s) \longrightarrow (F', n'_s)$$

is defined to be a natural transformation $t\ :\ F \longrightarrow F'$ of functors satisfying the conditions

$$(\alpha'_s * t) \circ n_s = n'_s \circ (t * \alpha_s) \qquad (s \in G)$$

We can also define the descended category $\Delta_G C$ only from the datum in Definition 1.1 .

Definition 1.3. For a category with a G-descent datum (a G-category) $(C;\alpha_s, a_{s,t})$ the descended category $\Delta_G C$ is defined as follows; An object of $\Delta_G C$ is a pair $(X,\ (\lambda_s)_{s\in G})$ consisting of an object X of C and for every $s\in G$ an isomorphism of C

$$\lambda_s : X \longrightarrow \alpha_s X$$

such that

$$\lambda_e = id_X$$

$$\alpha_s(\lambda_t)\circ\lambda_s = (a_{s,t})_X\circ\lambda_{st} \qquad (s,t\in G)$$

A morphism of $\Delta_G C$

$$f : (X,(\lambda_s)) \longrightarrow (X',(\lambda'_s))$$

is a morphism $f : X \longrightarrow X'$ of C such that for any $s\in G$

$$\lambda'_s\circ f = \alpha_s(f)\circ\lambda_s \ .$$

Thus we have got the abstract formulation for Galois descent.

Another problem required for theory of G-categories is that of representation theory. For a ring A what relations are there between a category C of A-modules and a category $\tilde{C}$ of module over the group ring $A[G]$ of a group G over A ? Under certain a nice circumstance $\tilde{C}$ turns to be the functor category $Fun(G,C)$. On the other side when the G-descent datum $(\alpha_s, a_{s,t})$ of a G-category C is trivial say

$$\alpha_s = Id_C$$

for any $s\in G$, the descended category $\Delta_G C$ is equivalent to the functor category $Fun(G,C)$. Thus this second problem is reduced as a special case to the first problem about Galois descent. At the same time theory of G-categories gets some advantage from the techniques in theory of representations. For instance induction theory (= abstract formalism of representation theory) will be generalized to theory of G-categories for finite G in § 6.

The other problem is about an usual G-category C *i.e.* a group

G actig on C as a strict functor. We call it a split G-category in order to distinguish it from the notion of our G-category in which G acts on C as a pseudo functor. As a split G-category provides a G-space under the classfying space functor this notion of G-categories has been treated so far by many authors in equivariant algebraic K-theory.[5],[11],[19] *etc.*・・・・

I shall give in § 4. the procedure of constructing a split G-category from our (pseudo) G-category. This is carried out by using the Giraud construction [7] through the notion of fibered categories over G. (The resulting one is called the split version.) This construction makes G-categories the ones to which theory of G-spaces in algebraic topology can be used effectively. I must note that even though we are handling split G-categories from the first it is important to apply the above construction to them by thinking of them as our G-categories. Because the subject of theory of G-categories is not the relation of a G-category C with the H-fixed category C^H for a subgroup H of G, but it is that of C with the descended category $\Delta_H C$. I show also in § 4 that for C^{sp} the split version of C the both categories $\Delta_H C^{sp}$ and $(C^{sp})^H$ are equivalent to the original descended category $\Delta_H C$. The useful ness of such procedure is found for instance in the work of Shimakawa [19] about the construction of infinite G-deloopings of symmetric monoidal (split) G-categories.

I shall now give the organization of this paper. First we show in § 2 that the notions of our G-categories and G-functors are equivalent to those of fibered categories over G and cartesian functors over G. The idea of using fibered categories over G in equivariant algebraic K-theory is due to Fröhlich-Wall [6][26]. They used the term of stable G-graded categories. But I think that to formulate G-categories in theory of fibered categories is appropriate for the nature of the theory. Moreover when a fibered category D over G is associated with a G-category $(C;\alpha_s, a_{s,t})$ it is shown that the representation category $Rep(G,D)$ of Fröhlich-Wall (*ibid.*) is equivalent to our descended category $\Delta_G C$.

If the situation is exchanged to a subgroup H of G, the above representation categories (relative to H) become delicate to deal with. In particular this is the case when applied to induction theory. So I introduce in § 3 a new representation category which is natural. It is shown that this new representation category is equivalent to the old one. This result turns out to be important later. (§ 4, § 8)

G-categories could be interpreted as pseudo functors from the category G to the 2-category *Cat* of small categories. We get the notions of lax G-categories and split G-categories by replacing the pseudo functors by weaker lax functor and by stronger (strict) functors. After noting that the descended category of a G-category is equivalent to a lax limit over G, I investigate the relation of G-categories with split G-categories. The content is such as mentioned in the above third problem. (§ 4)

As a next topic we shall study G-categories with further structures. The equivariant version of a category with some structures (*e.g.* an exact G-category) goes through in the simple and natural form in our theory of G-categories. For a pseudo functor we may only exchange the target *Cat* with relevant 2-category. The inheritance of structure in question to the limit categories (*e.g.* the descended category) can be proven in a natural way. By way of illustration exact G-categories are explained in § 5. Fundamental process is similar for symmetric monoidal G-categories and simplicial G-categories which are not dealt with in this paper.

It is shown in § 6 that for finite G a fibered category over G induces a Mackey functor in representation theory. By means of the notion of the new representation category in § 3 our formulation such as Mackey property and projection formula becomes much simpler. This result generalizes the work of Dress-Kuku [3].

In § 7 we shall discuss about the connection between G-categories and O_G-categories. The latter is the other notion like a G-category. An O_G-category is the one given a category for each subgroup H of G in a compatible manner and it is a main question to rediscover them up to homotopy as the H-fixed category of a certain G-category. This shall be done parallel to the work of Elmendorf [4] about the relation between G-spaces and O_G-spaces. The classifying space of our construction U from O_G-categories to split G-categories reduces to the Elmendorf construction C.

In the last section we shall study adjoint relations between functors connecting various notions of G-categories. This resarch seems to become usefull for applications of G-categories.

Finally note that I establish definite relations between all G-categories which appeared or unappeared in papers subject to equivariant algebraic K-theory. Our theory of G-categories provides a comprehensive and natural treatment of categories with group actions.

It is expected that this theory is used not only for the equivariant theory of algebraic K-theory, but also for many fields pertaining to group actions on categories. But it is the point at issue to face a difficult problem called *homotopy limit problem* by R.Thomason [24] when we are going to use rich tools in algebraic topology. This is one of the most important questions following this paper in theory of G-categories.

Fix some notations. Let G be a group. We can regard G as a category, denoted the same letter G. The category G has only one object $\cdot$ and the morphisms of G are elements of the group G. For two elements s, t of G the composition of s with t is denoted by ts. The opposite category G^{op} in which the composition of s with t is st is equivalent to G by the correspondence $s \leftrightarrow s^{-1}$.

§ 2. Categories with G-descent data and fibered categories over G

Regarding a group G as a category G with only one object $\cdot$ and with elements of G as morphisms, we can think of a G-category (= a category with a G-descent datum) in § 1 as a pseudo functor from G to the 2-category Cat of small categories. Then by the classical relation between pseudo functors and fibered categories following Grothendieck (Cf. SGA[21]) a G-category provides a fibered category over G. The notion of fibered categories is fruitfull and becomes a key stone of the development of our theory. Further we show that G-functors are correspondent to cartesian functors of fibered categories. Since our interest is in descent theory, it is more important to establish the correspondence between those limit categories; the descended category of a G-category and the representation category of a fibered category over G. The notion of representation categories is a generalization of that of categories of representation modules and is due to F.Fröhlich-C.T.C.Wall [6].

Since a group is a groupoid (= a category whose morphisms are all isomorphisms) as a category, the isomorphism of categories between G and G^{op} given by $s \leftrightarrow s^{-1}$ gives us the transition of the discusion below to pseudo opfunctor and cofibered categories and all the arguments become equivalent.

Definition 2.1 (SGA [21]). For a category F over E, $\pi : F \longrightarrow E$ a morphism $m : X \longrightarrow Y$ of F is said to be a cartesian morphism if for any object Z over $\pi(X)$ the assigment $q \longmapsto m \circ q$ provides a bijection

$$\mathrm{Hom}_{F,id}(Z,X) \cong \mathrm{Hom}_{F,\pi(m)}(Z,Y)$$

where id denotes the identity morphism of $\pi(m)$ and

$$\mathrm{Hom}_{F,g}(X,Y) = \{ f \in \mathrm{Hom}_F(X,Y) \ ; \ \pi(f) = g \}$$

Note that every isomorphism is obviously a cartesian morphism.

A category F is prefibered over E if for any morphism $g : \xi \longrightarrow \eta$ of E and any object Y of F over η there are an object X of F over ξ and a cartesian morphism $f : X \longrightarrow Y$ of F over g.

F or $\pi : F \longrightarrow E$ is said to be a fibered category over E if it is prefibered over E and the compositions of cartesian morphisms are cartesian.

Definition 2.2. Given two fibered categories over E, $\pi : F \longrightarrow E$ and $\pi' : F' \longrightarrow E$, a functor $u : F \longrightarrow F'$ is said to be a cartesian functor over E if it is a functor over E *i.e.* $\pi = \pi' \circ u$ and it sends cartesian morphisms of F to cartesian morphisms of F'. Note that if F is a groupoid the latter condition is unneccesary.

$Cart_E(F, F')$ denotes the category of cartesian functors from F to F' over E and natural transformations $t : u \longrightarrow u'$ satisfying

$$\pi'(t_X) = id_{\pi(X)}$$

for any object X in F.

Given a fibered category $\pi : F \longrightarrow E$ and an object ξ of E the *fiber* $\pi^{-1}(\xi)$ of π at ξ is defined to be a category whose objects are objects X of F such that $\pi(X) = \xi$ and whose morphisms are morphisms f of F such that $\pi(f) = id_\xi$. Consider the correspondence which specifies the fiber $\pi^{-1}(\xi)$ for an object ξ of E and the functor $\pi^{-1}(\eta) \longrightarrow \pi^{-1}(\xi)$ determined by the prefiberedness of F for a morphism $\xi \longrightarrow \eta$ of E. Then it follows from the fiberedness of F that there is a natural isomorphism satisfying certain coherence conditions between that functor correspondent to the composition of morphisms of E and the composition of those functors. Thus we obtain a pseudo functor from E^{op} to *Cat*. (Such datum is called a cleavage in SGA[21].) Conversely the Grothendieck construction makes a fibered category over E from a pseudo functor $E^{op} \longrightarrow Cat$.

We will observe precisely on these details in the case of $E = G$.

Definition 2.3 (Fröhlich-Wall[6]). Take a fibered category $\gamma : D \longrightarrow G$ over G. A morphism f of D such that $\gamma(f) = s$ ($s \in G$) is called a morphism of grade s. *Ker D* denotes the unique fiber $\gamma^{-1}(\cdot)$ of γ. This category is equivalent to $Cart_G(1, D)$ where 1 denotes the ponctual category (= the category with only one object and one morphism). To be more precise the objects of *Ker D* are equal to the objects of *D* and the morphisms of *Ker D* are morphisms of *D* of grade e where e is the identity element of G. Let $(\gamma_s, c_{s,t})$ be the normalized cleavage defined by the fibered structure of γ. (*Cf*. [21]) Explicitly for every $s \in G$ there is an equivalence of categories

$$\gamma_s : Ker\ D \longrightarrow Ker\ D$$

such that $\gamma_e = Id_{Ker\ D}$ and for every pair $s,t \in G$ there is a natural isomorphism of grade e

$$c_{s,t} : \gamma_t \circ \gamma_s \longrightarrow \gamma_{st}$$

such that

$$c_{s,e} = id_{\gamma_s} = c_{e,s} \qquad (s \in G)$$

$$c_{s,tu} \circ (c_{t,u} * \gamma_s) = c_{st,u} \circ (\gamma_u * c_{s,t}) \qquad (s,t,u \in G)$$

A set of morphisms of transport $\{ \xi_{s,X} \}$ is given as follows. For an object X of D and $s \in G$ one has a cartesian morphism of grade s

$$\xi_{s,X} : \gamma_s X \longrightarrow X$$

corresponding to a morphism of grade e $id : \gamma_s X \longrightarrow \gamma_s X$. And they satisfy the following properties.

a) For any object X of D

$$\xi_{e,X} = id_X$$

b) For any morphism $v : X \longrightarrow X'$ of *Ker D* and $s \in G$

$$v \circ \xi_{s,X} = \xi_{s,X'} \circ \gamma_s(v)$$

c)For any object X of D and $s,t \in G$

$$\xi_{s,X} \circ \xi_{t,\gamma_s X} = \xi_{st,X} \circ (c_{s,t})_X$$

Note in general that for a category F over E a fibered structure determines a normalized cleavage and a set of morphisms of transport, and conversely that one of a normalized cleavage and a set of morphisms of transport determines the other and a fibered category structure. The correspondence between pseudo functors and fibered categories is well-known, but in order to describe the equivalence between limit categories cartesian morphisms $\xi_{s,X}$'s turn to be usefull.

Definition 2.4 (Fröhlich-Wall[6]). The limit category of a fibered category $\gamma : D \longrightarrow G$ is given by the category of cartesian sections ;

$$Rep\ D\ (= Rep(G,D)\ ) = Cart_G(G,D)$$

This is called the representation category of a fibered category D over C. The terminology comes from the following fact. When γ (or D) is trivial *i.e.* $\gamma = pr_1 : D = G \times C \longrightarrow G$,

$$Rep\ D \approx Fun(G,C)$$

where $Fun(G,C)$ is the functor category from G to C whose objects are G-representations in C.

Let us write down explicitly the category $Rep\ D$. An object of $Rep\ D$ is a pair (X, φ) where X is an object of D and φ is a group homomorphism $G \longrightarrow Aut_D X$ such that $\varphi(s)$ is an automorphism of grade s for any $s \in G$. A morphism $(X, \varphi) \longrightarrow (X', \varphi')$ of $Rep\ D$ is given by a morphism $f : X \longrightarrow X'$ of D of grade e such that

$$\varphi'(s)\circ f = f\circ\varphi(s) \qquad (s \in G)$$

Under those definitions we have

Theorem 2.5. (1) Let C be a category with a G-descent datum $(\alpha_s, a_{s,t})$ (= a G-category in §1). Then there is a fibered category D over G satisfying equivalences of categories

$$Ker\ D \approx C\ , \qquad Rep\ D \approx \Delta_G D$$

(2) If $\gamma : D \longrightarrow G$ is a fibered category over G then $Ker\ D$ has a

G-descent datum satisfying an equivalence

$$\Delta_G Ker\ D \approx Rep\ D$$

Proof. (1) We will construct a category D with the desired properties. Take $ob\ D = ob\ C$. For two objects X, Y

$$\mathrm{Hom}_D(X, Y) = \coprod_{s \in G} \mathrm{Hom}_{D,s}(X, Y)$$

$$\mathrm{Hom}_{D,s}(X, Y) = \mathrm{Hom}_C(X, \alpha_{s^{-1}} Y) \qquad \text{(the morphisms of grade } s\text{)}$$

The composition of morphisms

$$\mathrm{Hom}_{D,s}(X, Y) \times \mathrm{Hom}_{D,t}(Y, Z) \longrightarrow \mathrm{Hom}_{D,ts}(X, Z), \ (f, g) \longmapsto g \circ f$$

is defined as follows. For $f : X \longrightarrow \alpha_{s^{-1}} Y$ and $g : Y \longrightarrow \alpha_{t^{-1}} Z$

$$g \circ f : X \xrightarrow{f} \alpha_{s^{-1}} Y \xrightarrow{\alpha_{s^{-1}}(g)} \alpha_{s^{-1}} \alpha_{t^{-1}} Z \xrightarrow{(a_{s^{-1}, t^{-1}})^{-1}} \alpha_{s^{-1} t^{-1}} Z$$
$$= \alpha_{(ts)^{-1}} Z$$

Thus one has a category D over G.

Next $id \in \mathrm{Hom}_C(\alpha_{s^{-1}} X, \alpha_{s^{-1}} X)$ defines a morphism of grade s in D

$$\xi_{s,X} : \alpha_{s^{-1}} X \longrightarrow X$$

and it is easily shown that a set $\{ \xi_{s,X} \}$ satisfies the conditions similar to (a) ~ (c) which are satisfied by $\{ \xi_{s,X} \}$ in Definition 2.3.

So put

$$\gamma_s = \alpha_{s^{-1}}, \ c_{s,t} = (a_{t^{-1}, s^{-1}})^{-1}, \ \xi_{s,X} = \xi_{s,X}$$

then it is verified that $(\gamma_s, c_{s,t})$ is a normalized cleavage of D and $\{\xi_{s,X}\}$ is a set of morphisms of transport. It is clear that $G = Ker\ D$.

To construct an equivalence between $\Delta_G C$ and $Rep\ D$ take an object $(X, (\lambda_s))$ of $\Delta_G C$. For $s \in G$ put

$$\varphi(s) = \xi_{s,X} \circ \lambda_s{}^{-1}$$

then $\varphi(s)$ is an autmorphism of X of grade s and $\varphi : G \longrightarrow \mathrm{Aut}_D X$ is a group homomorphism as follows;

$$\begin{aligned}\varphi(st) &= \xi_{st,X} \circ \lambda_{(st)}{}^{-1}\\ &= \xi_{st,X} \circ (c_{s,t})_X \circ (c_{s,t})_X^{-1} \circ \lambda_{t^{-1}s^{-1}}\\ &= \xi_{s,X} \circ \xi_{t,\gamma_s X} \circ (a_{t^{-1},s^{-1}})_X \circ \lambda_{t^{-1}s^{-1}}\\ &= \xi_{s,X} \circ \xi_{t,\gamma_s X} \circ \alpha_{t^{-1}}(\lambda_{s^{-1}}) \circ \lambda_{t^{-1}}\\ &= \xi_{s,X} \circ \lambda_s{}^{-1} \circ \xi_{t,X} \circ \lambda_t{}^{-1}\\ &= \varphi(s) \circ \varphi(t)\end{aligned}$$

Thus we have an object (X, φ) of *Rep D*. As regards morphisms if $(X, (\lambda_s)), (X', (\lambda'_s))$ are objects of $\Delta_G C$ and $\upsilon : X \longrightarrow X'$ is a morphism of grade e such that

$$\lambda'_s \circ \upsilon = \alpha_s(\upsilon) \circ \lambda_s \qquad (s \in G),$$

putting

$$\varphi(s)\ (\textit{resp.}\ \varphi'(s)) = \xi_{s,X} \circ \lambda_s{}^{-1}\ (\textit{resp.}\ \xi_{s,X'} \circ \lambda'_s{}^{-1})$$

we have

$$\begin{aligned}\varphi'(s) \circ \upsilon &= \xi_{s,X'} \circ \lambda'_s{}^{-1} \circ \upsilon\\ &= \xi_{s,X'} \circ \gamma_s(\upsilon) \circ \lambda_s{}^{-1}\\ &= \upsilon \circ \xi_{s,X} \circ \lambda_s{}^{-1}\\ &= \upsilon \circ \varphi(s)\end{aligned}$$

Thus we obtain a functor $\Delta_G C \longrightarrow$ *Rep D*. It is possible to follow up the converse of the above construction and so we have an equivalence of categories

$$\textit{Rep } D \approx \Delta_G C$$

(2) Given a fibered category $\gamma : D \longrightarrow G$ with a normalized cleavage $(\gamma_s, c_{s,t})$ put

$$\alpha_s = \gamma_{s^{-1}}, \quad a_{s,t} = (c_{t^{-1}, s^{-1}})^{-1}$$

Then one has a G-descent datum $(\alpha_s, a_{s,t})$ on *Ker* D. In this case an equivalence between Δ_G*Ker* D and *Rep* D goes through as in (1). *q.e.d.*

We shall now show that G-functors correspond to cartesian functors under the correspondence between G-categories and fibered categories over G in Theorem 2.5.

Theorem 2.6. (1) Let $b : B \longrightarrow G$ a category over G. A cartesian functor

$$\mathcal{F} : (\gamma : D \longrightarrow G) \longrightarrow (\gamma' : D' \longrightarrow G)$$

of fibered categories over G induces naturally a functor

$$Cart_G(B, \mathcal{F}) : Cart_G(B, D) \longrightarrow Cart_G(B, D')$$

In particular we have natural functors

$$Ker\ \mathcal{F} : Ker\ D \longrightarrow Ker\ D' \quad \text{and}$$

$$Rep\ \mathcal{F} : Rep\ D \longrightarrow Rep\ D'$$

(2) When providig a G-descent datum on *Ker* D (*resp.* *Ker* D') by Theorem 2.5 (2), *Ker* $\mathcal{F}$ of (1) turns to be a G-functor of G-categories.

(3) Given a G-functor

$$F : (C; \alpha_s, a_{s,t}) \longrightarrow (C'; \alpha'_s, a'_{s,t})$$

of G-categories, F extends to a cartesian functor

$$F : (\gamma : D \longrightarrow G) \longrightarrow (\gamma' : D' \longrightarrow G)$$

where $\gamma : D \longrightarrow G$ (*resp.* $\gamma' : D' \longrightarrow G$) is a fibered category over G associated to $(C; \alpha_s, a_{s,t})$ (*resp.* $(C'; \alpha'_s, a'_{s,t})$) by Theorem 2.5.

Proof. (1) The functor $Cart_G(B, \mathcal{F})$ is given by

$$(B \xrightarrow{\rho} D) \longmapsto (B \xrightarrow{\mathcal{F} \circ \rho} D').$$

Take $B = 1$ (*resp.* G), and the result for *Ker* (*resp.* *Rep*) follows.

(2) We can write down the conditions of being a cartesian functor by

using the cleavages of fibered categories in theory of fibered categories (*Cf.* Gray [9]p33). This implies that the restriction of a cartesian functor to the fibers *Ker* satisfies the conditions of the definition of a G-functor.

(3) Let us construct a cartesian functor $\tilde{F} : D \longrightarrow D'$ from a G-functor $F : C \longrightarrow C'$. For an object X of D put $\tilde{F}X = FX$. If $m : X \longrightarrow Y$ is a morphism of grade s in D there is a morphism $n : X \longrightarrow \alpha_s{-1}Y$ of C. Let $\tilde{F}m : FX \longrightarrow \tilde{F}Y$ be a morphism of grade s of D' correspondent to the following composition in C'

$$FX \xrightarrow{Fn} F\circ\alpha_s{-1}\ Y \xrightarrow{(\eta_s{-1})_X} \alpha'_s{-1}\circ FY$$

It follows easily from the conditions of the definition of a G-functor and the fact used in (2) that the functor $\tilde{F}$ defined above becomes a cartesian functor. *q.e.d.*

Fib(G) denotes the 2-category of fibered categories and cartesian functors together with 2-arrows which are defined to correspond to 2-arrows of *Des*(G) in § 1 under the correspondence of Theorems 2.5, 2.6. Then *Fib*(G) is 2-equivalent to *Des*(G) and we may identify the two 2-categories.

§ 3. Change of groups

In this section I shall give the definitions and properties of the representation categories for a group H exchanged from G. At first we state the definition of $Rep(H, D)$ given by Fröhlich-Wall [6]. But this is inconvenient from the lack of functoriality. So we adopt a new definition of the representation categories

$$\overline{Rep}(H, D) = Cart_G(\underline{\underline{G/H}}, D)$$

For $H = G$, $\overline{Rep}(G, D) = Rep(G, D)$ and for general H there is an equivalence

$$\overline{Rep}(H, D) \approx Rep(H, D)$$

For $H = \{e\}$ this equivalence has the form

$$Ker\ D \approx Cart_G(\underline{\underline{G/e}}, D)$$

and plays an important role in the next section.

Lemma and Definition 3.1. (1) For a fibered category $\gamma : D \longrightarrow G$ over G with a normalized cleavage $(\gamma_s, c_{s,t})$ and a group homomorphism $h : H \longrightarrow G$ the category over H

$$pr_1 : H \times_G D \longrightarrow H$$

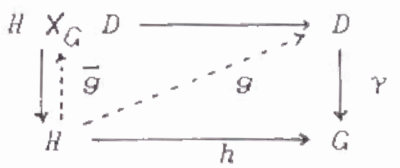
is a fibered category over H and

$$Rep(H \times_G D) \approx Cart_G(H, D)$$

holds. This category is denoted by $Rep(H, D)$.

(2) For a G-category $(C; \alpha_s, a_{s,t})$ and homomorphism $h : H \longrightarrow G$ $(C; \alpha_{h(s)}, a_{h(s),h(t)})$ is a H-category. This descended category is denoted by $\Delta_H C$.

Proof. (1) We shall discribe in term of cleavages. We may take $(\gamma_{h(s)}, c_{h(s),h(t)})$ as a normalized cleavage for $H \times_G D \longrightarrow H$. To show the latter category equivalence consider the diagram



The correspondence between functors

$$g : H \longrightarrow D \quad \text{and} \quad \bar{g} : H \longrightarrow H \times_G D$$

such that $\bar{g}(\cdot) = (\cdot,\ g(\cdot))$, $\bar{g}(s) = (s,\ g(s))$ gives the desired equivalence.

2) Trivial. *q.e.d.*

Theorems 2.5, 2.6 in the previous section can be immediately generalized to the present case.Let H be a subgroup of G.

Proposition 3.2. (1) When a G-category $(C;\ \alpha_s,\ a_{s,t})$ and a fibered category $D \longrightarrow G$ over G are under the correspondence in Theorem 2.5 there is an equivalence

$$\Delta_H C \approx Rep(H, D)$$

(2) A cartesian functor $F : D \longrightarrow D'$ over G of fibered categories over G (*resp.* a G-functor $F : C \longrightarrow C'$) induces functors in a natural way

$$Rep(H,\ F) : Rep(H,\ D) \longrightarrow Rep(H, D')$$

$$(\textit{resp.}\ \Delta_H F : \Delta_H C \longrightarrow \Delta_H C')$$

We shall here provide some notations of categories which occur from a group G. These categories play a central role from now on.

Definition 3.3. For a (left) G-set S the category $\underline{\underline{S}}$ has elements of S as objects and the morphisms of S from x to x' are elements a of G such that $ax = x'$; *i.e.* $ob\ \underline{\underline{S}} = S$, $mor\ \underline{\underline{S}} = G \times S$. There is a functor

$$\sigma : \underline{\underline{S}} \longrightarrow G$$

on objects $\sigma(x) = \cdot$ for $x \in S = ob\ \underline{\underline{S}}$

on morphisms $\sigma(x \xrightarrow{(a,x)} ax) = a$ for $x \in S$, $a \in G$.

Thus we have a category over G. (Not fibered!) We often use the categories $\underline{\underline{G/H}}$ for subgroups H of G. In special cases

$\underline{\underline{G/G}} = G$

$\underline{\underline{G/e}} = \cdot/G$ (= the comma category of G under $\cdot$)

The latter is also equal to the one called the translation category of G.

We use these notions to define a new representation category.

Lemma and Definition 3.4. For a fibered category $\gamma : D \longrightarrow G$ over G and a (left) G-set S there is an equivalence

$$Cart_G(\underline{\underline{S}}, D) \approx Cart\ Sec(\underline{\underline{S}} \times_G D)$$

where $Cart\ Sec(\underline{\underline{S}} \times_G D)$ is the category of cartesian sections of a fibered category $pr_1 : \underline{\underline{S}} \times_G D \longrightarrow \underline{\underline{S}}$. Define for $S = G/H$

$$\overline{Rep}(H, D) = Cart_G(\underline{\underline{G/H}}, D).$$

Proof. Similar to Lemma 3.1.

Since $\underline{\underline{G/G}} = G$ we have $Rep(G, D) = \overline{Rep}(G, D)$. In general we will show $Rep(H, D) \approx \overline{Rep}(H, D)$.

Theorem 3.5. For a fibered category $\gamma : D \longrightarrow G$ over G and a subgroup H of G there is an equivalence of categories

$$\begin{array}{ccc} Cart_G(\underline{\underline{G/H}}, D) & \approx & Cart_G(H, D) \\ \| & & \| \\ \overline{Rep}(H, D) & & Rep(H, D) \end{array}$$

Proof. We shall first define a canonical functor

$$\varphi : Cart_G(\underline{\underline{G/H}}, D) \longrightarrow Cart_G(H, D)$$

Assign on objects

$$\left(\begin{array}{ccc} \underline{\underline{G/H}} & \xrightarrow{k} & D \\ & \searrow \; G \; \swarrow & \end{array}\right) \stackrel{\varphi}{\longmapsto} \left(\begin{array}{ccc} H & \overset{\varphi(k)}{\dashrightarrow} & D \\ & \searrow \; G \; \swarrow & \end{array}\right)$$

$$\varphi(k) \;:\; \left(\begin{array}{c} \cdot \\ \downarrow h \\ \cdot \end{array}\right) \longmapsto \left(\begin{array}{c} k(eH) \\ \downarrow k(h, eH) \\ k(eH) \end{array}\right)$$

where $(h,\ eH) : eH \longrightarrow heH = eH$ for $h \in H$. Also on morphisms

$$\left(\underline{\underline{G/H}} \;\underset{k'}{\overset{k}{\underset{\downarrow\lambda}{\rightrightarrows}}}\; D\right) \stackrel{\varphi}{\longmapsto} \left(H \;\underset{\varphi(k')}{\overset{\varphi(k)}{\underset{\downarrow\varphi(\lambda)}{\rightrightarrows}}}\; D\right)$$

$$\varphi(\lambda) = \lambda_{eH} : k(eH) \longrightarrow k'(eH)$$

On the other side a functor

$$\psi : Cart_G(H,\ D) \longrightarrow Cart_G(\underline{\underline{G/H}},\ D)$$

is not canonical. This is determined for each choice of representatives $\{\ g_i\ \}_{i \in I}$ of G/H. Assign on objects

$$\left(\begin{array}{ccc} H & \xrightarrow{u} & D \\ & \searrow \; G \; \swarrow & \end{array}\right) \stackrel{\psi}{\longmapsto} \left(\begin{array}{ccc} \underline{\underline{G/H}} & \overset{\psi(u)}{\dashrightarrow} & D \\ & \searrow \; G \; \swarrow & \end{array}\right)$$

$$\psi(u) \;:\; \left(\begin{array}{c} g_iH \\ \downarrow g \\ g_jH \end{array}\right) \longmapsto \left(\begin{array}{c} \gamma_{g_i^{-1}}u(\cdot) \\ \vdots \\ \gamma_{g_j^{-1}}u(\cdot) \end{array}\right)$$

where $(g,\ g_iH) : g_iH \longrightarrow gg_iH = g_jH$, hence $g_j^{-1}gg_i \in H$ and the morphism $\psi(u)(g, g_iH) : \gamma_{g_i^{-1}}u(\cdot) \longrightarrow \gamma_{g_j^{-1}}u(\cdot)$ is determined by the following commutative square

$$\begin{array}{ccc} \gamma_{g_i^{-1}}u(\cdot) & \xrightarrow{\xi_{g_i^{-1},u(\cdot)}} & u(\cdot) \\ \vdots & & \downarrow u(g_j^{-1}gg_i) \\ \gamma_{g_j^{-1}}u(\cdot) & \xrightarrow{\xi_{g_j^{-1},u(\cdot)}} & u(\cdot) \end{array}$$

accomplished by the fact that $\xi_{g_j^{-1},u(\cdot)}$ is a cartesian morphism.

Assign on morphisms

$$\left(H \;\begin{array}{c}\xrightarrow{\;u\;}\\[-4pt] \downarrow\mu \\[-4pt] \xrightarrow[\;u'\;]{}\end{array}\; D\right) \stackrel{\psi}{\longmapsto} \left(\underline{\underline{G/H}} \;\begin{array}{c}\xrightarrow{\;\psi(u)\;}\\[-4pt] \downarrow\psi(\mu) \\[-4pt] \xrightarrow[\;\psi(u')\;]{}\end{array}\; D\right)$$

where for each $g_iH \in ob\ \underline{\underline{G/H}}$

$$\psi(\mu)_{g_iH} = \gamma_{g_i^{-1}}(\mu) : \gamma_{g_i^{-1}}u(\cdot) \longrightarrow \gamma_{g_i^{-1}}u\ (\cdot)$$

Then it is clear that $\varphi \circ \psi = Id_{Cart_G(H,\ D)}$. Also there is a natural transformation

$$\eta : Id_{Cart_G(\underline{\underline{G/H}},\ D)} \longrightarrow \psi \circ \varphi$$

For $(\underline{\underline{G/H}} \xrightarrow{\;k\;} D) \in ob\ Cart_G(\underline{\underline{G/H}},\ D)$ a morphism

$$\eta_k : k \longrightarrow \psi\circ\varphi(k)$$

of $Cart_G(\underline{\underline{G/H}},\ D)$ is given as follows. For $g_iH \in ob\ \underline{\underline{G/H}}$

$$(\eta_k)_{g_iH} : k(g_iH) \longrightarrow \gamma_{g_i^{-1}}k(eH) = (\psi\circ\varphi(k))(g_iH)$$

is defined by the commutative triangle

$$\begin{array}{ccc}
k(g_iH) & \xrightarrow{\;k(g_i^{-1},\,g_iH)\;} & \\
\vdots\downarrow & & k(eH) \\
\gamma_{g_i^{-1}}k(eH) & \xrightarrow{\;\xi_{g_i^{-1},\,k(eH)}\;} &
\end{array}$$

obtained by the cartesianness of $\xi_{g_i^{-1},k(eH)}$.

Since $k(g_i^{-1},\ g_iH)$ is an isomorphism η_k is an isomorphism for any k. The naturality of η is shown from the following commutative diagram

$$\begin{array}{ccccc}
k(g_iH) & \xrightarrow{\;k(g_i^{-1})\;} & k(eH) & \xleftarrow{\;\xi_{g_i^{-1}}\;} & \gamma_{g_i^{-1}}k(eH) \\
\downarrow \lambda_{g_iH} & & \downarrow \lambda_{eH} & & \downarrow \gamma_{g_i^{-1}}(\lambda_{eH}) = \psi\circ\varphi(\lambda) \\
k'(g_iH) & \xrightarrow[\;k'(g_i^{-1})\;]{} & k'(eH) & \xleftarrow[\;\xi_{g_i^{-1}}\;]{} & \gamma_{g_i^{-1}}k'(eH)
\end{array}$$

deduced from any morphism $\lambda : k \longrightarrow k'$ of $Cart_G(\underline{\underline{G/H}},\ D)$.

Note that since $(\eta_k)_{eH} = id$ the image of η by φ is the identity and that η is the identity on the image of ψ. These show that φ is left adjoint to ψ. Further as η is a natural isomorphism φ is inverse to ψ. Therefore we have the desired equivalence of categories. *q.e.d.*

It is more important when $H = e$. Though it is a special case of the above theorem we shall here renote it to take advantage in the next section.

Corollary 3.6. Let $(C;\ \alpha_s,\ a_{s,t})$ be a G-category and $\gamma : D \longrightarrow G$ the associated fibered category over G. Then there exists an equivalence of categories

$$\varphi : Cart_G(\underline{\underline{G/e}},\ D) \xrightarrow{\approx} C$$

Proof. We shall write explicitly the functor φ and the quasi-inverse functor ψ in spite of a special case of the theorem. The functor

$$\varphi : Cart_G(\underline{\underline{G/e}},\ D) \longrightarrow C$$

is given by

$$\left(\underline{\underline{G/e}} \ \underset{k'}{\overset{k}{\underset{\downarrow\lambda}{\rightrightarrows}}}\ D\right) \longmapsto \begin{pmatrix} k(e) \\ \downarrow\lambda_e \\ k'(e) \end{pmatrix}$$

and the functor

$$\psi : C \longrightarrow Cart_G(\underline{\underline{G/e}},\ D)$$

is given by

$$\begin{pmatrix} X \\ \downarrow\mu \\ Y \end{pmatrix} \longmapsto \left(\underline{\underline{G/e}} \ \underset{\psi(Y)}{\overset{\psi(X)}{\underset{\downarrow\psi(\mu)}{\rightrightarrows}}}\ D\right)$$

where

$$\psi(X)\ (resp.\ \psi(Y)) : \begin{pmatrix} a \\ \downarrow\ g \\ ga \end{pmatrix} \longmapsto \begin{pmatrix} \gamma_a^{-1}X \\ \downarrow \\ \gamma_a^{-1}g^{-1}X \end{pmatrix} \quad \left(resp. \begin{pmatrix} \gamma_a^{-1}Y \\ \downarrow \\ \gamma_a^{-1}g^{-1}Y \end{pmatrix}\right)$$

$$\psi(\mu)_a = \gamma_a^{-1}(\mu)$$

Remark that the choice of representatives does not occur, hence ψ is also canonical. The rest of the proof is as in Theorem 3.5. *q.e.d.*

§ 4. Split G-categories

I mentioned in § 2 that a G-category was considered as a pseudo functor from G to Cat. 2-functors from a category to a 2-category are classified primarily to three classes —— lax, pseudo and strict functors by means of conditions relative to compositions (*e.g.* $a_{s,t}$ in a G-category). It is a main object in this section to give the relations between those notions of 2-functors on G. First it is shown that a lax colimit of a G-category is equivalent to the descended category. And we shall state the relation of G-categories with split G-categories (= strict functors from G to Cat). For this we may use the Giraud construction (Giraud [7]) which associates a strict functor on G with a fibered category over G. Remark that applied to a pseudo functor the one called the Street first construction or Kleisli rectification which sends lax functors to strict functors is equivalent to the Giraud construction. A split G-category has two kinds of limit categories; the descended category considered as a pseudo G-category and the category which consists of G-fixed objects and G-fixed morphisms. It is also shown that those are equivalent only for the G-category deduced from a fibered category over G. We begin by defining various 2-functors

Definition 4.1. For a category E and 2-category $\mathbb{C}$, a lax functor

$$\alpha : E \longrightarrow \mathbb{C}$$

is a pair of functions which assign an object $\alpha(a)$ of $\mathbb{C}$ to each object a of E and an 1-arrow $\alpha(t) : \alpha(a) \longrightarrow \alpha(b)$ of $\mathbb{C}$ to each morphism $t : a \longrightarrow b$ of E together with 2-arrows of $\mathbb{C}$

$$\iota_a : \alpha(id_a) \longrightarrow id_{\alpha(a)}$$

$\mu_{s,t} : \alpha(s \circ t) \longrightarrow \alpha(s)\circ\alpha(t)$

for each identity morphism $id_a : a \longrightarrow a$ of E and each composition $a \xrightarrow{t} b \xrightarrow{s} c$ of morphisms of E such that the following diagrams of 1-arrows of $\mathbf{C}$ commute;

$$\begin{array}{ccccc} \alpha(id_b)\circ\alpha(t) & \xleftarrow{\mu_{id,t}} & \alpha(t) & \xrightarrow{\mu_{t,id}} & \alpha(t)\circ\alpha(id_a) \\ & {\scriptstyle \iota_b * \alpha(t)} \searrow & \downarrow = & \swarrow {\scriptstyle \alpha(t) * \iota_a} & \\ & & \alpha(t) & & \end{array}$$

$$\begin{array}{ccc} \alpha(\upsilon\circ s\circ t) & \xrightarrow{\mu_{\upsilon, s\circ t}} & \alpha(\upsilon)\circ\alpha(s\circ t) \\ \downarrow {\scriptstyle \mu_{\upsilon\circ s, t}} & & \downarrow {\scriptstyle \alpha(\upsilon) * \mu_{s,t}} \\ \alpha(\upsilon\circ s)\circ\alpha(t) & \xrightarrow{\mu_{\upsilon,s} * \alpha(t)} & \alpha(\upsilon)\circ\alpha(s)\circ\alpha(t) \end{array}$$

Further a lax functor α is called a pseudo functor if $\iota_a = id$ for any object a of E and $\mu_{s,t}$ is an isomorphism for every composable pair (s,t) of morphisms of E. And also a pseudo functor α is called a strict functor if $\mu_{s,t} = id$ for any composable pair (s,t) of morphisms of E. Regarding the 2-category $\mathbf{C}$ as a category by forgetting the 2-arrows this turns to be an usual functor from E to $\mathbf{C}$.

Now Cat denotes an 2-category in which objects are small categories, 1-arrows are functors and 2-arrows are natural transformations.

Definition 4.2. A G-category was a pseudo functor from G to Cat. A lax functor $G \longrightarrow Cat$ is called a lax G-category and a strict functor $G \longrightarrow Cat$ is called a split G-category. A strict functor satisfying $\alpha(s) = id$ for all $s \in G$ is called a trivial G-category.

Then

a trivial G-category $\Longrightarrow$ a split G-category

$\Longrightarrow$ a G-category $\Longrightarrow$ a lax G-category and further by the result of § 2

a G-category $\Longleftrightarrow$ a fibered category over G

Now we will define various G-functors. Though they corespond to lax natural transformations, pseudo natural transformations and (usual) natural transformations we write down explicitly

Definition 4.3. Let α, $\alpha' : G \longrightarrow Cat$ be two lax G-categories. A lax G-functor $t : \alpha \longrightarrow \alpha'$ is a functor

$$F = t_{.} : \alpha(\cdot) \longrightarrow \alpha'(\cdot)$$

of categories together with a natural transformation

$$\eta_s : F\circ\alpha(s) \longrightarrow \alpha'(s)\circ F$$

to each $s \in G$ and a natural trasformation

$$i : F * \iota_{.} \longrightarrow (\iota'_{.} * F)\circ \eta_e$$

such that the following diagrams of functors commute;

$$\begin{array}{ccccc} F\circ\alpha(s\cdot t) & \xrightarrow{F*\mu_{s,t}} & F\circ\alpha(s)\circ\alpha(t) & \xrightarrow{\eta_s * \alpha(t)} & \alpha'(s)\circ F\circ\alpha(t) \\ \downarrow \eta_{st} & & & & \downarrow \alpha'(s)*\eta_t \\ \alpha'(s\cdot t)\circ F & & \xrightarrow{\mu'_{s,t} * F} & & \alpha'(s)\circ\alpha'(t)\circ F \end{array}$$

When $i = id$ and η_s is a natural isomorphism for every $s \in G$, such functor was called a G-functor (*Cf.* § 1). Further when $\eta_s = id$ for every $s \in G$, it's called a split G-functor. This is a (usual) natural transformation between (usual) functors from G to Cat.

At first we shall see that the limit categories for lax and pseudo G-categories coincide.

Theorem 4.4. Let $\alpha : G \longrightarrow Cat$ be a pseudo functor *i.e.* putting $\alpha(\cdot) = C$, $\alpha(s) = \alpha_s$ and $\mu_{s,t} = a_{s,t}$ $(C;\ \alpha_s,\ a_{s,t})$ is a G-category. Then we have an equivalence

$$lax\ \underset{G}{limit}\ \alpha \quad \approx \quad \Delta_G C$$

where $\Delta_G C$ is the descended category of a G-category C.

Proof. Consider the forgetful functor

$$j : \Delta_G C \longrightarrow C, \ (X, (\lambda_s)) \longmapsto X$$

Then for each $t \in G$ a natural isomorphism

$$j(t) : \alpha_t \circ j \longrightarrow j, \qquad j(t)_{(X, \ (\lambda_s))} = \lambda_t^{-1}$$

is defined and the following a),b) hold.

a) $j(e) = id_j$

b) $j(tu) \circ (a_{t,u}^{-1} * j) = j(t) \circ (\alpha_t * j(u))$

a) is followed by $\lambda_e = id$. b) is deduced from the facts;

$$\{j(tu) \circ (a_{t,u}^{-1} * j)\}_{(X, \ (\lambda_s))} = (\alpha_t \alpha_u X \xrightarrow{(a_{t,u})_X^{-1}} \alpha_{tu} X \xrightarrow{\lambda_{tu}^{-1}} X)$$

$$\{j(t) \circ (\alpha_t * j(u))\}_{(X, \ (\lambda_s))} = (\alpha_t \alpha_u X \xrightarrow{\alpha_t(\lambda_u^{-1})} \alpha_t X \xrightarrow{\lambda_t^{-1}} X)$$

and the conditions with respect to (λ_s) in the definition of $\Delta_G C$ (see 1.8).

$$\alpha_s(\lambda_t) \circ \lambda_s = (a_{s,t})_X \circ \lambda_{st}.$$

Now given any category C' and a functor $k : C' \longrightarrow C$ together with a natural isomorphism

$$k(t) : \alpha_s \circ k \longrightarrow k$$

to each $t \in G$ such that conditions

a) $k(e) = id_k$

b) $k(tu) \circ (a_{t,u}^{-1} * k) = k(t) \circ (\alpha_t * k(u))$

are satisfied. Then define a functor

$$l : C' \longrightarrow \Delta_G C$$

on objects $l(Y) = (k(Y), (k(s)_Y^{-1}))$, $\quad Y \in ob\ C$

on morphisms $l(Y \xrightarrow{g} Y') = k(g)$, $\quad g \in$ mor C

then one has

$k = j \circ l$

$k(s) = j(s) * l$

These facts show that the descended category $\Delta_G C$ is a lax limit for a lax functor $\alpha : G \longrightarrow Cat$. *q.e.d.*

We shall now describe the relation between G-categories (or equivalently fibered categories over G in the view of § 2) and split G-categories. This is the main theme of this section.

Let $Split(G)$ (*resp.* $Pseudo(G)$) denotes the category of split G-categories (*resp.* G-categories) and split G-functors (*resp.* G-functors). We will regard the 2-category $Fib(G)$ in § 2 (*resp.* $Des(G)$ in § 1) as a category by forgetting the 2-arrows. We know

$Pseudo(G) = Des(G)$ and $Split(G) \subset Pseudo(G)$

We verified in § 2

$Des(G) \approx Fib(G)$

Remark that the equivalence $Des(G) \xrightarrow{\approx} Fib(G)$ constructed essentially in 2.5, 2.6 is got from the usual Grothendieck construction $Pseudo(G^{op}) \longrightarrow Fib(G)$ by exchanging the compositions $G^{op} \leftrightarrow G$. Restricting this functor to $Split(G)$ we have a functor

$\Phi : Split(G) \longrightarrow Fib(G)$

An object of the essential image of Φ is called a split fibered category over G. (all $c_{s,t}$ are identities. *Cf.* § 2) We shall construct a functor opposite to Φ.

Begin with a G-category $(C;\ \alpha_s, a_{s,t})$ or equivalently a pseudo functor $\alpha : G \longrightarrow Cat$ such that $\alpha(\cdot) = C$, $\alpha(s) = \alpha_s$. This corresponds to a fibered category $\gamma : D \longrightarrow G$ with a normalized cleavage $(\gamma_s,\ c_{s,t})$ by Theorem 2.5. Recall that $\gamma_s = \alpha_{s^{-1}}$, $c_{s,t} =$

$(a_{t^{-1},s^{-1}})^{-1}$. Define

$$C^{sp} = Cart_G(\underline{\underline{G/e}}, D)$$

$$\alpha^{sp} : G \longrightarrow Cat$$

on objects $\alpha^{sp}(\cdot) = C^{sp}$

on morphisms $\alpha^{sp}(s) : (\underline{\underline{G/e}} \xrightarrow{\eta} D) \longmapsto (\underline{\underline{G/e}} \xrightarrow{\eta'} D)$

where $\eta' : \begin{pmatrix} u \\ \downarrow(a,u) \\ au \end{pmatrix} \longmapsto \begin{pmatrix} \eta(us) \\ \downarrow \eta(a,us) \\ \eta(aus) \end{pmatrix}$

for a morphism $(a,u) : u \longrightarrow au$ of $\underline{\underline{G/e}}$

Take another G-category $(C';\alpha'_s, a'_{s,t})$ and the associated fibered category $\gamma' : D' \longrightarrow G$. For a cartesian functor $F : D \longrightarrow D'$ over G a functor $F^{sp} : C^{sp} \longrightarrow C'^{sp}$ is defined by taking the composition with F. Then

Lemma 4.5. α^{sp} is a strict functor, hence $(C^{sp};\ \alpha^{sp}(s))$ is a split G-category and F^{sp} is a split G-functor.

Proof. For any $s,t \in G$, any $k : \underline{\underline{G/e}} \longrightarrow D$ and $u \in ob\ \underline{\underline{G/e}}$

$$(\alpha^{sp}(st)k)(u) = k(ust) = (\alpha^{sp}(t)k)(us)$$
$$= \{\alpha^{sp}(s)(\alpha^{sp}(t)k)\}(u)$$

hence

$$\alpha^{sp}(st)k = \alpha^{sp}(s)(\alpha^{sp}(t)k)$$

Further the actions on morphisms are similar, therefore

$$\alpha^{sp}(st) = \alpha^{sp}(s) \circ \alpha^{sp}(t)$$

On the other hand for $(k : \underline{\underline{G/e}} \longrightarrow D) \in ob\ C^{sp}$ and any $s \in G$

$$\alpha'^{sp}(s) \circ F^{sp}(k) = \alpha'^{sp}(s)(F \circ k) = F \circ k(s) = F^{sp}(k(s))$$
$$= F^{sp} \circ \alpha^{sp}(s)(k)$$

which implies F^{sp} is a split G-functor. *q.e.d.*

Therefore one has a functor

$$S : Fib(G) \longrightarrow Split(G)$$

on objects $\quad S(\gamma : D \longrightarrow G) = (C^{sp};\ \alpha^{sp}(s))$

on morphisms $\quad S(\mathcal{F} : D \longrightarrow D') = (F^{sp} : C^{sp} \longrightarrow C'^{sp})$

Also $\gamma^{sp} : D^{sp} \longrightarrow G$ denotes the (split) fibered category over G associated to the split G-category $(C^{sp};\ \alpha^{sp}(s))$ by Theorem 2.5.

Definition 4.6. C^{sp}, α^{sp}, $\gamma^{sp} : D^{sp} \longrightarrow G$ and $F^{sp} : C^{sp} \longrightarrow C'^{sp}$ are called respectively the split version of C, α, $\gamma : D \longrightarrow G$ and $F : C \longrightarrow C'$ where F is the restriction of $\mathcal{F}$ to the fibers.

For those objects the following theorem is fundamental. This is essentially due to Giraud[7]. Here we shall use the results of § 3 to give another proof over the base G.

Theorem 4.7. (1) There is a functor

$$\tilde{\varphi} : \Phi \circ S(D) = D^{sp} \longrightarrow D$$

which is a fiber equivalence over G.

(2) S is right adjoint to Φ.

Proof. (1) Consider the equivalence φ in the proof of Corollary 3.6.

$$C^{sp} = Cart_G(\underline{\underline{G/e}},\ D) \xrightarrow[\varphi]{\approx} C$$

Put $\gamma_s = \alpha_s{-1} : C \longrightarrow C$ and $\overline{\gamma}_s = \alpha^{sp}(s^{-1}) : C^{sp} \longrightarrow C^{sp}$. For any object $(k : \underline{\underline{G/e}} \longrightarrow D)$ of C^{sp} one has

$$\gamma_s \circ \varphi(k) = \gamma_s k(e)$$
$$\varphi \circ \overline{\gamma}_s(k) = k(s^{-1})$$

However there are morphisms of D

$$\xi_{s,k(e)} : \gamma_s k(e) \longrightarrow k(e)$$
$$k(s,s^{-1}) : k(s^{-1}) \longrightarrow k(e)$$

which are isomorphisms. Since $\xi_{s,k(e)}$ is a cartesian morphism there is an isomorphism

$$\xi_{s,k} : k(s^{-1}) \longrightarrow \gamma_s k(e)$$

To verify that $\xi_{s,k}$'s define a natural isomorphism

$$\xi_s : \gamma_s \circ \varphi \longrightarrow \varphi \circ \overline{\gamma}_s$$

take a morphism $\lambda : k \longrightarrow k'$ of $C^{sp} = Cart_G(\underline{\underline{G/e}}, D)$ and see the assignements by $\gamma_s \circ \varphi$ and $\varphi \circ \overline{\gamma}_s$

$$\gamma_s \circ \varphi(\lambda) = (\gamma_s \lambda_e : \gamma_s k(e) \longrightarrow \gamma_s k'(e))$$
$$\varphi \circ \overline{\gamma}_s(\lambda) = (\lambda_{s^{-1}} : k(s^{-1}) \longrightarrow k'(s^{-1}))$$

Then it follows from the commutative diagram

$$\begin{array}{ccccc} \gamma_s k(e) & \longrightarrow & k(e) & \longleftarrow & k(s^{-1}) \\ \downarrow \gamma_s \lambda_e & & \downarrow \lambda_e & & \downarrow \lambda_{s^{-1}} \\ \gamma_s k'(e) & \longrightarrow & k'(e) & \longleftarrow & k'(s^{-1}) \end{array}$$

that ξ_s is a natural transformation.

Thus (φ, ξ_s) is a G-functor from C^{sp} to C. It follows from Theorem 2.6 that there is a cartesian functor

$$\tilde{\varphi} : D^{sp} \longrightarrow D$$

But the restriction of $\tilde{\varphi}$ to the fibers is φ which is an equivalence of categories by Corollary 3.6. It follows from [21] Proposition 6.10 that the cartesian functor $\tilde{\varphi}$ is a fiber equivalence.

(2) (1) implies that there is a natural transformation

$$\Phi \circ S \longrightarrow Id_{Fib(G)}$$

which is denoted by the same letter $\tilde{\varphi}$.

Consider ψ in the proof of Corollary 3.6. Take γ_s and $\overline{\gamma}_s$ as in (1). Here we assume that C (or α) is split. That is

$$\gamma_{st} = \gamma_t \circ \gamma_s$$

For any object X of C and any $u \in ob\ G/e$

$$\overline{\gamma}_s \circ \psi(X) = (u \longmapsto \gamma_{(su)^{-1}} X)$$
$$\psi \circ \gamma_s(X) = (u \longmapsto \gamma_{u^{-1}} \gamma_{s^{-1}} X)$$

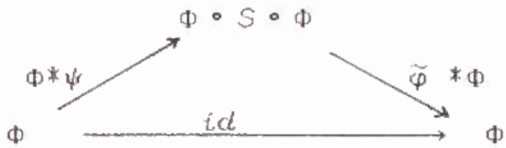
Similarily for morphisms. So

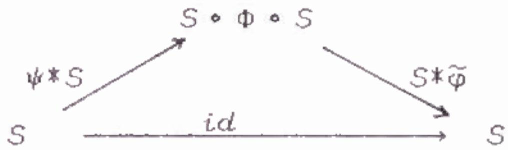
$$\overline{\gamma}_S \circ \psi = \psi \circ \gamma_S$$

Thus we have a natural transformation

$$Id_{Split(G)} \longrightarrow S \circ \Phi$$

All that remains is to show the commutativity of the following two triangles





Both of them are obtained from the fact

$$\varphi \circ \psi = Id : C \longrightarrow C$$

by reducing the problem to that on fibers. *q.e.d.*

Corollary 4.8. Let the notations be as above. There are equivalences

$$C^{sp} \approx C$$

$$\Delta_G C^{sp} \approx \Delta_G C$$

$$Rep(G, D^{sp}) \approx Rep(G, D)$$

Proof. Theorem 4.7 (1) implies

$$(\gamma^{sp})^{-1}(\cdot) \approx \gamma^{-1}(\cdot)$$

$$Cart\ Sec(G, D^{sp}) \approx Cart\ Sec(G, D)$$

However we know

$$C^{sp} = Ker\ D^{sp} = (\gamma^{sp})^{-1}(\cdot)$$

$$C = Ker\ D = \gamma^{-1}(\cdot)$$

$$\Delta_G C^{sp} \approx Rep(G, D^{sp}) = Cart\ Sec(G, D^{sp})$$

$$\Delta_G C \approx Rep(G, D) = Cart\ Sec(G, D)$$

These show the desired results. *q.e.d.*

Next we shall state the results for subgroups H of G. To do this for a split G-category $(C; \alpha_s)$ *i.e.* a functor $\alpha : G \longrightarrow Cat$, $\alpha(\cdot) = C$, $\alpha(s) = \alpha_s$ we define another limit category different from the descended category $\Delta_G C$.

Definition 4.9. For a split G-category $(C; \alpha_s)$ and a subgroup H of G the H-fixed category C^H is defined to be a category which consists of H-fixed objects and H-fixed morphisms of C.

A key result is following.

Proposition 4.10. For a fibered category $\gamma : D \longrightarrow G$ over G and a subgroup H of G the natural functor

$$Cart_G(\underline{\underline{G/H}}, D) \longrightarrow Cart_G(\underline{\underline{G/e}}, D)$$

induced by the natural projection $G/e \longrightarrow G/H$ provides an isomorphism of categories

$$Cart_G(\underline{\underline{G/H}}, D) \xrightarrow{\cong} Cart_G(\underline{\underline{G/e}}, D)^H$$

Proof. It is clear from the definitions.

Corollary 4.11. Let $(G; \alpha_s, a_{s,t})$ be a (not necessarily split) G-categoy and $\gamma : D \longrightarrow G$ be the associated fibered category over G. C^{sp}, α^{sp}, D^{sp} and γ^{sp} denote as above. Then there are equivalences of categories for a subgroup H of G

$$\Delta_H C \approx (C^{sp})^H \approx \Delta_H(G^{sp})$$

Proof. It follows from Proposition 3.2, Theorem 3.5 and Proposition 4.10 that

$$\Delta_H C \approx Rep(H, D) \approx \overline{Rep}(H, D) = Cart_G(\underline{\underline{G/H}}, D) \cong (C^{sp})^H$$

On the other side since $\tilde{\varphi} : D^{sp} \longrightarrow D$ was a fiber equivalence we have

$$Cart_G(H,\ D^{sp}) \approx Cart_G(H,\ D)$$

which implies as in Corollary 4.8

$$\Delta_H C^{sp} \approx \Delta_H C \qquad q.e.d.$$

Remark 4.12. Even though $(C;\ \alpha_s,\ a_{s,t})$ is split an equivalence

$$\Delta_H C \approx C^H$$

does not hold generally, because $\varphi : C^{sp} \longrightarrow C$ does not become a split G-functor. The above corollary shows if a split G-category C comes from a fibered category over G *i.e.* $C = Cart_G(\underline{\underline{G/e}},\ D)$ an equivalence

$$\Delta_H C \approx C^H$$

holds.

§ 5. Exact G-categories

We have cosidered actions of G on any categories until the last section. The definition of G-categories for categories with certain additional structures is as follows.

As a G-category is regarded as a pseudo functor from G to *Cat*, we may replace only *Cat* by an adquate 2-category consisting of categories with certain additional structures, functors and natural transformations preserving the additional structures. Other objects which we don't handle in this paper, but which are important for algebraic K-theory; symmetric monoidal G-categories, simplicial

G-categories and categories with actions of two kinds of groups *etc.* all are done well by this method. In this section we deal in exact G-categories and state the commutativity of Quillen's Q-construction with descended categories as a main result.

Definition 5.1. A G-category $(C;\ \alpha_s,\ a_{s,t})$ is an additive (*resp.* abelian *resp.* exact) G-category if C is an additive (*resp.* abelian *resp.* exact) category and for every $s \in G$ α_s is an additive (*resp.* exact *resp.* exact) functor.

Then an old limit category of a G-category taken in Cat is also a new limit category taken in the 2-category of categories with certain additional structures.

Proposition 5.2. Let $(C;\ \alpha_s,\ a_{s,t})$ be an additive (*resp.* abelian *resp.* exact) G-category, $\gamma : D \longrightarrow G$ be the fibered category over G associated to $(C;\ \alpha_s, a_{s,t})$ by Theorem 2.5 and $\varepsilon : E \longrightarrow G$ be any groupoid over G. Then $Cart_G(E,\ D)$ turns to be an additive (*resp.* abelian *resp.* exact) category in the natural manner.

Proof. The category $Cart_G(E,\ D)$ has as objects functors $\eta : E \longrightarrow D$ satisfying $\gamma \circ \eta = \varepsilon$ and as morphisms natural transformations $t : \eta \longrightarrow \eta'$ of grade e between those functors.

To show that $Cart_G(E,\ D)$ becomes an additive or abelian category according to D additive or abelian, we have to check the abelian group structure of hom sets, the existence of a 0-object and coproducts, the existence of kernels and cokernels, an isomorphism of coimage with image and so on. The definitions of the desired objects may work in obvious way by applying the correspondent constructions to the images of E. This procedure of proofs is long but routine, so we *will* omit the details.

With respect to an exact G-category C our proof is as follows. Embed C into A (= the category of left exact functors on C^{op} to the category Ab of abelian groups) as a full subcategory. Then A has a G-descent datum which is an extension of the one of C. $B \longrightarrow C$ denotes the fibered category over G associated to the G-category A. The former result shows $Cart_G(E, B)$ is an abelian category. Further it is easy to show the existence of an embedding

$$Cart_G(E, D) \longrightarrow Cart_G(E, B)$$

and that the category in the left side is closed under extensions in the category in the right side. It follows from Quillen [17] that $Cart_G(E, D)$ is an exact category. *q.e.d.*

By the results of § 4 we have easily

Corollary 5.3. Let $(C; \alpha_s, a_{s,t})$ be an additive (*resp.* abelian *resp.* exact) G-category, $\gamma : D \longrightarrow C$ be the associated fibered category over G and H be a subgroup of G. Then various limit categories

$$\Delta_H C,\ Rep(H, D),\ \overline{Rep}(H, D),\ (C^{\mathrm{sp}})^H \text{ and } \Delta_H C^{\mathrm{sp}}$$

are all additive (*resp.* abelian *resp.* exact) categories in natural manners.

Now we shall state the relation with Quillen's Q-construction for exact categories.

Theorem 5.4. Let $(C; \alpha_s, a_{s,t})$ be an exact G-category, $\gamma : D \longrightarrow G$ be the associated fibered category over G by means of Theorem 2.5 and $\varepsilon : E \longrightarrow G$ be any groupoid over G. Then

(1) The category QC has a natural structure of G-category.

(2) If $\tilde{\gamma} : Q_f D \longrightarrow G$ denotes the fibered category over G associated

to the G-category QC, then there exists an equivalence of categories

$$Q\ Cart_G(E,\ D) \approx Cart_G(E,\ Q_f D)$$

Proof. (1) We shall define a natural G-descent datum $(\alpha_s, a_{s,t})$ on QC. Note that the category QC has the same objects as C and morphisms $X \longrightarrow X'$ in QC are isomorphism classes of $(X \twoheadleftarrow Z \rightarrowtail X')$, where $\twoheadrightarrow$ (*resp.* $\rightarrowtail$) denotes an admissible epimorphism (*resp.* an admissible monomorphism). Here define an endofunctor for each $s \in G$

$$\overline{\alpha}_s : QC \longrightarrow QC$$

whose function on objects is the same as α_s and which sends a morphism

$$\text{an isomorphism class of } (X \overset{j}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} X')$$

of QC to a morphism

$$\text{an isomorphism class of } (\alpha_s X \overset{\alpha_s(j)}{\twoheadleftarrow} \alpha_s Z \overset{\alpha_s(i)}{\rightarrowtail} \alpha_s X')$$

of QC. We should remark that as α_s is an exact functor for every $s \in G$ the image by α_s of an admissible epi (*resp.* an admissible mono) is so. Also for $s, t \in G$

$$(\overline{a}_{s,t})_X : \alpha_{st}X \longrightarrow \alpha_s(\alpha_t X)$$

on $X \in ob\ QC$ is given by an isomorphism class of

$$(\alpha_{st}X \overset{=}{\twoheadleftarrow} \alpha_{st}X \overset{a_{s,t}}{\rightarrowtail} \alpha_s(\alpha_t X))$$

and it is seen immediately that $\overline{a}_{s,t}$'s satisfy the conditions of a G-descent datum.

(2) The objects of the category $Q\ Cart_G(E,\ D)$ are functors $\eta : E \longrightarrow D$ over G and a morphism $t : \eta \longrightarrow \eta'$ of $Q\ Cart_G(E,\ D)$ is an isomorphism class of

$$(\eta \twoheadleftarrow \xi \rightarrowtail \eta')$$

in $Cart_G(E,\ D)$ namely for each object a of E an isomorphism class of diagrams

$$(\eta(a) \twoheadleftarrow \xi(a) \rightarrowtail \eta'(a))$$

such that for any morphism $m : a \longrightarrow b$ of E a diagram

$$\begin{array}{ccccc}
\eta(a) & \twoheadleftarrow & \xi(a) & \rightarrowtail & \eta'(a) \\
\downarrow{\scriptstyle \eta(m)} & & \downarrow{\scriptstyle \xi(m)} & & \downarrow{\scriptstyle \eta'(m)} \\
\eta(b) & \twoheadleftarrow & \xi(b) & \rightarrowtail & \eta'(b)
\end{array}$$

is commutative. On the other hand the objects of $Cart_G(E,\ Q_fD)$ consist of functors $\tilde{\eta} : E \longrightarrow Q_fD$. The objects of Q_fD are same as the objects of D and an isomorphism in Q_fD reduces to an isomorphism in D. As E is a groupoid the functor $\tilde{\eta}$ is identified with a functor $E \longrightarrow D$.

Next letus cosider a morphism $\tilde{\tau} : \tilde{\eta} \longrightarrow \tilde{\eta}'$ of $Cart_G(E,\ Q_fD)$. This consists of morphisms of QC

$$\tilde{\tau}(a) : \tilde{\eta}(a) \longrightarrow \tilde{\eta}'(a)$$

for all $a \in ob\ E$, which are compatible for every morphism of E. That is to say it is given by an isomrphsm class of diagrams

$$(\tilde{\eta}(a) \twoheadleftarrow X_{\tilde{\tau}(a)} \rightarrowtail \tilde{\eta}\ (a))$$

for each $a \in ob\ E$ such that a diagram

$$\begin{array}{ccccc}
\tilde{\eta}(a) & \twoheadleftarrow & X_{\tilde{\tau}(a)} & \rightarrowtail & \tilde{\eta}'(a) \\
\downarrow{\scriptstyle \tilde{\eta}(m)} & & \downarrow & & \downarrow{\scriptstyle \tilde{\eta}'(m)} \\
\tilde{\eta}(b) & \twoheadleftarrow & X_{\tilde{\tau}(b)} & \rightarrowtail & \tilde{\eta}'(b)
\end{array}$$

is commutative for every morphism $m : a \longrightarrow b$ of E. Since E is a groupoid the morphisms which appeared above are all isomorphisms, hence we can use $X_{\tilde{\tau}(a)}$'s to make a functor $\chi : E \longrightarrow D$ such that $\chi(a) = X_{\tilde{\tau}(a)}$ for any $a \in ob\ E$. Thus the morphisms of $Cart_G(E,\ Q_fD)$ reduce to morphisms of $Q\ Cart_G(E,\ D)$, too. The desired equivalence of categories will follows. *q.e.d.*

From this theorem we deduce the commutativity of Q-construction with taking various limit categories of various G-categories cosidered so far.

Corollary 5.5. Let $(C; \alpha_s, a_{s,t})$, $\gamma : D \longrightarrow G$, QC and $\gamma_f : Q_fD \longrightarrow G$ be as in Therorem 5.4. If H is a subgroup of G then

(1) $QRep(H, D) \approx Rep(H, Q_fD)$

(2) $Q\Delta_H C \approx \Delta_H QC$

(3) $Q\overline{Rep}(H, D) \approx \overline{Rep}(H, Q_fD)$

(4) $(QC^{\mathrm{sp}})^H \approx Q(C^{\mathrm{sp}})^H$

Proof. (1) Take $E = H$ in Theorem 5.4.

(2) follows from equivalence $\Delta_H C \approx Rep(H, D)$ and (1).

(3) Take $E = G/H$ in Theorem 5.4.

(4) From (3), Proposition 4.10 and Theorem 4.12 we obtain

$$\begin{aligned}(QC^{\mathrm{sp}})^H &= (QCart_G(\underline{\underline{G/e}}, D))^H \\ &\approx (Cart_G(\underline{\underline{G/e}}, Q_fD))^H \\ &\cong Cart_G(\underline{\underline{G/H}}, Q_fD) \\ &\approx QCart_G(\underline{\underline{G/H}}, D) \\ &\cong QCart_G(\underline{\underline{G/e}}, D)^H \\ &= Q(C^{\mathrm{sp}})^H\end{aligned}$$

q.e.d.

§ 6. Induction theory deduced from a G-category

The representation category $Rep(H,\ D)$ cosidered in § 3 was an analogue to the category of H-representations in $Ker\ D$. So we can chase the analogous formulation of representation theory of finite groups.

We can proceed with the arguments by defining restriction and induction functors for $Rep(H,\ D)$. The approach of Fröhlich-Wall[6] was in such manner. But in this approach the definitions depend to choices of representatives of cosets with respect to subgroups, hence it is troublesome to check naturality. Therefore we will use

$$\overline{Rep}(H,\ D) = Cart_G(\underline{\underline{G/H}},\ D).$$

This makes the argument functorial and formal.

We shall only refer to Mackey property and projection formula which are fundamental tools in representation theory. These results generalize the results for trivial G-categories of Dress-Kuku[3] to general G-categories which is not neccesarily split.

In this section we assume the group G is finite or profinite. We consider the category S_G^{fin} of finite G-sets and G-maps. To an object S of S_G^{fin} *i.e.* a finite G-set S, we assign a category $\underline{\underline{S}}$ whose objects are elements of S and whose morphisms $x \longrightarrow y$ $(x, y \in S)$ are represented by pairs $(g,\ x)$ such that $gx = y$ as in § 4. Further to a morphism (a G-map) $\phi : S \longrightarrow T$ of S_G^{fin} we assign a functor

$$\underline{\underline{\phi}} : \underline{\underline{S}} \longrightarrow \underline{\underline{T}}$$

$$\begin{pmatrix} x \\ \big\downarrow (g, x) \\ x' \end{pmatrix} \longmapsto \begin{pmatrix} \phi(x) \\ \big\downarrow (g, \phi(x)) \\ \phi(x') \end{pmatrix}$$

where $gx = x'$ hence $g\,\phi(x) = \phi(gx) = \phi(x')$.

Let $(C;\ \alpha_S,\ a_{S,t})$ be an exact G-category and $\gamma : D \longrightarrow G$ the associated fibered category over G by means of Theorem 2.5. Extend the definition of $\overline{Rep}(H,\ D)$ for subgroups H of G to objects of S_G^{fin}.

$$\overline{Rep}(S,\ D) = Cart_G(\underline{\underline{S}},\ D)$$

It follows from Proposition 5.2 that the category $\overline{Rep}(S,\ D)$ is an exact category for any S. We shall now define restriction and induction functors between them.

Definition 6.1. For a morphism $\phi : S \longrightarrow T$ of S_G^{fin} there are exact functors between $\overline{Rep}(S,\ D)$ and $\overline{Rep}(T,\ D)$

$$\overline{Rep}(T,\ D) \underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}} \overline{Rep}(S,\ D)$$

$$\phi_*(\underline{\underline{T}} \xrightarrow{\ \eta\ } D) = (\underline{\underline{S}} \xrightarrow{\ \eta\circ\phi\ } D)$$

namely

$$\phi_*(\underline{\underline{T}} \xrightarrow{\ \eta\ } D)\ :\ \left(\begin{array}{c} x \\ \big\downarrow (g, x) \\ x' \end{array}\right) \longmapsto \left(\begin{array}{c} \eta(\phi(x)) \\ \big\downarrow \eta(g, \phi(x)) \\ \eta(\phi(x')) \end{array}\right)$$

where $x, x \in S$ and $g \in G$ such that $gx = x$.

$$\phi^*(S \xrightarrow{\ \xi\ } D)\ :\ \left(\begin{array}{c} y \\ \big\downarrow (g, y) \\ y \end{array}\right) \longrightarrow \left(\begin{array}{c} \bigoplus_{x \in \phi^{-1}(y)} \xi(x) \\ \big\downarrow \bigoplus \xi(g, x) \\ \bigoplus_{x' \in \phi^{-1}(y')} \xi(x') \end{array}\right)$$

where $y, y' \in T$ and $g \in G$ such that $gy = y'$. We can take $x, x' \in S$ such that $gx = x'$ by exchanging appropriately an order of $\oplus$ if neccesary. Hence ϕ^* is determined up to a natural isomorphism.

We note that ϕ_* (*resp.* ϕ^*) is corresponding to restriction (*resp.* induction).

Proposition 6.2. (1) For a composition $S \xrightarrow{\phi} T \xrightarrow{\psi} U$ of morphisms of S_G^{fin}

$$(\psi \circ \phi)_* = \phi_* \circ \psi_*$$

$$(\psi \circ \phi)^* \cong \psi^* \circ \phi^*$$

(2) If $S \amalg T$ denotes the coproduct (= disjoint union) of S and T in S_G^{fin} there is an equivalence

$$\overline{Rep}(S \amalg T,\ D) \approx \overline{Rep}(S,\ D) \times \overline{Rep}(T,\ D)$$

(3) If a morphism $\phi : S \longrightarrow T$ is an isomorphism of S_G^{fin} ϕ_+ induces an isomorphism of categories

$$\phi_+ : \overline{Rep}(T,\ D) \xrightarrow{\cong} \overline{Rep}(S,\ D)$$

Proof. (1) is clear.

(2) The assignement

$$(\eta : \underline{\underline{S \amalg T}} \longrightarrow D) \longmapsto (\eta | \underline{\underline{S}} : \underline{\underline{S}} \longrightarrow D,\ \eta | \underline{\underline{T}} : \underline{\underline{T}} \longrightarrow D)$$

gives the desired equivalence.

(3) follows from the first formula of (1).

Proposition 6.3 (Pull-Back Formula). Let S_1, S_2 and T be objects of S_G^{fin} and $\phi_1 : S_1 \longrightarrow T$ and $\phi_2 : S_2 \longrightarrow T$ be maps of S_G^{fin}. Make a pull-back diagram

$$\begin{array}{ccc} S_1 \times_T S_2 & \xrightarrow{\psi_2} & S_2 \\ \downarrow \psi_1 & & \downarrow \phi_2 \\ S_1 & \xrightarrow{\phi_1} & T \end{array}$$

where $S_1 \times_T S_2 = \{(a,b) |\ a \in S_1,\ b \in S_2,\ \phi_1(a) = \phi_2(b)\}$

$$\psi_1 : S_1 \times_T S_2 \longrightarrow S_1, \quad (a,b) \longmapsto a$$

$$\psi_2 : S_1 \times_T S_2 \longrightarrow S_2, \quad (a,b) \longmapsto b$$

Then there is a natural isomorphism

$$(\phi_2)_* \circ (\phi_1)^* \cong (\psi_2)^* \circ (\psi_1)_*$$

of functors from $\overline{Rep}(S_1,\ D)$ to $\overline{Rep}(S_2,\ D)$. In other words the diagram

$$\begin{array}{ccc}
\overline{Rep}(S_1 \times_T S_2,\ D) & \xrightarrow{(\psi_2)^*} & \overline{Rep}(S_2,\ D) \\
\uparrow (\psi_1)_* & & \uparrow (\phi_2)_* \\
\overline{Rep}(S_1,\ D) & \xrightarrow{(\phi_1)^*} & \overline{Rep}(T,\ D)
\end{array}$$

is commutative up to natural isomorphism.

Proof. For $g \in G$, $y, y' \in S_2$ such that $gy = y'$ and $(\underline{\underline{S_1}} \xrightarrow{\eta} D) \in ob\ Rep(S_1,\ D)$ two kinds of assignements are given as follows;

$$((\phi_2)_* \circ (\phi_1)^*)(\underline{\underline{S_1}} \xrightarrow{\eta} D) : \left(\begin{array}{c} y \\ \Big\downarrow (g, y) \\ y' \end{array} \right) \longmapsto \left(\begin{array}{c} \bigoplus_{x \in \phi_1^{-1}(\phi_2(y))} \eta(x) \\ \Big\downarrow \oplus\ \eta(g, x) \\ \bigoplus_{x' \in \phi_1^{-1}(\phi_2(y'))} \eta(x') \end{array} \right)$$

$$((\psi_2)^* \circ (\psi_1)_*)(\underline{\underline{S_1}} \xrightarrow{\eta} D) : \left(\begin{array}{c} y \\ \Big\downarrow (g, y) \\ y' \end{array} \right) \longmapsto \left(\begin{array}{c} \bigoplus_{z \in \psi_2^{-1}(y)} \eta(\psi_1(z)) \\ \Big\downarrow \oplus\ \eta(g, z) \\ \bigoplus_{z' \in \psi_2^{-1}(y')} \eta(\psi_1(z')) \end{array} \right)$$

Now for $y \in S_2$

$$x \in \phi_1^{-1}(\phi_2(y)) \iff \phi_1(x) = \phi_2(y) \iff (x, y) \in S_1 \times_T S_2$$
$$\iff x = \psi_1(z),\ z \in \psi_2^{-1}(y)$$

hence there is a natural isomorphism

$$\bigoplus_{x \in \phi_1^{-1}(\phi_2(y))} \eta(x) \xrightarrow{\cong} \bigoplus_{z \in \psi_2^{-1}(y)} \eta(\psi_1(z)) \qquad q.e.d.$$

Remark 6.4. Propositions 6.2 and 6.3 show that the functor

$$\overline{Rep}(-,\ D)\ :\ S_G^{fin} \longrightarrow \textit{Exact categories}$$

makes a Mackey functor.

If subgroups H, K of G satisfy

$$H < K,\ (K : H) < \infty$$

and a G-map $\phi : G/H \longrightarrow G/K$ is the natural projection then we write

$$res(H,\ K) = \phi_*$$

$$ind(H,\ K) = \phi^*$$

Also if

$$\psi = \text{conjugation by } s : G/(sHs^{-1}) \longrightarrow G/H$$

we write $c_s = \psi_*$

Under these notations we have

Corollary 6.5 (Double Coset Formula). If H, K are subgroups of finite index in G then there is an isomorphism of functors

$$res(H,\ G) \circ ind(K,\ G) \cong$$

$$\bigoplus_s ind(sKs^{-1} \cap H,\ H) \circ c_s \circ res(K \cap s^{-1}Hs,\ K)$$

where s varies on a set of representatives of double cosets of $H \backslash G/K$.

Proof We may only apply Proposition 6.3 to the following pull-back diagram

$$\begin{array}{ccc} G/H \underset{G/G}{\times} G/K & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ G/K & \longrightarrow & G/G \end{array}$$

in which

$$G/H \underset{G/G}{\times} G/K = \coprod_{s \in H \backslash G/K} G/(sKs^{-1} \cap H) \qquad q.e.d.$$

Finally we shall express projection formula. For $i = 1,2,3$ let C_i be an exact G-category and $\gamma_i : D_i \longrightarrow G$ be the associated fibered category over G. An exact pairing

$$C_1 \times C_2 \longrightarrow C_3$$

compatible with the respective G-descent data defines a fiber pairing over G

$$D_1 \times_G D_2 \longrightarrow D_3$$

which induces an exact pairing

$$\overline{Rep}(S, D_1) \times \overline{Rep}(S, D_2) \longrightarrow \overline{Rep}(S, D_3)$$

for each object S of S_G^{fin}. Then

Proposition 6.6 (Projection Formula). For any morphism $\phi : S \longrightarrow T$ of S_G^{fin} the following diagram commutes up to natural isomorphism

$$\begin{array}{ccc}
\overline{Rep}(T, D_1) \times \overline{Rep}(S, D_2) & \xrightarrow{\phi_* \times id} & \overline{Rep}(S, D_1) \times \overline{Rep}(S, D_2) \\
 & & \downarrow \text{pairing} \\
id \times \phi^* \downarrow & & \overline{Rep}(S, D_3) \\
 & & \downarrow \phi^* \\
\overline{Rep}(T, D_1) \times \overline{Rep}(T, D_2) & \xrightarrow{\text{pairing}} & \overline{Rep}(T, D_3)
\end{array}$$

Proof. $(X, Y) \longrightarrow X \otimes Y$ denotes the pairings. Then we may express the induced pairing as

$$\overline{Rep}(S, D_1) \times \overline{Rep}(S, D_2) \longrightarrow \overline{Rep}(S, D_3)$$

$$(\xi : \underline{\underline{S}} \longrightarrow D_1,\ \eta : \underline{\underline{S}} \longrightarrow D_2) \longmapsto (\xi : \underline{\underline{S}} \longrightarrow D_3)$$

$$\xi : \begin{pmatrix} x \\ \downarrow (g, x) \\ x' \end{pmatrix} \longmapsto \begin{pmatrix} \xi(x) \otimes \eta(x) \\ \downarrow \xi(g, x) \otimes \eta(g, x) \\ \xi(x') \otimes \eta(x') \end{pmatrix}$$

where $g \in G$ $x, x' \in S$ such that $gx = x'$.

Now for an object $(\xi : \underline{\underline{T}} \longrightarrow D_1,\ \eta : \underline{\underline{S}} \longrightarrow D_2)$ of $\overline{Rep}(T, D_1) \times \overline{Rep}(S, D_2)$ let $\xi_1 : \underline{\underline{T}} \longrightarrow D_3$ (*resp.* $\xi_2 : \underline{\underline{T}} \longrightarrow D_3$) denotes the result of left round (*resp.* right round) in the diagram of (ξ, η). Then ξ_1, ξ_2 are described as follows;

$$\xi_1 : \begin{pmatrix} y \\ \downarrow (g, y) \\ y' \end{pmatrix} \longmapsto \begin{pmatrix} \xi(y) \otimes \left(\bigoplus_{x \in \phi^{-1}(y)} \eta(x) \right) \\ \downarrow \\ \xi(y') \otimes \left(\bigoplus_{x' \in \phi^{-1}(y')} \eta(x') \right) \end{pmatrix}$$

$$\xi_2 : \begin{pmatrix} y \\ \big\downarrow (g, y) \\ y' \end{pmatrix} \longmapsto \begin{pmatrix} \bigoplus_{x \in \phi^{-1}(y)} (\xi(\phi(x)) \otimes \eta(x)) \\ \big\downarrow \\ \bigoplus_{x' \in \phi^{-1}(y')} (\xi(\phi(x')) \otimes \eta(x')) \end{pmatrix}$$

for a morphism $(g, y) : y \longrightarrow y'$ of $\underline{\underline{I}}$. But it follows from the bilinearity of the pairings that there is a natural isomorphism from ξ_1 to ξ_2. *q.e.d.*

§ 7. O_G-categories

We shall consider in this section the last notion of G-categories which we should handle. The one providing a category for each subgroup H of G in a compatible manner, which is called an O_G-category, is also related with various notions of G-categories studied in § 1 ~ § 4. A fundamental problem is to construct from given O_G-category β a split G-category C such that the H-fixed category C^H is homotopy equivalent to the given category $\beta(G/H)$ on G/H. This construction from O_G-categories to split G-categories is sent by the classfying space functor B to Elmendorf construction which is a functor from O_G-spaces to G-spaces with analogous properties. (*Cf.* [4])

Definition 7.1 ([4]). The category O_G of canonical orbits has as objects canonical orbits G/H where H varies on subgroups of G and as morphisms G-maps between them. A morphism

$$\phi : G/H \longrightarrow G/K \qquad (H,\ K < G)$$

corresponds to an element $fK \in (G/K)^H$ *i.e.*

$$\mathrm{Hom}_{O_G}(G/H,\ G/K) = (G/K)^H$$

By an O_G-category we shall mean a (strict) functor

$$O_G{}^{op} \longrightarrow Cat$$

An O_G-functor of O_G-categories is a natural transformation of functors $O_G^{op} \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} Cat$. O_G-Cat denotes the category of O_G-categories and O_G-functors. When we want to consider $Fib(G)$, $Split(G)$ *etc.* as categories we should forget the 2-arrows of them.

O_G-categories can be obtained as follows.

Definition 7.2. We define a functor

$$\Psi : Fib(G) \longrightarrow O_G\text{-}Cat$$

as follows; for an object $\gamma : D \longrightarrow G$ of $Fib(G)$ an object

$$\Psi(\gamma) : O_G{}^{op} \longrightarrow Cat$$

of O_G-Cat is given by

on objects $\quad G/H \longmapsto \overline{Rep}(H,\ D) = Cart_G(\underline{\underline{G/H}},\ D)$

on morphisms
$$\begin{pmatrix} G/H \\ \big\downarrow \phi \\ G/K \end{pmatrix} \longmapsto \begin{pmatrix} \overline{Rep}(H,\ D) \\ \big\uparrow \phi_* \\ \overline{Rep}(K,\ D) \end{pmatrix}$$

where ϕ_* is the restriction functor of Definition 6.1

$$\phi_*(\underline{\underline{G/K}} \xrightarrow{\ \xi\ } D) = (\underline{\underline{G/H}} \xrightarrow{\ \xi \circ \underline{\underline{\phi}}\ } D)$$

explicitly if $\phi(aH) = afK$ then $\phi_*(\xi)(aH) = \xi(afK)$. Since the restriction functors ϕ_* are natural with respect to cartesian functors of fibered categories over G we can get the correspondence on morphisms of Ψ. Further L denotes the composition of functors

$$L = \Psi \circ \Phi : Split(G) \longrightarrow O_G\text{-}Cat$$

We have also another O_G-categories from split G-categories.

Definition 7.3. Define a functor

$$I : Split(G) \longrightarrow O_G\text{-}Cat$$

For a (strict) functor $\alpha : G \longrightarrow Cat$ an O_G-category

$$I(\alpha) : {O_G}^{op} \longrightarrow Cat$$

is given as follows;

on objects $\quad G/H \longmapsto (\alpha(\cdot))^H$ = the H-fixed category of $\alpha(\cdot)$

on morphisms
$$\begin{pmatrix} G/H \\ \downarrow \phi \\ G/K \end{pmatrix} \longmapsto \begin{pmatrix} \alpha(\cdot)^H \\ \uparrow I(\alpha)(\phi) \\ \alpha(\cdot)^K \end{pmatrix}$$

where the functor $I(\alpha)(\phi)$ assignes

$$\begin{pmatrix} X \\ \downarrow u \\ Y \end{pmatrix} \longmapsto \begin{pmatrix} \alpha(f)X \\ \downarrow \alpha(f)u \\ \alpha(f)Y \end{pmatrix}$$

Note that $f \in G$ is given by $\phi(aH) = afK$, hence f is determined modulo K, but since objects and morphisms of $\alpha(\cdot)^K$ are K-fixed the functor $I(\alpha)(\phi)$ does not depend on a choice of representative f. Further for any K-fixed object or morphism x an object or morphism $\alpha(f)x$ is H-fixed as follows;

$$\alpha(h)\alpha(f)x = \alpha(hf)x = \alpha(fk)x = \alpha(f)\alpha(k)x = \alpha(f)x$$

where $h \in H$ and k $(\in K)$ is given by $f^{-1}Hf \subset K$.

We obtain by Corollary 3.6 an equivalence

$$\varphi : Cart_G(\underline{\underline{G/e}}, \Phi(\alpha)) \xrightarrow{\approx} \alpha(\cdot)$$

and by Proposition 4.10

$$Cart_G(\underline{\underline{G/H}}, \Phi(\alpha)) \cong Cart_G(\underline{\underline{G/e}}, \Phi(\alpha))^H$$

But as φ is not a split G-functor the functor I is different from the functor $L = \Psi \circ \Phi$.

Next we shall consider to construct split G-categories from O_G-categories.

Definition 7.4. Given an O_G-category

$$\mathcal{B} : O_G{}^{op} \longrightarrow Cat$$

Since one has

$$\mathrm{Hom}_{O_G}(G/e, G/e) = (G/e)^e = G$$

the category $\mathcal{B}(G/e)$ has a structure of a split G-category. Thus we have a functor

$$K : O_G\text{-}Cat \longrightarrow Split(G)$$
$$K(\mathcal{B}) = (\mathcal{B}(G/e); \mathcal{B}(s))$$

The functor

$$K \circ \Psi : Fib(G) \longrightarrow Split(G)$$

is nothing but the modified Giraud construction S (see § 4). In spite of very nice properties of the functor S the solitary K does not go well through. For instance

$$K(\mathcal{B})^H \quad \text{and} \quad \mathcal{B}(G/H)$$

have no relation. So we need another funtor from O_G-Cat to $Split(G)$. I constructed a functor from $Fib(G)$ to $Split(G)$ different from the Giraud construction S in my ealier paper [16] § 3. This was the one which decomposes through O_G-Cat. We will use the modified one to construct the desired functor.

Definition 7.5. A functor

$$U : O_G\text{-}Cat \longrightarrow Split(G)$$

is defined as follows. Given a O_G-category

$$\mathcal{B} : O_G{}^{op} \longrightarrow Cat$$

take a functor

$$\tilde{\beta} : \tilde{O}_G^{\,op} \longrightarrow Cat$$

where $\tilde{O}_G = (G/e)/O_G$ is the comma category of O_G under G/e,

$p : (G/e)/O_G \longrightarrow O_G$ is the fogetfull functor and

$\tilde{\beta}$ is the composition $\beta \circ p^{op} : \tilde{O}_G^{\,op} \longrightarrow O_G^{\,op} \longrightarrow Cat$

and a lax colimit over $\tilde{O}_G^{\,op}$

$$U_\beta = \underset{\tilde{O}_G^{\,op}}{lax\ colimit}\ \tilde{\beta}$$

Note that there is a fibered category U_β over $\tilde{O}_G$ associated to the functor $\tilde{\beta} : \tilde{O}_G^{\,op} \longrightarrow Cat$ by Grothendioeck construction. We shall write down explicitly the split G-category

$$U(\beta) = (U_\beta;\ \delta_s)$$

The category U_β has as objects triples $(X,\ G/H,\ x)$ where $G/H \in ob\ O_G$, $x \in G/H$ (a pair $(G/H,\ x)$ represents an object of $\tilde{O}_G$ such that $G/e \xrightarrow{\ \phi\ } G/H$, $\phi(a) = ax$) and $X \in ob\ \beta(G/H)$.A morphism

$$(X,\ G/H,\ x) \longrightarrow (Y,\ G/K,\ y)$$

of U_β is given by a pair $(\sigma,\ q)$ where $\sigma : G/H \longrightarrow G/K$ is a morphism of O_G such that $\sigma(x) = y$ (such σ provides a morphism of $\tilde{O}_G$) and

$$q : X \longrightarrow \beta(\sigma)Y$$

is a morphism of the category $\beta(G/H)$. The action δ_s on U_β $(s \in G)$ is given as follows;

$$\delta_s : \begin{pmatrix} (X,\ G/H,\ x) \\ \downarrow (\sigma,q) \\ (Y,\ G/K,\ y) \end{pmatrix} \longmapsto \begin{pmatrix} (X,\ G/H,\ s^{-1}x) \\ \downarrow (\sigma,q) \\ (Y,\ G/K,\ s^{-1}y) \end{pmatrix}$$

The assignment on morphisms of the functor U is defined in obvious manner by means of the naturality of lax colimits.

The functor U provides the desired homotopy property.

Theorem 7.6. For an O_G-category $\beta : O_G^{\,op} \longrightarrow Cat$ and a subgroup H

of G there is a homotopy equivalence of categories

$$(U_\beta)^H \simeq \beta(G/H)$$

Proof. We shall first define a subcategory V_H of $(U_\beta)^H$. The objects of V_H consist of triples $(X, G/H, eH)$ where $X \in ob\ \beta(G/H)$ and the morphisms of V_H consist of pairs $(id_{G/H}, q)$ where $q : X \longrightarrow Y$ is a morphism of $\beta(G/H)$. Then it is clear that the category V_H is isomorphic to $\beta(G/H)$ and we identify V_H with $\beta(G/H)$. We shall next construct a right adjoint functor

$$k : (U_\beta)^H \longrightarrow V_H = \beta(G/H)$$

to the inclusion functor

$$i : \beta(G/H) = V_H \longrightarrow (U_\beta)^H$$

Let $(X, G/K, y)$ be any object of $(U_\beta)^H$. Since $X \in ob\ \beta(G/K)$ and

$$y \in (G/K)^H = \mathrm{Hom}_{O_G}(G/H, G/K)$$

we have

$$\beta(\phi_y)X \in ob\ \beta(G/H)$$

where $\phi_y : G/H \longrightarrow G/K$ is a morphism of O_G corresponding to y. Hence define

$$k(X, G/K, y) = (\beta(\phi_y)X, G/H, eH)$$

Then it's clear $k \circ i = Id$. Also there is a natural transformation

$$\eta : i \circ k \longrightarrow Id$$

$$\eta(X, G/H, y) = (\phi_y, id_{\beta(\phi_y)X}) : (\beta(\phi_y)X, G/H, eH) \longrightarrow (X, G/K, y)$$

It is easily verified that k is right adjoint to i. Thus we have the desired homotopy equivalence. *q.e.d.*

Finally we shall note the relation with the work of Elmendorf [4].

Consider the classifying space functor

$$\mathrm{B} : Cat \longrightarrow Top$$

where *Top* denotes the category of certain nice topological spaces as usual.

Proposition 7.7. Suppose G is a finite group.

(1) The classifying space functor B sends a split G-category (*resp.* an O_G-category) to a G-space (*resp.* an O_G-space), hence there is a functor

$$\mathrm{B} : \mathit{Split}(G) \longrightarrow G\text{-}\mathit{spaces}$$

$$(\mathit{resp.}\ \mathrm{B} : O_G\text{-}\mathit{Cat} \longrightarrow O_G\text{-}\mathit{spaces})$$

(2) The image of our I (*resp.* K, *resp.* U) by the classifying space functor B is Elmendorf's Φ (*resp.* D, *resp.* C), hence there are commutative diagrams (up to homotopy for U)

$$\begin{array}{ccc} \mathit{Split}(G) & \overset{I}{\underset{K,\ U}{\rightleftarrows}} & O_G\text{-}\mathit{Cat} \\ \downarrow \mathrm{B} & & \downarrow \mathrm{B} \\ G\text{-}\mathit{spaces} & \overset{\Phi}{\underset{D,\ C}{\rightleftarrows}} & O_G\text{-}\mathit{spaces} \end{array}$$

Proof. (1) and the statement for K of (2) are trivial. Since

$$\mathrm{B}(C^H) = (\mathrm{B}C)^H$$

for any split G-category C and any subgroup H of G, one has the statement for I of (2). The remaining fact that U becomes a categorical Elmendorf construction *i.e.*

$$\mathrm{B} \circ U \simeq C \circ \mathrm{B}$$

is verified as follows. By Thomason's Homotopy Colimit Theorem [23]

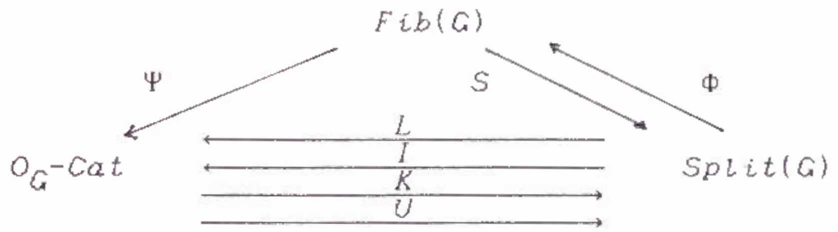
$$\mathrm{B}(U_\beta) \simeq \operatorname*{hocolim}_{\mathcal{O}_G^{op}} \mathrm{B}(\beta(G/H))$$

because U_β is given by a lax colimit over $\mathcal{O}_G^{op}$. On the other hand Elmendorf's definition of C using the two-sided bar construction is shown to be nothing but a homotopy colimit over $\mathcal{O}_G^{op}$ on simplicial sets level. Since taking a homotopy colimit commutes the geometric realization functor by Bousfield-Kan[1] the result will follow. *q.e.d.*

§ 8. Properties of functors connecting various notions of G-categories

In this last section we are going to study properties especially adjoint properties of functors providing the relations between various notions of G-categories which have been treated with until the preceeding section.

Since $Des(G)$ and $Pseudo(G)$ are equivalent to $Fib(G)$ it is sufficient to study the following triangle;



Recall the functors

Φ = the modified Grothendieck construction (see § 4 and Theorem 2.5)

$S = K \circ \Psi$ = the modified Giraud construction (see § 4)

$\Psi : (D \xrightarrow{\gamma} G) \longmapsto (G/H \longmapsto Cart_G(G/H,\ D)$ (see § 4)

$L = \Psi \circ \Phi$

$I : (G \xrightarrow{\alpha} Cat) \longmapsto (G/H \longmapsto \alpha(\cdot)^H)$ (see § 7)

$K : ({O_G}^{op} \xrightarrow{\beta} Cat) \longmapsto (\beta(G/e);\ \beta(s))$ (see § 7)

$U : ({O_G}^{op} \xrightarrow{\beta} Cat) \longmapsto (U_\beta;\ \delta_s),\ U_\beta = \underset{O_G}{lax\ colimit}\ \beta$ (see § 7)

First we note that the functor S is right adjoint to the functor Φ as shown in Theorem 4.7 (2). Next we shall show that the functor K is left adjoint and left inverse to the functor I.

Proposition 8.1. (1) $K \circ I = Id_{Split(G)}$

(2) There is a natural transformation

$$\mu : Id_{O_G\text{-}Cat} \longrightarrow I \circ K$$

(3) The functor K is left adjoint to the functor I.

Proof. (1) Trivial

(2) For a O_G-category $\beta : {O_G}^{op} \longrightarrow Cat$ the functor

$$\beta(p_H) : \beta(G/H) \longrightarrow \beta(G/e)$$

correspondent to the natural projection $p_H : G/e \longrightarrow G/H$, $p_H(a) = aH$ induces a functor

$$\mu_{G/H} : \beta(G/H) \longrightarrow \beta(G/e)^H$$

which is natural for $G/H \in ob\ {O_G}^{op}$. Thus we have a natural transformation

$$\mu : Id_{O_G\text{-}Cat} \longrightarrow I \circ K$$

(3) It follows from the definition of $\mu_{G/H}$ that $\mu_{G/e} = id_{\beta(G/e)}$, hence

$$K * \mu = id_K$$

On the other hand if $\beta = I(\alpha)$ for a split G-category $\alpha : G \longrightarrow Cat$ then

$$\mu_{G/H} = id : \alpha(\cdot)^H \longrightarrow \alpha(\cdot)^H$$

hence

$$\mu * I = id_I$$

These facts imply that K is left adjoint to I. *q.e.d.*

Putting two adjoint properties together we obtain

Proposition 8.2. (1) There is a natural isomorphism

$$\Psi \xrightarrow{\cong} I \circ S$$

of functors from $Fib(G)$ to $O_G\text{-}Cat$.

(2) The composed functor $\Phi \circ K$ is left adjoint to the functor Ψ.

Proof. (1) Let $\gamma : D \longrightarrow G$ be an object of $Fib(G)$. It follows from Proposition 1.10 that there is an isomorphism in Cat

$$\Psi(\gamma) \longrightarrow I \circ S(\gamma)$$

which is natural for $\gamma \in ob\ Fib(G)$. The result follows.

(2) Consider the compositions of two kinds of adjoint functors

$$Fib(G) \underset{S}{\overset{\Phi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} Split(G) \underset{I}{\overset{K}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} O_G\text{-}Cat$$

which imply that $\Phi \circ K$ is left adjoint to $I \circ S$. Together with (1) we obtain the result. *q.e.d.*

We shall now state the properties relating to the functor U which are obtained from Theorem 7.6.

Proposition 8.3. (1) There is a natural transformation

$$\eta : I \circ U \longrightarrow Id_{O_G\text{-}Cat}$$

(2) There is a natural transformation

$$\eta_{G/e} : U \longrightarrow K$$

(3) There is a natural transformation

$$\xi : U \circ I \longrightarrow Id_{Split(G)}$$

(4) $\eta * I = I * \xi : I \circ U \circ I \rightrightarrows I$

Proof. (1) Consider a functor

$$k : (U_\beta)^H \longrightarrow \beta(G/H)$$

in the proof of Theorem 7.6, which sends an object $(Y,\ G/H,\ y)$ of

$(U_\beta)^H$ to an object $\beta(\phi_y)Y$ of $\beta(G/H)$ where $\phi_y : G/H \longrightarrow G/K$ is the morphism of O_G corresponding to $y \in (G/K)^H$. Putting $k = \eta_{G/H}$, it's easily seen that $\eta_{G/H}$ is natural for $G/H \in ob\ O_G{}^{op}$, hence we have a natural transformation

$$\eta : I \circ U \longrightarrow Id_{O_G\text{-}Cat}$$

(2) $\eta_{G/e} = K * \eta : U = K \circ I \circ U \longrightarrow K$

(3) $\xi = K * \eta * I : U \circ I = K \circ I \circ U \circ I \longrightarrow K \circ Id \circ I = K \circ I = Id$

(4) Consider

$$I * \eta_{G/e} : I \circ U \longrightarrow I \circ K$$

which is of the form

$$((I * \eta_{G/e})_\beta)_{G/H} : (U_\beta)^H \longrightarrow \beta(G/e)^H$$

for $\beta \in ob\ O_G\text{-}Cat$ and $G/H \in ob\ O_G$. it is clear that if $\beta = I(\alpha)$ for $\alpha \in ob\ Split(G)$ then

$$((I * \eta_{G/e})_\beta)_{G/H} = \eta_{G/H}$$

This implies the result. *q.e.d.*

We note here that there are no adjoint relations containing the functor U such as the functor K in Propositions 8.1 and 8.3. We need to turn to the arguments up to homotopy. We shall finally state a part of outline of homotopy theory of G-categories.

The category $O_G\text{-}Cat$ has a structure of 2-category. Let β, β' be O_G-categories $O_G{}^{op} \rightrightarrows Cat$ and t, t' be O_G-functors from β to β'. A 2-arrow $\lambda : t \longrightarrow t'$ of $O_G\text{-}Cat$ is defined as follows; To each object G/H of O_G and each object X of $\beta(G/H)$ assign a morphism

$$\lambda_{G/H,\ X} : t_{G/H}(X) \longrightarrow t'_{G/H}(X)$$

of $\beta'(G/H)$. They satisfy the following conditions. For any morphism $u : X \longrightarrow Y$ of $\beta(G/H)$

$$t'_{G/H}(u) \circ \lambda_{G/H,\ X} = \lambda_{G/H,\ Y} \circ t_{G/H}(u)$$

and for any morphism $\phi : G/K \longrightarrow G/H$ of O_G a diagram

$$\begin{array}{ccc} \beta'(\phi)t_{G/H}(X) & \xrightarrow{\beta'(\phi)(\lambda_{G/H,X})} & \beta'(\phi)t'_{G/H}(X) \\ \| & & \| \\ t_{G/K}\beta(\phi)(X) & \xrightarrow{\lambda_{G/K,\ \beta(\phi)X}} & t'_{G/K}\beta(\phi)(X) \end{array}$$

commutes.

When regarding the categories $Fib(G)$, $Split(C)$ and O_G-Cat as 2-categories one has the following facts about 2-arrows.

Lemma 8.4. The functors Φ, Ψ, S, L, I, K and U preserve the respective 2-arrows.

Suppose the group G is finite from now on.

Let t, t' be morphisms from α to α' of $Split(G)$ (*resp.* of O_G-Cat). t is said to be G-homotopic (*resp.* O_G-homotopic) to t' if Bt is G-homotopic (*resp.* O_G-homotopic) to Bt'. $[\alpha, \alpha']_G$ (*resp.* $[\alpha, \alpha']_{O_G}$) denotes the set of G-homotopy classes (*resp.* O_G-homotopy classes) of morphisms from α to α' of $Split(G)$ (*resp.* of O_G-Cat).

Lemma 8.5. (1) The functors I, K and U preserve G- or O_G-homotopies.
(2) A morphism $\lambda : \alpha \longrightarrow \alpha'$ of $Split(G)$ is a G-homotopy equivalence if and only if $I(\lambda) : I(\alpha) \longrightarrow I(\alpha')$ is a O_G-homotopy equivalence.

Proof. (1) is obtained from Proposition 7.7 and [4].
(2) is obtained from the well-known fact that a G-map f of certain nice G-spaces is a G-homotopy equivalence if and only if the restriction f^H of f to the H-fixed spaces is a homotopy equivalence for each $H < G$. *q.e.d.*

Proposition 8.6. (1) For any object α of $Split(G)$ a morphism

$$\xi_\alpha : U \circ I(\alpha) \longrightarrow \alpha$$

of $Split(G)$ is a G-homotopy equivalence.

(2) For any object β of O_G-*Cat* two morphisms

$$(U * \eta)_\beta,\ (\xi * U)_\beta : U \circ I \circ U(\beta) \rightrightarrows U(\beta)$$

of *Split*(G) are G-homotopic.

(3) There is a bijection

$$[\alpha, U(\beta)]_G \cong [I(\alpha), \beta]_{O_G}$$

for $\alpha \in ob$ *Split*(G) and $\beta \in ob\ O_G$-*Cat*.

Proof. (1) is obtained from Proposition 7.7, Proposition 8.3 (4) and Lemma 8.5 (2).

(2) is obtained from (1), Theorem 7.6, Proposition 7.7 and Proposition 8.3 (4).

(3) can be verified as in [4] Theorem 2 by using (1), (2), Propositon 8.3 (4) and Lemma 8.5 (1). *q.e.d.*

Let t, t' be morphisms (*i.e.* cartesian morphisms) from γ to γ' of *Fib*(G). t is said to be fiber homotopic to t' over G if Bt is fiber homotopic to Bt'over BG.

By the classifying space functor B a fibered category $\gamma : D \longrightarrow G$ over G provides a Dold fibration (= a map fiber homotopic to a Hurewicz fibration)

$$\mathrm{B}\gamma : \mathrm{B}D \longrightarrow \mathrm{B}G$$

(see [15]Ⅳ2) ,hence the functor Φ preserves homotopies. But in general Ψ and S don't preserve homotopies because B$Cart_G(\underline{\underline{G/H}}, D)$ isn't homotopy equivalent to the space of fiber maps from B(G/H) to BD over BG. (Thomason's Homotopy Limit Problem [24]) Therefore on the present stage we have no more statements about homotopy theory of *Fib*(G).

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