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DIFFERENCES OF THE SELBERG TRACE FORMULA AND SELBERG TYPE ZETA FUNCTIONS FOR HILBERT MODULAR SURFACES

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ABSTRACT. We study analytic properties of a certain kind of Selberg type zeta functions attached to Hilbert modular surfaces. The method is based on considering the differences among the Selberg trace formula with several weights.

1. INTRODUCTION

In this article, we consider Selberg type zeta functions attached to the Hilbert modular group of a real quadratic field. First of all, we recall the definition of Selberg zeta function for a comapct Riemann surface. Let $G = \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ and $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ be the upper half plane. Then G acts on $\mathbb{H}$ by the fractional linear transformation $g.z = \frac{az+b}{cz+d}$. Let Γ be a co-compact torsion-free discrete subgroup of G, then the quotient space $X = \Gamma\backslash\mathbb{H}$ is a compact Riemann surface of genus $g \geq 2$.

Let $\gamma \in \Gamma$ is hyperbolic, that is $|\mathrm{tr}(\gamma)| > 2$, then the centralizer of γ in Γ is infinite cyclic and γ is conjugate in G to

$$\gamma \sim \begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix} \quad \text{with } N(\gamma) > 1.$$

Put $\mathrm{Prim}(\Gamma)$ be the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ. (i.e, not a power of other hyperbolic elements)

The Selberg zeta function for Γ (or X) is defined by the following Euler product:

$$Z_\Gamma(s) := \prod_{p\in\mathrm{Prim}(\Gamma)} \prod_{k=0}^{\infty} \Big(1 - N(p)^{-(k+s)}\Big) \qquad \text{for } \mathrm{Re}(s) > 1.$$

Selberg proved the following theorem on $Z_\Gamma(s)$:

Theorem 1.1 (Selberg 1956, [14]). (1) *$Z_\Gamma(s)$ defined for $\mathrm{Re}(s) > 1$ extends meromorphically over $\mathbb{C}$ (actually entire).*

(2) *$Z_\Gamma(s)$ has zeros at $s = -k$ ($k \in \mathbb{N}$) of order $(2g-2)(2k+1)$, at $s = 0$ of order $2g-1$ and at $s = 1$ of order 1 : trivial zeros.*

(3) *$Z_\Gamma(s)$ has zeros at $s = \frac{1}{2} \pm ir_n$: nontrivial zeros.*

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Here, $\{\lambda_n = 1/4 + r_n^2\}$ is the eigenvalues of the Laplacian $\Delta_0 = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ acting on $L^2(\Gamma\backslash\mathbb{H})$. This theorem is proved by using the Selberg trace formula for the compact Riemann surface $\Gamma\backslash\mathbb{H}$.

This zeta function $Z_\Gamma(s)$ satisfies the following functional equation:

Theorem 1.2 (Functional equation by Selberg 1956, [14]).

$$Z_\Gamma(1-s) = Z_\Gamma(s)\exp\Big(-4(g-1)\pi\int_0^{s-\frac{1}{2}} r\tan(\pi r)\,dr\Big).$$

The above functional equation is rewritten to a symmetric functional equation by using the double gamma function.

$$\hat{Z}_\Gamma(1-s) = \hat{Z}_\Gamma(s) := Z_\Gamma(s)\big(\Gamma_2(s)\Gamma_2(s+1)\big)^{2g-2}.$$

Here, $\Gamma_2(z) = \exp(\zeta_2'(0,z))$ is the double gamma function and $\zeta_2(s,z) = \sum_{n,m\geq 0}(n+m+z)^{-s}$ is the double Hurwitz zeta function. The theory of Selberg zeta functions for locally symmetric spaces of rank one is evolved by Gangolli [4] (compact case) and Gangolli-Warner [5] (noncompact case). Multiple gamma functions also appear in functional equation for these Selberg zeta functions. We refer to [11] for multiple gamma functions and [6], [7] for gamma factors of Selberg zeta functions of rank one locally symmetric spaces. Therefore, our concern is "Selberg type zeta functions" for *higher rank* locally symmetric spaces such as Hilbert modular varieties etc.

In this article, we consider the following problems:

(1) Construct Selberg type zeta functions for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$.

(2) Study analytic properties of the above Selberg type zeta functions for $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$.

In the next section, we introduce Selberg type zeta functions for Hilbert modular surfaces and study analytic properties of them.

2. Selberg type zeta functions for Hilbert modular surfaces

2.1. **Notation and definition.** Let $K/\mathbb{Q}$ be a real quadratic field with class number one and $\mathcal{O}_K$ be the ring of integers of K. Put D be the discriminant of K and $\varepsilon > 1$ be the fundamental unit of K. We denote the generator of $\mathrm{Gal}(K/\mathbb{Q})$ by σ and put $a' = \sigma(a)$ for $a \in K$. We also put $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathcal{O}_K)$.

Let $\Gamma_K = \{(\gamma,\gamma') \mid \gamma \in \mathrm{PSL}(2,\mathcal{O}_K)\}$ be the Hilbert modular group.

It is known that Γ_K is an irreducible discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})^2$ and Γ_K acts on the product of two upper half planes $\mathbb{H}^2$ by linear fractional transformation every component. Γ_K have only one cusp (∞,∞), i.e. Γ_K-inequivalent parabolic fixed point. $X_K = \Gamma_K\backslash\mathbb{H}^2$ is called the Hilbert modular surface.

Let $(\gamma,\gamma') \in \Gamma_K$ be hyperbolic-elliptic, i.e, $|\mathrm{tr}(\gamma)| > 2$ and $|\mathrm{tr}(\gamma')| < 2$. Then the centralizer of hyperbolic-elliptic (γ,γ') in Γ_K is infinite cyclic.

Fix an even integer $m \geq 6$.

Definition 2.1 (Selberg type zeta function for Γ_K).

$$Z_K(s;m) := \prod_{(p,p')}\prod_{k=0}^{\infty}\Big(1 - e^{i(m-2)\omega}\,N(p)^{-(k+s)}\Big)^{-\kappa} \quad \text{for } \mathrm{Re}(s) \gg 0$$

Here, (p,p') run through the set of primitive hyperbolic-elliptic Γ_K-conjugacy classes of Γ_K, and (p,p') is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$(p,p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \right).$$

Here, $N(p) > 1$, $\omega \in (0,\pi)$ and $\omega \notin \pi\mathbb{Q}$. We define the smallest natural number κ such that $2\kappa\,\zeta_K(-1) \in \mathbb{Z}$ and $\kappa\,\nu_j^{-1} \in \mathbb{Z}$ $(1 \le j \le N)$, where $\zeta_K(s)$ is the Dedekind zeta function of K and $\{\nu_1, \nu_2, \cdots, \nu_N\}$ is the set of the orders of primitive elliptic elements in Γ_K.

2.2. **Analytic properties of $Z_K(s;m)$.** Our main theorems on analytic properties of $Z_K(s;m)$ are followings.

Theorem 2.2. *For an even integer $m \ge 6$, $Z_K(s;m)$ a priori defined for* $\mathrm{Re}(s) > 1$ *has a meromorphic extension over the complex plane $\mathbb{C}$.*

Theorem 2.3. *$Z_K(s,m)$ has the following "essential" zeros and poles at*

- $s = \frac{1}{2} \pm i\rho_j \quad j = 0,1,2,\cdots$ *: zeros*
- $s = \frac{1}{2} \pm i\mu_k \quad k = 0,1,2,\cdots$ *: poles*

Here,

- $\{\frac{1}{4} + \rho_j^2 \,|\, j = 0,1,2,\cdots\} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_m^{(2)})})$
- $\{\frac{1}{4} + \mu_k^2 \,|\, k = 0,1,2,\cdots\} = \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_{m-2}^{(2)})})$

are the sets of eigenvalues of the Laplacian $\Delta_0^{(1)}$ acting on "Hilbert-Maass forms" of weight $(0,m)$ or $(0,m-2)$ and $\Lambda_m^{(2)}$, $\Lambda_{m-2}^{(2)}$ are "Maass operators".

2.3. **Functional equation of $Z_K(s;m)$.** $Z_K(s,m)$ has another series of zeros and poles coming from the identity, elliptic, "type 2 hyperbolic" conjugacy classes of Γ_K and the scattering terms. (See Definition 3.2 for type 2 hyperbolic element.)

Theorem 2.4. *$Z_K(s,m)$ satisfies the following functional equation*

$$\hat{Z}_K(s;m) = \hat{Z}_K(1-s;m).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;m) := Z_K(s;m)\Big(Z_{\mathrm{id}}(s)\,Z_{\mathrm{ell}}(s)\,Z_{\mathrm{sct/hyp2}}(s)\Big)^{\kappa}$$

with

$$Z_{\mathrm{id}}(s) := \Big(\Gamma_2(s)\Gamma_2(s+1)\Big)^{2\zeta_K(-1)}$$

$$Z_{\mathrm{ell}}(s) := \prod_{j=1}^{N}\prod_{l=0}^{\nu_j-1}\Gamma\big(\tfrac{s+l}{\nu_j}\big)^{\frac{\nu_j-1-\alpha_l(m,j)-\overline{\alpha_l}(m,j)}{\nu_j}}$$

$$Z_{\mathrm{sct/hyp2}}(s) := \zeta_\varepsilon(s+\tfrac{m}{2}-1)\,\zeta_\varepsilon(s+\tfrac{m}{2}-2)^{-1}$$

Here, $\{\nu_1, \nu_2, \cdots, \nu_N\}$ is the set of the orders of primitive elliptic elements in Γ_K and the definition of $\alpha_l(m,j)$, $\overline{\alpha_l}(m,j) \in \{0, 1, \cdots, \nu_j - 1\}$ will be given in the next subsection. We define $\zeta_\varepsilon(s) := (1-\varepsilon^{-2s})^{-1}$ and ε is the fundamental unit of K. The zeros and poles of $Z_{\mathrm{id}}(s)$, $Z_{\mathrm{ell}}(s)$ and $Z_{\mathrm{sct/hyp2}}(s)$ are easily calculated. Therefore, all zeros and poles of $Z_K(s;m)$ are determined.

These analytic properties and functional equation of $Z_K(s;m)$ are obtained by using the "differences" of the Selberg trace formula for Hilbert modular surfaces. In the next subsection, we introduce and investigate the Selberg trace formula for our case and their differences.

3. Differences of the Selberg trace formula for Hilbert modular surfaces

3.1. Notation. Let $G = \mathrm{PSL}(2,\mathbb{R})^2 = \Big(\mathrm{SL}(2,\mathbb{R})/\{\pm I\}\Big)^2$.

G acts on $\mathbb{H}^2$ by $(g_1, g_2).(z_1, z_2) = \big(\frac{a_1z_1+b_1}{c_1z_1+d_1}, \frac{a_2z_2+b_2}{c_2z_2+d_2}\big) \in \mathbb{H}^2$. $\Gamma \subset G$ is called irreducible discrete subgroup if it is not commensurable with any direct product $\Gamma_1 \times \Gamma_2$ of two discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

We have classification of the elements of irreducible Γ.

(1) $\gamma = (I, I)$ is the identity
(2) $\gamma = (\gamma_1, \gamma_2)$ is hyperbolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| > 2$ and $|\mathrm{tr}(\gamma_2)| > 2$
(3) $\gamma = (\gamma_1, \gamma_2)$ is elliptic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| < 2$ and $|\mathrm{tr}(\gamma_2)| < 2$
(4) $\gamma = (\gamma_1, \gamma_2)$ is hyperbolic-elliptic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| > 2$ and $|\mathrm{tr}(\gamma_2)| < 2$
(5) $\gamma = (\gamma_1, \gamma_2)$ is elliptic-hyperbolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| < 2$ and $|\mathrm{tr}(\gamma_2)| > 2$
(6) $\gamma = (\gamma_1, \gamma_2)$ is parabolic $\Leftrightarrow$ $|\mathrm{tr}(\gamma_1)| = |\mathrm{tr}(\gamma_2)| = 2$

Note that there are no other types in Γ. (parabolic-elliptic etc.) (Cf. Shimizu [16])

We consider the Hilbert modular group,

$$\Gamma_K := \Big\{(\gamma,\gamma') = \Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\Big)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathcal{O}_K)\Big\}.$$

Γ_K is an irreducible discrete subgroup of $G = \mathrm{PSL}(2,\mathbb{R})^2$ with the only one cusp $\infty := (\infty, \infty)$.

Lemma 3.1 (Stabilizer of the cusp $\infty = (\infty,\infty)$). *The stabilizer of $\infty = (\infty,\infty)$ in Γ_K is given by*

$$\Gamma_\infty := \Big\{\Big(\begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} u' & \alpha' \\ 0 & u'^{-1} \end{pmatrix}\Big)\Big| u \in \mathcal{O}_K^\times, \alpha \in \mathcal{O}_K\Big\}.$$

Definition 3.2 (Types of hyperbolic elements). For a hyperbolic element γ, we define that

- γ is type 1 hyperbolic $\Leftrightarrow$ whose all fixed points are not fixed by parabolic elements.
- γ is type 2 hyperbolic $\Leftrightarrow$ not type 1 hyperbolic.

Lemma 3.3. *Any type 2 hyperbolic elements of Γ_K are conjugate to an element of*

$$\left\{\gamma_{k,\alpha} = \begin{pmatrix} \varepsilon^k & \alpha \\ 0 & \varepsilon^{-k} \end{pmatrix} \,\middle|\, k \in \mathbb{N},\ \alpha \in \mathcal{O}_K\right\}$$

in Γ_K. The centralizer of $\gamma_{k,\alpha}$ in Γ_K is an infinite cyclic group.

By the above lemma, we may take a generator of the centralizer $Z_{\Gamma_K}(\gamma_{k,\alpha})$ as $\gamma_{k_0,\beta}$ with $k_0 \in \mathbb{N}$ and $\beta \in \mathcal{O}_K$. We also write k_0 as $k_0(\gamma_{k,\alpha})$.

Let $R_1, R_2, \cdots, R_N$ be a complete system of representatives of the Γ_K-conjugacy classes of primitive elliptic elements of Γ_K. $\nu_1, \nu_2, \cdots, \nu_N$ $(\nu \in \mathbb{N},\ \nu \geq 2)$ denote the orders of $R_1, R_2, \cdots, R_N$. We may assume that R_j is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$R_j \sim \left(\begin{pmatrix} \cos\frac{\pi}{\nu_j} & -\sin\frac{\pi}{\nu_j} \\ \sin\frac{\pi}{\nu_j} & \cos\frac{\pi}{\nu_j} \end{pmatrix}, \begin{pmatrix} \cos\frac{t_j\pi}{\nu_j} & -\sin\frac{t_j\pi}{\nu_j} \\ \sin\frac{t_j\pi}{\nu_j} & \cos\frac{t_j\pi}{\nu_j} \end{pmatrix} \right), \quad (t_j, \nu_j) = 1.$$

For even natural number $m \geq 4$ and $l \in \{0, 1, \cdots, \nu_j - 1\}$, we define $\alpha_l(m,j)$, $\overline{\alpha_l}(m,j) \in \{0, 1, \cdots, \nu_j - 1\}$ by

$$l + t_j\left(\frac{m-2}{2}\right) \equiv \alpha_l(m,j) \pmod{\nu_j}$$
$$l - t_j\left(\frac{m-2}{2}\right) \equiv \overline{\alpha_l}(m,j) \pmod{\nu_j}$$

We denote by Γ_{H1}, Γ_{E}, Γ_{HE}, Γ_{EH} and Γ_{H2}, type 1 hyperbolic Γ_K-conjugacy classes, elliptic Γ_K-conjugacy classes, hyperbolic-elliptic Γ_K-conjugacy classes, elliptic-hyperbolic Γ_K-conjugacy classes and type 2 hyperbolic Γ_K-conjugacy classes of Γ_K respectively.

3.2. **Selberg trace formula for Hilbert modular surfaces.** Fix the weight $(m_1, m_2) \in (2\mathbb{Z}_{\geq 0})^2$. Set the automorphic factor $j_\gamma(z_j) = \frac{cz_j+d}{|cz_j+d|}$ for $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ $(j = 1, 2)$. Let $\Delta_{m_j}^{(j)} := -y_j^2\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}\right) + im_j\, y_j \frac{\partial}{\partial x_j}$ $(j = 1, 2)$.

Definition 3.4.

$$L^2(\Gamma_K\backslash\mathbb{H}^2;\,(m_1,m_2)) := \Big\{ f\colon \mathbb{H}^2 \to \mathbb{C},\ C^\infty \,\Big|$$
$$(i)\ f((\gamma,\gamma')(z_1,z_2)) = j_\gamma(z_1)^{m_1} j_{\gamma'}(z_2)^{m_2} f(z_1,z_2) \quad \forall(\gamma,\gamma') \in \Gamma_K$$
$$(ii)\ \exists(\lambda^{(1)},\lambda^{(2)}) \in \mathbb{R}^2 \quad \Delta_{m_1}^{(1)} f(z_1,z_2) = \lambda^{(1)} f(z_1,z_2), \quad \Delta_{m_2}^{(2)} f(z_1,z_2) = \lambda^{(2)} f(z_1,z_2)$$
$$(iii)\ ||f||^2 = \int_{\Gamma_K\backslash\mathbb{H}^2} f(z)\overline{f(z)}\, d\mu(z) < \infty. \Big\}$$

Here, $d\mu(z) = \frac{dx_1dy_1}{y_1^2}\frac{dx_2dy_2}{y_2^2}$ for $z = (z_1, z_2) \in \mathbb{H}^2$.

Let $(m_1, m_2) \in (2\mathbb{Z}_{\geq 0})^2$, $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$, and $(s_1, s_2) \in \mathbb{C}^2$ with $\mathrm{Re}(s_1), \mathrm{Re}(s_2) \gg 0$. We define,

$$E_{(m_1,m_2)}(z, s_1, s_2) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_K} \frac{y_1^{s_1}}{|cz_1 + d|^{2s_1}} \frac{y_2^{s_2}}{|c' z_2 + d'|^{2s_2}} \frac{|cz_1 + d|^{m_1}}{(cz_1 + d)^{m_1}} \frac{|c' z_2 + d'|^{m_2}}{(c' z_2 + d')^{m_2}}.$$

Definition 3.5 (Family of Eisenstein series). For $(m_1, m_2) \in (2\mathbb{Z}_{\geq 0})^2$, $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $k \in \mathbb{Z}$ we define

$$E_{(m_1,m_2)}(z, s; k) := E_{(m_1,m_2)}\Big(z, s + \frac{\pi i k}{2 \log \varepsilon}, s - \frac{\pi i k}{2 \log \varepsilon}\Big).$$

Proposition 3.6. *For* $\mathrm{Re}(s) > 1$, *the Eisenstein series* $E_{(m_1,m_2)}(z, s; k)$ *is absolutely convergent and*

$$E_{(m_1,m_2)}(\gamma z, s; k) = E_{(m_1,m_2)}(z, s; k)$$

for any $\gamma \in \Gamma_K$. $E_{(m_1,m_2)}(z, s; k)$ *is a common eigenfunction of* $\Delta_{m_1}^{(1)}$ *and* $\Delta_{m_2}^{(2)}$.

Proposition 3.7. *We have a direct sum decomposition:*

$$L^2(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2)) = L^2_{dis}(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2)) \oplus L^2_{con}(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2))$$

and there is an orthonormal basis $\{\phi_j\}_{j=0}^\infty$ *of* $L^2_{dis}(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2))$.

To subtract continuous spectrum on $L^2_{\mathrm{con}}(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2))$, we introduce the scattering determinant $\varphi_{(m_1,m_2)}(s, k)$.

Proposition 3.8. *The constant term of* $E_{(m_1,m_2)}(z, s; k)$ *is given by*

$$y_1^{s + \frac{\pi i k}{2 \log \varepsilon}} y_2^{s - \frac{\pi i k}{2 \log \varepsilon}} + \varphi_{(m_1,m_2)}(s, k)\, y_1^{1 - s - \frac{\pi i k}{2 \log \varepsilon}} y_2^{1 - s + \frac{\pi i k}{2 \log \varepsilon}}$$

with

$$\varphi_{(m_1,m_2)}(s, k) = \frac{(-1)^{\frac{m_1 + m_2}{2}} \pi}{2\sqrt{D}} \frac{L(2s - 1, \chi_{-k})}{L(2s, \chi_{-k})} \frac{\Gamma(s + \frac{\pi i k}{2 \log \varepsilon} - \frac{1}{2}) \Gamma(s + \frac{\pi i k}{2 \log \varepsilon})}{\Gamma(s + \frac{\pi i k}{2 \log \varepsilon} + \frac{m_1}{2}) \Gamma(s + \frac{\pi i k}{2 \log \varepsilon} - \frac{m_1}{2})}$$
$$\times \frac{\Gamma(s - \frac{\pi i k}{2 \log \varepsilon} - \frac{1}{2}) \Gamma(s - \frac{\pi i k}{2 \log \varepsilon})}{\Gamma(s - \frac{\pi i k}{2 \log \varepsilon} + \frac{m_2}{2}) \Gamma(s - \frac{\pi i k}{2 \log \varepsilon} - \frac{m_2}{2})}$$

for $k \in \mathbb{Z}$, where $L(s, \chi_{-k})$ is defined by $L(s, \chi_{-k}) := \sum_{(c) \subset \mathcal{O}_K} \left| \frac{c}{c'} \right|^{-\frac{i\pi k}{\log \varepsilon}} N(c)^{-s}$.

Let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal basis of $L^2_{\mathrm{dis}}(\Gamma_K \backslash \mathbb{H}^2\,;\, (m_1, m_2))$ and $(\lambda_j^{(1)}, \lambda_j^{(2)}) \in \mathbb{R}^2$ such that

$$\Delta_{m_1}^{(1)} \phi_j = \lambda_j^{(1)} \phi_j \quad \text{and} \quad \Delta_{m_2}^{(2)} \phi_j = \lambda_j^{(2)} \phi_j.$$

Put $\mathrm{Spec}(m_1, m_2) := \big\{(r_j^{(1)}, r_j^{(2)})\big\}_{j=0}^\infty \subset \mathbb{R}^2$. (discrete subset), where, we write $\lambda_j^{(l)} = \frac{1}{4} + (r_j^{(l)})^2$. $(l = 1, 2)$.

Now we can state on the Selberg trace formula,

- $h(r_1, r_2) = h(\pm r_1, \pm r_2)$: test function (satisfying certain analytic conditions)

- $g(u_1,u_2) := \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r_1,r_2)e^{-i(r_1u_1+r_2u_2)}\,dr_1dr_2$: the Fourier transform of h

Hereafter, we assume that $h(r_1,r_2)=h_1(r_1)\,h_2(r_2)$ and also write $g(u_1,u_2)=g_1(u_1)g_2(u_2)$.

Theorem 3.9 (Selberg trace formula for $L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m))$ with $m\in 2\mathbb{Z}_{\geq 0}$). *Let $g(u_1,u_2)$ be an even function in $C_c^\infty(\mathbb{R}^2)$ and put $h(r_1,r_2) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u_1,u_2)e^{i(r_1u_1+r_2u_2)}\,du_1du_2$. Then we have,*

$$\begin{aligned}
&\sum_{j=0}^{\infty} h(r_j^{(1)},r_j^{(2)}) - \frac{1}{4\pi}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} h(r+\frac{\pi k}{2\log\varepsilon}, r-\frac{\pi k}{2\log\varepsilon})\,\frac{\varphi'_{(0,m)}}{\varphi_{(0,m)}}(\frac{1}{2}+ir,k)\,dr\\
&\quad+\frac{1}{4}h(0,0)\,\varphi_{(0,m)}(\frac{1}{2},0)\\
&=\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2}\iint_{\mathbb{R}^2}\frac{\frac{\partial^2}{\partial u_1\partial u_2}g(u_1,u_2)}{\sinh(u_1/2)\sinh(u_2/2)}e^{-\frac{m}{2}u_2}\,du_1du_2\\
&\quad+\sum_{(\gamma,\gamma')\in\Gamma_{H1}}\frac{\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\,g(\log N(\gamma),\log N(\gamma'))}{(N(\gamma)^{1/2}-N(\gamma)^{-1/2})(N(\gamma')^{1/2}-N(\gamma')^{-1/2})}\\
&\quad+\sum_{R(\theta_1,\theta_2)\in\Gamma_E}\frac{-e^{-i\theta_1+i(m-1)\theta_2}}{16\nu_R\sin\theta_1\sin\theta_2}\iint_{\mathbb{R}^2}g(u_1,u_2)e^{-\frac{u_1}{2}+\frac{(m-1)}{2}u_2}\prod_{j=1}^{2}\Big[\frac{e^{u_j}-e^{2i\theta_j}}{\cosh u_j-\cos 2\theta_j}\Big]du_1du_2\\
&\quad+\sum_{(\gamma,\omega)\in\Gamma_{HE}}\frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}\frac{ie^{i(m-1)\omega}}{4\sin\omega}\int_{-\infty}^{\infty}g(\log N(\gamma),u)e^{\frac{m-1}{2}u}\Big[\frac{e^u-e^{2i\omega}}{\cosh u-\cos 2\omega}\Big]du\\
&\quad+\sum_{(\omega',\gamma')\in\Gamma_{EH}}\frac{\log N(\gamma'_0)}{N(\gamma')^{1/2}-N(\gamma')^{-1/2}}\frac{ie^{-i\omega'}}{4\sin\omega'}\int_{-\infty}^{\infty}g(u,\log N(\gamma'))e^{\frac{-1}{2}u}\Big[\frac{e^u-e^{2i\omega'}}{\cosh u-\cos 2\omega'}\Big]du\\
&\quad+\Big[2\sqrt{D}A_0-4\log\varepsilon(\log 2+C_E)\Big]g(0,0)+\log\varepsilon\int_{-\infty}^{\infty}[g(u,0)+g(0,u)]du\\
&\quad-\frac{\log\varepsilon}{2\pi^2}\iint_{\mathbb{R}^2}\Big[\frac{\Gamma'}{\Gamma}(1+ir_1)+\frac{\Gamma'}{\Gamma}(1+ir_2)\Big]h(r_1,r_2)\,dr_1dr_2\\
&\quad+2\log\varepsilon\int_0^{\infty}\frac{g(0,u)}{e^{u/2}-e^{-u/2}}\Big[1-\cosh\frac{m}{2}u\Big]du\\
&\quad-4\log\varepsilon\sum_{k=1}^{\infty}\sum_{\gamma_{k,\alpha}\in\Gamma_{H2}}\frac{k_0(\gamma_{k,\alpha})\log(N(\alpha,\varepsilon^k-\varepsilon^{-k}))}{|N(\varepsilon^k-\varepsilon^{-k})|}g(2k\log\varepsilon,2k\log\varepsilon)\\
&\quad+2\log\varepsilon\sum_{k=1}^{\infty}\int_{2k\log\varepsilon}^{\infty}\Big[g(u,2k\log\varepsilon)+g(2k\log\varepsilon,u)\Big]\frac{\cosh(u/2)}{\sinh(u/2)+\sinh(k\log\varepsilon)}du\\
&\quad+2\log\varepsilon\sum_{k=1}^{\infty}\int_{2k\log\varepsilon}^{\infty}g(2k\log\varepsilon,u)\frac{1-\cosh\big(m(u/2-k\log\varepsilon)\big)}{\sinh(u/2-k\log\varepsilon)}du.
\end{aligned}$$

Here, A_0 is the constant term of the Laurent expansion of $\zeta_K(s)$ at $s=1$ and C_E is the Euler constant. The case of $(0,m)=(0,0)$ is proved by Zograf [17] and Efrat [1].

Next we consider the following Maass operator

$$\Lambda_m^{(2)} := iy_2\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_2} + \frac{m}{2} : L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m)) \to L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2)).$$

Let $\{\frac{1}{4}+\rho_j^2\}_{j=0}^{\infty} := \mathrm{Spec}(\Delta_0^{(1)}|_{\mathrm{Ker}(\Lambda_m^{(2)})})$ and recall that

$$\mathrm{Ker}(\Lambda_m^{(2)}) = L^2\Big(\Gamma_K\backslash\mathbb{H}^2\,;\,(*,\frac{m}{2}(1-\frac{m}{2})),(0,m)\Big).$$

i.e. $\lambda^{(2)} = \frac{m}{2}(1-\frac{m}{2})$-eigenspace.

Theorem 3.10 (Differences of STF for $L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m)) - L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2))$). *Let $m \in 2\mathbb{N}$ and $m \geq 4$. We have*

$$\begin{aligned}
&\sum_{j=0}^{\infty} h_1(\rho_j)\,h_2(\tfrac{i(m-1)}{2})\\
&= (m-1)h_2(\tfrac{i(m-1)}{2})\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2}\int_{-\infty}^{\infty} r_1 h_1(r_1)\tanh(\pi r_1)\,dr_1\\
&\quad+\sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}}\frac{ie^{i(m-1)\theta_2}}{8\nu_R\sin\theta_1\sin\theta_2}h_2(\tfrac{i(m-1)}{2})\int_{-\infty}^{\infty}\frac{\cosh((\pi-2\theta_1)r_1)}{\cosh\pi r_1}h_1(r_1)\,dr_1\\
&\quad+\sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}}\frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}g_1(\log N(\gamma))\,\frac{ie^{i(m-1)\omega}}{2\sin\omega}h_2(\tfrac{i(m-1)}{2})\\
&\quad-\log\varepsilon\,g_1(0)\,h_2(\tfrac{i(m-1)}{2}) - 2\log\varepsilon\,h_2(\tfrac{i(m-1)}{2})\sum_{k=1}^{\infty}g_1(2k\log\varepsilon)\,\varepsilon^{-k(m-1)}.
\end{aligned}$$

We write the above formula as $L(m)-L(m-2)$ for $m \geq 4$.

We assume that $h_2(\frac{i(m-1)}{2}) \neq 0$ and $h_2(\frac{i(m-3)}{2}) \neq 0$. Next we consider (for $m \geq 6$) : $\big(L(m)-L(m-2)\big)h_2(\frac{i(m-1)}{2})^{-1} - \big(L(m-2)-L(m-4)\big)h_2(\frac{i(m-3)}{2})^{-1}$.

Theorem 3.11 (Double differences of STF for $L^2\big(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m)\big)$). *Let $m \in 2\mathbb{N}$ and $m \geq 6$. We have*

$$\begin{aligned}
\sum_{j=0}^{\infty} h_1(\rho_j) - \sum_{k=0}^{\infty} h_1(\mu_k) &= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}\int_{-\infty}^{\infty} r h_1(r)\tanh(\pi r)\,dr\\
&\quad-\sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}}\frac{e^{i(m-2)\theta_2}}{4\nu_R\sin\theta_1}\int_{-\infty}^{\infty}\frac{\cosh((\pi-2\theta_1)r)}{\cosh\pi r}h_1(r)\,dr\\
&\quad-\sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}}\frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}g_1(\log N(\gamma))\,e^{i(m-2)\omega}\\
&\quad-2\log\varepsilon\sum_{k=1}^{\infty}g_1(2k\log\varepsilon)\,(\varepsilon^{-k(m-1)}-\varepsilon^{-k(m-3)}).
\end{aligned}$$

4. Proof of Theorems 2.2, 2.3 and 2.4

4.1. **Test function.** Theorem 3.9, the Selberg trace formula for $L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m))$, holds for the test function $h(r_1,r_2)$ which satisfies the following condition:

(1) $h(\pm r_1,\pm r_2)=h(r_1,r_2)$,
(2) h is analytic in the domain $|\operatorname{Im}(r_1)|<\frac{1}{2}+\delta$, $|\operatorname{Im}(r_2)|<\frac{m-1}{2}+\delta$ for some $\delta>0$,
(3) $h(r_1,r_2)=O\big((1+|r_1|^2+|r_2|^2)^{-2-\delta}\big)$ for some $\delta>0$ in this domain.

Let us consider the following test function: Firstly, we fix real numbers $\beta_1,\beta_2\geq 2$, $\beta_1\neq\beta_2$. For $s\in\mathbb{C}$, $\operatorname{Re}(s)>1$, We set

$$h_1(r):=\frac{\big((\beta_1^2-(s-\frac{1}{2})^2\big)\big((\beta_2^2-(s-\frac{1}{2})^2\big)}{(r^2+(s-\frac{1}{2})^2)(r^2+\beta_1^2)(r^2+\beta_2^2)}=\frac{1}{r^2+(s-\frac{1}{2})^2}+\frac{c_1(s)}{r^2+\beta_1^2}+\frac{c_2(s)}{r^2+\beta_2^2}$$

with

$$c_1(s)=\frac{(s-\frac{1}{2})^2-\beta_2^2}{\beta_2^2-\beta_1^2},\quad c_2(s)=-\frac{(s-\frac{1}{2})^2-\beta_1^2}{\beta_2^2-\beta_1^2}.$$

(See [13] for this type test functions.) Then the Fourier transform of h_1 is given by

$$g_1(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}h_1(r)e^{-iru}\,dr=\frac{1}{2s-1}e^{-(s-\frac{1}{2})|u|}+\frac{c_1(s)}{2\beta_1}e^{-\beta_1|u|}+\frac{c_2(s)}{2\beta_2}e^{-\beta_2|u|}.$$

Secondly, we take $g_2(u)\in C_c^\infty(\mathbb{R})$ such that its Fourier inverse transform $h_2(r)$ satisfies $h_2(\frac{i(m-1)}{2})\neq 0$ and $h_2(\frac{i(m-3)}{2})\neq 0$. Then we can easily check that our test function $h(r_1,r_2):=\kappa\,h_1(r_1)\,h_2(r_2)$ satisfies the above sufficient condition for Theorem 3.9. (κ is defined in Definition 2.1.)

Finally, we consider Theorem 3.11, the double difference of the Selberg trace formula, for the above our test function.

We recall that $\{\rho_j\}$ is given by

$$\Big\{\frac{1}{4}+\rho_j^2\Big\}_{j=0}^{\infty}=\operatorname{Spec}\big(\Delta_0^{(1)}|_{\operatorname{Ker}(\Lambda_m^{(2)})}\big)$$

with

$$\Lambda_m^{(2)}\colon L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m))\to L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2)).$$

Note that

$$\operatorname{Ker}(\Lambda_m^{(2)})=L^2\Big(\Gamma_K\backslash\mathbb{H}^2\,;\,(*,\frac{m}{2}(1-\frac{m}{2})),(0,m)\Big).$$

i.e. $\lambda^{(2)}=\frac{m}{2}(1-\frac{m}{2})$-eigenspace.

And $\{\mu_k\}$ is given by

$$\Big\{\frac{1}{4}+\mu_k^2\Big\}_{k=0}^{\infty}=\operatorname{Spec}\big(\Delta_0^{(1)}|_{\operatorname{Ker}(\Lambda_{m-2}^{(2)})}\big)$$

with

$$\Lambda_{m-2}^{(2)}\colon L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-2))\to L^2(\Gamma_K\backslash\mathbb{H}^2\,;\,(0,m-4)).$$

Note that

$$\operatorname{Ker}(\Lambda_{m-2}^{(2)})=L^2\Big(\Gamma_K\backslash\mathbb{H}^2\,;\,(*,\frac{m-2}{2}(2-\frac{m}{2})),(0,m-2)\Big).$$

i.e. $\lambda^{(2)} = \frac{m-2}{2}(2 - \frac{m}{2})$-eigenspace.

Theorem 4.1 (DD-STF for the above test function h_1 and h_2).

$$\kappa \sum_{j=0}^{\infty} \Big[\frac{1}{\rho_j^2 + (s-\frac{1}{2})^2} + \sum_{l=1}^{2} \frac{c_l(s)}{\rho_j^2 + \beta_l^2} \Big] - \kappa \sum_{k=0}^{\infty} \Big[\frac{1}{\mu_k^2 + (s-\frac{1}{2})^2} + \sum_{l=1}^{2} \frac{c_l(s)}{\mu_k^2 + \beta_l^2} \Big]$$

$$= 2\kappa \zeta_K(-1) \sum_{k=0}^{\infty} \Big[\frac{1}{s+k} + \sum_{l=1}^{2} \frac{c_l(s)}{\beta_l + \frac{1}{2} + k} \Big]$$

$$+ \frac{1}{2s-1} \frac{Z_K'(s)}{Z_K(s)} + \sum_{l=1}^{2} \frac{c_l(s)}{2\beta_l} \frac{Z_K'(\frac{1}{2}+\beta_l)}{Z_K(\frac{1}{2}+\beta_l)} + \frac{\kappa}{2s-1} \frac{Z_{ell}'(s)}{Z_{ell}(s)} + \sum_{l=1}^{2} \frac{\kappa c_l(s)}{2\beta_l} \frac{Z_{ell}'(\frac{1}{2}+\beta_l)}{Z_{ell}(\frac{1}{2}+\beta_l)}$$

$$+ \frac{\kappa}{2s-1} \frac{d}{ds} \log \Big\{ \frac{(1-\varepsilon^{-(2s+m-4)})}{(1-\varepsilon^{-(2s+m-2)})} \Big\} + \sum_{l=1}^{2} \frac{\kappa c_l(s)}{2\beta_l} \frac{d}{d\beta_l} \log \Big\{ \frac{(1-\varepsilon^{-(2\beta_l+m-3)})}{(1-\varepsilon^{-(2\beta_l+m-1)})} \Big\}.$$

Note that $c_l(1-s) = c_l(s)$ $(l = 1, 2)$, $c_1(s) + c_2(s) = -1$ and $\kappa \frac{\mathrm{vol}(\Gamma_K \backslash \mathbb{H}^2)}{4\pi^2} = 2\kappa \zeta_K(-1) \in \mathbb{N}$. By using the above formula, we can easily obtain Theorems 2.2, 2.3 and 2.4.

4.2. **Final remark.** We remark that scattering and type 2 hyperbolic components of $Z_K(s; m)$ are local Selberg zeta functions for $\mathrm{PSL}(2, \mathbb{Z})$:

$$Z_{\mathrm{sct/hyp2}}(s) = \zeta_\varepsilon(s + \tfrac{m}{2} - 1)\zeta_\varepsilon(s + \tfrac{m}{2} - 2)^{-1}$$
$$\text{with} \quad \zeta_\varepsilon(s) = (1 - \varepsilon^{-2s})^{-1}$$

Here, ε is the fundamental unit of K.

Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. The Selberg (Ruelle) zeta function for Γ is given by

$$\zeta_\Gamma(s) := \prod_{p \in \mathrm{Prim}(\Gamma)} (1 - N(p)^{-s})^{-1} \quad \text{then} \quad \zeta_\Gamma(s) = \prod_K (1 - \varepsilon(K)^{-2s})^{-h(K)},$$

where, K run through "all" real quadratic fields over $\mathbb{Q}$ and $\varepsilon(K)$ and $h(K)$ are the fundamental unit and the class number of K.

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