Mathematical Physics of the Navier-Stokes Turbulence

Tomomasa Tatsumi
Kyoto University (Professor Emeritus)

Abstract

Statistical mechanics of the Navier-Stokes turbulence is described briefly, making reference to the theory of homogeneous isotropic turbulence based on the cross-independence closure hypothesis (Tatsumi (2001), Tatsumi et al.(2004, 2007)). In particular, the relationship between the physical characteristics of turbulence and the mathematical structure of the governing equations is discussed.

Key words: Navier-Stokes turbulence, Cross independence, Inertial normality, Local non-normality

1. Mathematical Framework of Turbulence Theory

The velocity field of turbulence in an incompressible viscous fluid is governed by the Navier-Stokes equation.

1.1. The Navier-Stokes equation

If we denote the fluid velocity at a point $\mathbf{x}$ and time t by $\mathbf{u}(\mathbf{x},t)$ and the pressure by $p(\mathbf{x},t)$, the *Navier-Stokes equation* is expressed as

$$\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\partial/\partial\mathbf{x})\mathbf{u} - \nu|\partial\mathbf{x}|^2\mathbf{u} = -(1/\rho)\partial p/\partial\mathbf{x}, \qquad (1)$$

with the incompressibility condition,

$$(\partial/\partial\mathbf{x})\cdot\mathbf{u} = 0, \qquad (2)$$

where ρ and ν denote the density and the kinetic viscosity of the fluid respectively. Eqs.(1) and (2) constitute the *fundamental equations* for the fluid motion in general, including turbulence.

The pressure p can be eliminated from Eq.(1) using the relation,

$$\partial/\partial\mathbf{x}\cdot(\mathbf{u}\cdot\partial/\partial\mathbf{x})\mathbf{u} = -|\partial/\partial\mathbf{x}|^2(p/\rho) = -\Delta_{\mathbf{x}}(p/\rho), \qquad (3)$$

which is obtained by applying the condition (2) to Eq.(1), and its solution,

$$p/\rho = -\Delta_{\mathbf{x}}^{-1}\{\partial/\partial\mathbf{x}\cdot(\mathbf{u}\cdot\partial/\partial\mathbf{x})\mathbf{u}\} = (1/4\pi)\int|\mathbf{x}-\mathbf{x}'|^{-1}\{\partial/\partial\mathbf{x}'\cdot(\mathbf{u}'\cdot\partial/\partial\mathbf{x}')\mathbf{u}'\}d\mathbf{x}', \qquad (4)$$

where $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{x}}^{-1}$ denote the Laplacian and inverse Laplacian operators respectively and $\mathbf{u}'=\mathbf{u}(\mathbf{x}',t)$.

On substitution from Eq.(4), the equation of motion (1) is written as

$$\partial \mathbf{u}/\partial t + (\mathbf{u}\cdot\partial/\partial\mathbf{x})\mathbf{u} - \nu\,|\partial/\partial\mathbf{x}|^2\mathbf{u} = -(\partial/\partial\mathbf{x})(1/4\pi)\int|\mathbf{x}-\mathbf{x}'|^{-1}\{\partial/\partial\mathbf{x}'\cdot(\mathbf{u}'\cdot\partial/\partial\mathbf{x}')\mathbf{u}'\}d\mathbf{x}'. \tag{5}$$

Eq.(5) gives another expression of the *fundamental equation* for the fluid motion including turbulence.

These fundamental equations, either Eqs.(1) and (2) or Eq.(5), have common mathematical features, that is, the *nonlinearity* manifested by the transfer term $(\mathbf{u}\cdot\partial/\partial\mathbf{x})\mathbf{u}$ and the pressure term $-(1/\rho)\partial p/\partial\mathbf{x}$, and the *energy dissipation* due to the viscous term $-\nu\,|\partial/\partial\mathbf{x}|^2\mathbf{u}$. It will be shown below that these mathematical features give rise to physical difficulties in dealing with turbulent motions.

1.2. Statistics of turbulence

It is well known that the complete statistical description of turbulence is provided by the *probability distribution functional* of the turbulent velocity field $\mathbf{u}(\mathbf{x},t)$ at all possible values of $\mathbf{x}$ and t, and actually the equation for the characteristic functional of such a distribution functional has been obtained by Hopf (1952) using the Navier-Stokes equation and the probability conservation law.

Since, however, no general method is available at the moment for dealing with such a functional equation, an alternative description has to be employed. This is provided by an infinite set of the *joint-probability distributions* of the turbulent velocities at all possible space-time points, $(\mathbf{x}_1,t_1)$, $(\mathbf{x}_2,t_2)$ … $(\mathbf{x}_n,t_n)$, n being an arbitrary positive integer.

The equations for such multi-point velocity distributions have been obtained by Lundgren (1967) and Monin (1967) independently with each other. In this context, it should be noted that these equations for the multi-point velocity distributions are not closed since the equation for the n-point velocity distribution always includes the $(n+1)$-point velocity distribution. This is the outcome of the *nonlinearity* of the Navier-Stokes equation (1) or (5), and in order to make the equations solvable, we have to supplement the equations with a relation which expresses the $(n+1)$-point distribution in terms of the lower-order distributions. This is the well-known *closure problem* which commonly arises in the *nonlinear* physical systems.

1.3. Velocity distributions

If we denote the velocities at the two points by $\mathbf{u}_1 = \mathbf{u}(\mathbf{x}_1,t)$ and $\mathbf{u}_2 = \mathbf{u}(\mathbf{x}_2,t)$, the one- and two-point velocity distributions are defined respectively as

$$f(\mathbf{v}_1,\mathbf{x}_1,t) = \langle\delta(\mathbf{u}_1-\mathbf{v}_1)\rangle,$$

$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t) = \langle\delta(\mathbf{u}_1-\mathbf{v}_1)\delta(\mathbf{u}_2-\mathbf{v}_2)\rangle, \tag{6}$$

where $\mathbf{v}_i$ (i=1,2) denotes the probability variable corresponding to the velocities $\mathbf{u}_i$, δ the three-dimensional delta function, and ⟨ ⟩ the probability mean with respect to a certain initial distribution. The homogeneity of the distributions f and $f^{(2)}$ have been incorporated in their definitions. The higher-order distributions $f^{(n)}$, $n \geq 3$ can be defined accordingly.

These distributions must satisfy the reduction conditions,

$$\int f(\mathbf{v}_1,\mathbf{x}_1,t)\, d\mathbf{v}_1 = 1,$$
$$\int f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\, d\mathbf{v}_2 = f(\mathbf{v}_1,\mathbf{x}_1,t),$$
$$\int f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\, d\mathbf{v}_1 = f(\mathbf{v}_2,\mathbf{x}_2,t), \qquad (7)$$

where $d\mathbf{v}_i$ (i=1,2) represents the volume element in the velocity space.

In the following, we consider only *homogeneous turbulence* unless mentioned otherwise. Then, Eq.(6) is written as

$$f(\mathbf{v}_1,\mathbf{x}_1,t) = f(\mathbf{v}_1,t),$$
$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t) = f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t), \qquad (8)$$

where $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$ denotes the distance between the two points.

For homogeneous turbulence, we can generally assume the zero mean velocity,

$$\int \mathbf{v}_1 f(\mathbf{v}_1,t)\, d\mathbf{v}_1 = 0,$$
$$\iint \mathbf{v}_i f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t)\, d\mathbf{v}_1\, d\mathbf{v}_2 = 0 \qquad (i=1,2). \qquad (9)$$

Although there is no general relationship between the one- and two-point velocity distributions f and $f^{(2)}$, they are related in the definite forms for large and small distances $r = |\mathbf{r}|$ between the two points.

In the limit of large distance $r \to \infty$, the two velocities $\mathbf{u}(\mathbf{x}_1,t)$ and $\mathbf{u}(\mathbf{x}_2,t)$ become independent with each other, leading to the *separation condition*,

$$\lim_{r\to\infty} f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t) = f(\mathbf{v}_1,t) f(\mathbf{v}_2,t). \qquad (10)$$

In the opposite limit of vanishing distance $r \to 0$, the two velocities $\mathbf{u}(\mathbf{x}_1,t)$ and $\mathbf{u}(\mathbf{x}_2,t)$ become equivalent with each other, leading to the *coincidence condition*,

$$\lim_{r\to 0} f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t) = f(\mathbf{v}_1,t)\, \delta(\mathbf{v}_2 - \mathbf{v}_1). \qquad (11)$$

2. Equations of Turbulence

In principle, turbulence can be described in terms of the Lundgren-Monin equations for the multi-point velocity distributions. Since, however, these equations are not closed in the sense mentioned above, we have to introduce a *closure hypothesis* in order to make them solvable.

2.1. The Lundgren-Monin equations

The Lundgren-Monin equations for the one- and two-point velocity distributions are written for the general *inhomogeneous* case as

$$[\partial/\partial t + \mathbf{v}_1 \cdot \partial/\partial \mathbf{x}_1] f(\mathbf{v}_1,\mathbf{x}_1,t)$$
$$= \partial/\partial \mathbf{v}_1 \cdot [(1/4\pi)\, \partial/\partial \mathbf{x}_1 \iint |\mathbf{x}_2-\mathbf{x}_1|^{-1}(\mathbf{v}_2\cdot \partial/\partial \mathbf{x}_2)^2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\, d\mathbf{x}_2 d\mathbf{v}_2$$
$$- \nu \lim_{\mathbf{x}_2\to\mathbf{x}_1} |\partial/\partial \mathbf{x}_2|^2 \int \mathbf{v}_2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\, d\mathbf{v}_2], \quad (12)$$

$$[\partial/\partial t + \mathbf{v}_1 \cdot \partial/\partial \mathbf{x}_1 + \mathbf{v}_2 \cdot \partial/\partial \mathbf{x}_2] f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)$$
$$= \partial/\partial \mathbf{v}_1 \cdot [(1/4\pi)\, \partial/\partial \mathbf{x}_1 \iint |\mathbf{x}_3-\mathbf{x}_1|^{-1}(\mathbf{v}_3\cdot \partial/\partial \mathbf{x}_3)^2 f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t)\, d\mathbf{x}_3 d\mathbf{v}_3$$
$$- \nu \lim_{\mathbf{x}_3\to\mathbf{x}_1} |\partial/\partial \mathbf{x}_3|^2 \int \mathbf{v}_3 f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t) d\mathbf{v}_3]$$
$$+ \partial/\partial \mathbf{v}_2 \cdot [(1/4\pi)\, \partial/\partial \mathbf{x}_2 \iint |\mathbf{x}_3-\mathbf{x}_2|^{-1}(\mathbf{v}_3\cdot \partial/\partial \mathbf{x}_3)^2 f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t)\, d\mathbf{x}_3 d\mathbf{v}_3$$
$$- \nu \lim_{\mathbf{x}_3\to\mathbf{x}_2} |\partial/\partial \mathbf{x}_3|^2 \int \mathbf{v}_3 f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t) d\mathbf{v}_3], \quad (13)$$

with the three-point velocity distribution,

$$f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t) = \langle \delta(\mathbf{u}_1-\mathbf{v}_1)\delta(\mathbf{u}_2-\mathbf{v}_2)\delta(\mathbf{u}_3-\mathbf{v}_3)\rangle \quad (14)$$

The incompressibility condition (2) requires the following conditions for the velocity distributions:

$$\partial/\partial \mathbf{x}_1 \cdot \int \mathbf{v}_1 f(\mathbf{v}_1,\mathbf{x}_1,t)\, d\mathbf{v}_1 = 0,$$
$$\partial/\partial \mathbf{x}_i \cdot \int \mathbf{v}_i f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t)\, d\mathbf{v}_i = 0 \qquad (i=1,2).$$
$$\partial/\partial \mathbf{x}_i \cdot \int \mathbf{v}_i f^{(3)}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3;\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3;t)\, d\mathbf{v}_i = 0 \qquad (i=1,2,3), \quad (15)$$

the first of which is identically satisfied for homogeneous turbulence.

2.2. Cross-independence hypothesis

The easiest hypothesis for closing the Lundgren-Monin equations may be the *independence* relation between the two-point velocities, which is written for the two-point velocity distribution as

$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{x}_1,\mathbf{x}_2;t) = f(\mathbf{v}_1,\mathbf{x}_1,t)\, f(\mathbf{v}_2,\mathbf{x}_2,t). \quad (16)$$

Eq.(14) is exactly satisfied for the normal distribution f, but otherwise only valid for large distance $r = |\mathbf{x}_2-\mathbf{x}_1| \to\infty$ as seen from the condition (10). For an arbitrary distance r, Eq.(16) remains only to be a *quasi-normal* approximation.

On the other hand, if we take the sum and difference of the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$,

$$\mathbf{u}_+ = (1/2)(\mathbf{u}_1+\mathbf{u}_2), \qquad \mathbf{u}_- = (1/2)(\mathbf{u}_2-\mathbf{u}_1), \quad (17)$$

and introduce the distributions of these *cross-velocities*,

$$g_\pm(\mathbf{v}_\pm;\mathbf{x}_1,\mathbf{x}_2;t) = \langle \delta(\mathbf{u}_\pm-\mathbf{v}_\pm)\rangle,$$
$$g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t) = \langle \delta(\mathbf{u}_+-\mathbf{v}_+)\, \delta(\mathbf{u}_--\mathbf{v}_-)\rangle, \quad (18)$$

with the probability variables,

$$\mathbf{v}_+ = (1/2)(\mathbf{v}_1+\mathbf{v}_2), \quad \mathbf{v}_- = (1/2)(\mathbf{v}_2-\mathbf{v}_1). \tag{19}$$

we can consider a new relationship between the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$.

The cross-velocity distributions must satisfy the reduction conditions,

$$\int g_\pm(\mathbf{v}_\pm;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_\pm = 1,$$
$$\int g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_- = g_+(\mathbf{v}_+;\mathbf{x}_1,\mathbf{x}_2;t),$$
$$\int g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t)\,d\mathbf{v}_+ = g_-(\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t), \tag{20}$$

where $d\mathbf{v}_\pm$ represent the volume elements in the cross-velocity space.

For homogeneous turbulence, Eq.(18) is written as

$$g_\pm(\mathbf{v}_\pm;\mathbf{x}_1,\mathbf{x}_2;t) = g_\pm(\mathbf{v}_\pm,\mathbf{r},t),$$
$$g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{x}_1,\mathbf{x}_2;t) = g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t), \tag{21}$$

and the zero mean conditions Eq.(9) give the corresponding conditions,

$$\int \mathbf{v}_\pm g_\pm(\mathbf{v}_\pm,\mathbf{r},t)\,d\mathbf{v}_\pm = 0,$$
$$\iint \mathbf{v}_\pm g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t)\,d\mathbf{v}_+d\mathbf{v}_- = 0. \tag{22}$$

Like the velocity distributions f and $f^{(2)}$, the cross-velocity distributions $g_\pm$ and $g^{(2)}$ must satisfy the boundary conditions at large and small distances r between the two points.

In the limit of large distance $r \to \infty$, the cross-velocities $\mathbf{u}_+$ and $\mathbf{u}_-$ become independent with each other just like their constituent velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, and hence we have the *separation condition*,

$$\lim_{r\to\infty} g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t) = g_+(\mathbf{v}_+,\mathbf{r},t)\,g_-(\mathbf{v}_-,\mathbf{r},t). \tag{23}$$

In the opposite limit of vanishing distance $r \to 0$, the two velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ become equivalent with each other, and hence we have the *coincidence condition*,

$$\lim_{r\to 0} g_+(\mathbf{v}_+,\mathbf{r},t) = f(\mathbf{v}_1,t) = f(\mathbf{v}_2,t),$$
$$\lim_{r\to 0} g_-(\mathbf{v}_-,\mathbf{r},t) = \delta(\mathbf{v}_-),$$
$$\lim_{r\to 0} g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t) = f(\mathbf{v}_1,t)\delta(\mathbf{v}_-) = f(\mathbf{v}_2,t)\delta(\mathbf{v}_-). \tag{24}$$

Thus, if we define the *cross-independence* relation between the velocities $\mathbf{v}_+$ and $\mathbf{v}_-$ as

$$f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t) = 2^{-3} g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t), \tag{25}$$
$$g^{(2)}(\mathbf{v}_+,\mathbf{v}_-;\mathbf{r},t) = g_+(\mathbf{v}_+,\mathbf{r},t)g_-(\mathbf{v}_-,\mathbf{r},t), \tag{26}$$

it immediately follows from Eqs.(23) and (24) that this relation is valid for both the limits of the large and small distances $r \to \infty$ and 0. Thus we use the *cross-independence* relation (25) and (26) as the *closure hypothesis* in the present work.

2.3. Kolmogorov's hypotheses

It should be noted that there exists a close analogy between the *cross-independence* hypothesis, which assumes the independence of the velocity-difference $\mathbf{u}_-$ from the velocity-sum $\mathbf{u}_+$, with the *local similarity* hypothesis of Kolmogorov (1941), which implies the independence of the small eddies of turbulence represented by the velocity increment $\Delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1 (=2\mathbf{u}_-)$ from the large eddies represented by $\mathbf{u}_1$ and $\mathbf{u}_2$. The analogy is evident in this context and the both hypotheses are based upon the *scale-separation* of turbulence which is common at large Reynolds numbers.

However, the way of application of the hypothesis is rather different in these theories. While in Kolmogorov's theory, the *independence* of the velocity-increment has been assumed against the large-scale structures of turbulence and the argument has been succeeded by dimensional analyses, the *independence* is assumed specifically to the pair of the cross-velocities in the present theory and the hypothesis is incorporated into the exact (but open) set of the Lundgren-Monin equations. Concerning the more technical variance between the theories, reference will be made in case.

2.4. Equation for one-point velocity distribution

The Lundgren-Monin equtation (12) for the one-point velocity distribution is written for homogeneous turbulence as

$$[\partial/\partial t + \mathbf{v}_1 \cdot \partial/\partial\mathbf{x}_1] f(\mathbf{v}_1,t) = \partial/\partial\mathbf{v}_1 \cdot [(1/4\pi)\, \partial/\partial\mathbf{x}_1 \iint |\mathbf{r}|^{-1} (\mathbf{v}_2 \cdot \partial/\partial\mathbf{r}_2)^2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t)\, d\mathbf{r} d\mathbf{v}_2$$
$$- \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int \mathbf{v}_2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t)\, d\mathbf{v}_2].$$

Since, however, the transfer and the pressure terms vanish according to the homogeneity, this equation is reduced to

$$(\partial/\partial t) f(\mathbf{v}_1,t) = -\nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2\, \partial/\partial\mathbf{v}_1 \cdot \int \mathbf{v}_2 f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t)\, d\mathbf{v}_2. \tag{27}$$

On substitution from the cross-independence hypothesis (25) and (26), Eq.(27) is written as

$$(\partial/\partial t) f(\mathbf{v}_1,t) = -2^{-3} \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2\, \partial/\partial\mathbf{v}_1 \cdot \int \mathbf{v}_2\, g_+(\mathbf{v}_+,\mathbf{r},t) g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_2. \tag{28}$$

For small values of $|\mathbf{r}|$, the distribution g_+ on the right-hand side of Eq.(28) becomes

$$g_+(\mathbf{v}_+,\mathbf{r},t) \to f(\mathbf{v}_+,t) = f(\mathbf{v}_1 + \mathbf{v}_-,t) \to (1 + \mathbf{v}_- \cdot \partial/\partial\mathbf{v}_1) f(\mathbf{v}_1,t), \tag{29}$$

up to the linear term of $\mathbf{v}_-$. Then, on substitution from (29), the viscous term of Eq.(28) is written as

$$T_\nu = -2^{-3} \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2\, \partial/\partial\mathbf{v}_1 \cdot \int (\mathbf{v}_1 + 2\mathbf{v}_-)(1 + \mathbf{v}_- \cdot \partial/\partial\mathbf{v}_1) f(\mathbf{v}_1,t)\, g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_2$$
$$= -\nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int (3 + 2\mathbf{v}_- \cdot \partial/\partial\mathbf{v}_1)(1 + \mathbf{v}_- \cdot \partial/\partial\mathbf{v}_1) f(\mathbf{v}_1,t)\, g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_-$$
$$= -2\nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int (\mathbf{v}_- \cdot \partial/\partial\mathbf{v}_1)^2 f(\mathbf{v}_1,t)\, g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_-,$$

where other terms vanish according to Eqs.(20) and (22).

Taking account of the isotropic distribution of $\mathbf{v}_-$, the term T_v is rewritten as

$$T_v = -2\nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int (1/3)|\mathbf{v}_-|^2 |\partial/\partial\mathbf{v}_1|^2 f(\mathbf{v}_1,t)\, g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_-$$
$$= -\alpha(t)\, |\partial/\partial\mathbf{v}_1|^2 f(\mathbf{v}_1,t), \tag{30}$$

where

$$\alpha(t) = (2/3)\, \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int |\mathbf{v}_-|^2 g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_- \tag{31}$$

denotes a constant equivalent to $\varepsilon/3$, ε being the mean energy dissipation rate, and a function of time.

On substitution from (31), Eq.(28) gives the closed equation,

$$[\partial/\partial t + \alpha(t)\, |\partial/\partial\mathbf{v}|^2]\, f(\mathbf{v},t) = 0, \tag{32}$$

for the *one-point velocity distribution* f, where the suffix of $\mathbf{v}_1$ has been omitted for brevity.

3. One-Point Velocity Distribution

The one-point velocity distribution f is obtained by solving Eq.(32), but before that the parameter $\alpha(t)$ of the equation must be specified as the function of t.

3.1. Energy dissipation rate

On substitution from (18), the parameter $\alpha(t)$ defined by Eq.(31) is written as

$$\alpha(t) = (2/3)\, \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int |\mathbf{v}_-|^2 g_-(\mathbf{v}_-,\mathbf{r},t)\, d\mathbf{v}_-$$
$$= (2/3)\, \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \int |\mathbf{v}_-|^2 \langle \delta(\mathbf{u}_-(\mathbf{r},t) - \mathbf{v}_-)\rangle\, d\mathbf{v}_-$$
$$= (2/3)\, \nu \lim_{|\mathbf{r}|\to 0} |\partial/\partial\mathbf{r}|^2 \langle |\mathbf{u}_-(\mathbf{r},t)|^2\rangle$$
$$= (1/6)\, \nu \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} |\partial/\partial\mathbf{x}_2|^2 \langle |\mathbf{u}_2(\mathbf{x}_2,t) - \mathbf{u}_1(\mathbf{x}_1,t)|^2\rangle$$
$$= (1/6)\, \nu \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} \sum_{i,j=1}^{3} (\partial/\partial x_{2j})^2 \langle (u_{2i}(\mathbf{x}_2,t) - u_{1i}(\mathbf{x}_1,t))^2\rangle$$
$$= (1/3)\, \nu \lim_{|\mathbf{x}_2-\mathbf{x}_1|\to 0} \sum_{i,j=1}^{3} \langle (\partial u_{2i}(\mathbf{x}_2,t)/\partial x_{2j})^2\rangle$$
$$= (1/3)\, \nu \sum_{i,j=1}^{3} \langle (\partial u_{1i}(\mathbf{x}_1,t)/\partial x_{1j})^2\rangle. \tag{33}$$

On the other hand, the *energy* of turbulence is defined by the probability mean of the kinetic energy per unit mass of the fluid at the point $(\mathbf{x},t)$,

$$E(\mathbf{x},t) = (1/2)\langle|\mathbf{u}(\mathbf{x},t)|^2\rangle. \tag{34}$$

The equation of the energy E is obtained from the Navier-Stokes equation (1) as

$$\partial E/\partial t + \langle\mathbf{u}\rangle \cdot \partial/\partial\mathbf{x}\, \langle E + p/\rho)\rangle = -\varepsilon, \tag{35}$$

where

$$\varepsilon(\mathbf{x},t) = \nu \sum_{i,j=1}^{3} \langle(\partial u_i(\mathbf{x},t)/\partial x_j)^2\rangle \tag{36}$$

denotes the *energy-dissipation rate*.

For *homogeneous turbulence*, Eqs.(34)~(36) are written as

$$E(t) = (1/2)\langle|\mathbf{u}(\mathbf{x},t)|^2\rangle \tag{36}$$

$$dE/dt = -\,\varepsilon, \tag{37}$$

$$\varepsilon(t) = \nu \sum_{i,j=1}^{3} \langle(\partial u_i(\mathbf{x},t)/\partial x_j)^2\rangle. \tag{38}$$

Hence it follows from (33) and (38) that

$$\alpha(t) = (1/3)\,\varepsilon(t), \tag{39}$$

showing that the parameter $\alpha(t)$ of Eq.(32) is nothing but the *energy-dissipation rate* $\varepsilon(t)$.

3.2. Inviscid energy dissipation

Eq.(39) shows that time-dependence of $\alpha(t)$ is given by that of the energy dissipation rate $\varepsilon(t)$, and it will be shown below that the latter follows directly from the definition of *homogeneous turbulence*.

It is the principal hypothesis of Kolmogorov's theory (1941) of *local isotropic turbulence* that the small-scale eddies of turbulence are in an equilibrium state which depends upon the *energy dissipation rate* ε and the *viscosity* ν. Obviously, this hypothesis is preceded by the premise that the parameter ε is independent of the viscosity ν for its small values,

$$\varepsilon(t) = \nu \sum_{i,j=1}^{3} \langle(\partial u_i(\mathbf{x},t)/\partial x_j)^2\rangle \rightarrow \text{constant} > 0 \qquad \text{for} \quad \nu \rightarrow 0. \tag{40}$$

Eq.(40) implies the *inviscid catastrophe* of the velocity field of turbulence in the limit of vanishing viscosity. Although no mathematical proof is available yet for this premise, the existing large-scale numerical simulations of turbulent flows seem to support this result.

Thus, we shall maintain this principal hypothesis in our study and assume that the whole velocity field of *homogeneous isotropic turbulence* is statistically determined by the two parameters, the *energy dissipation rate* ε and the *viscosity* ν. It is hoped that this assumption is mathematically shown to be compatible with the Navier-Stokes equation.

In accordance with the fundamental hypothesis, the representative length L and time T of turbulence are expressed dimensionally in terms of ε and ν as

$$[L] = (\nu^3/\varepsilon)^{1/4}, \qquad [T] = (\nu/\varepsilon)^{1/2}, \tag{41}$$

which are known as the Kolmogorov length and time respectively.

On substitution from (41), Eq.(38) is expressed dimensionally as

$$[\varepsilon(t)] = [\nu]\ \{[L]/[T]\}^2/[L]^2 = [\nu]\ [T]^{-2}, \tag{42}$$

which gives the actual relation,

$$\varepsilon(t) = \varepsilon_0 t^{-2}, \qquad \varepsilon_0 = \nu c, \tag{43}$$

with a positive constant c. Eq.(43) gives the decay law of the *energy dissipation rate* $\varepsilon(t)$ in time for *homogeneous isotropic turbulence*. It also gives from Eq. (39) that

$$\alpha(t) = \alpha_0 t^{-2}, \qquad \alpha_0 = \varepsilon_0/3, \tag{44}$$

which is the required form of the function $\alpha(t)$.

On substitution from (43), Eq.(37) is integrated to give the relation,

$$E(t) = E_0 t^{-1}, \qquad E_0 = \varepsilon_0, \tag{45}$$

which is the *inverse linear energy decay law* for *homogeneous isotropic turbulence*.

The energy decay law (45) is fairly close but not quite equivalent to the existing laws given by several authors. The law $E(t) \sim t^{-10/7}$ was obtained by Loitsiansky (1939) assuming the existence of the integral $I_2 = -\int r^2\langle \mathbf{u}_1\cdot\mathbf{u}_2\rangle dr$, and the law $E(t) \sim t^{-6/5}$ was derived by Saffman (1967) from the invariance of the integral $I_0 = \int \langle \mathbf{u}_1\cdot\mathbf{u}_2\rangle dr$. Apparently, these laws are at variance with the present law (45) but, since the both integrals I_0 and I_2 vanish for homogeneous turbulence in the inviscid limit, there is no actual contradiction. The measured exponents of the time t due to the experiments and numerical simulations are reported to be in the range of $-1.0 \sim -1.4$, but nothing definite can be said until the measurements at higher Reynolds numbers are performed.

3.3. Inertial normal distribution (N1)

On substitution from the expression (44) for $\alpha(t)$, Eq.(32) for the velocity distribution f is written as

$$[\partial/\partial t + \alpha_0 t^{-2}|\partial/\partial \mathbf{v}|^2] f(\mathbf{v},t) = 0. \tag{46}$$

Generally speaking, Eq.(46), having a negative diffusion constant $-\alpha_0 t^{-2}$, may lead to an ill-posed initial value problem. In order to avoid such difficulty, we confine ourselves to the self-similar solutions in time. Introducing the similarity variables in accordance with Eq.(45) as

$$\mathbf{w} = \mathbf{v}t^{1/2}, \qquad \mathbf{w}_\pm = \mathbf{v}_\pm t^{1/2}, \qquad \mathbf{s} = \mathbf{r}\, t^{-1/2}, \tag{47}$$

we express the self-similar solutions as

$$F(\mathbf{w}) = t^{-3/2} f(\mathbf{v},t), \quad G_\pm(\mathbf{w}_\pm,\mathbf{s}) = t^{-3/2} g_\pm(\mathbf{v}_\pm,\mathbf{r},t), \tag{48}$$

with the parameter,

$$\alpha_0 = (2/3)\, \nu \lim_{|\mathbf{s}|\to 0} |\partial/\partial \mathbf{s}|^2 \int |\mathbf{w}_-|^2 G_-(\mathbf{w}_-,\mathbf{s})\, d\mathbf{w}_-. \tag{49}$$

On substitution from (47) and (48), Eq.(46) is expressed as

$$[(d/dw)^2 + (2/w)d/dw + (1/2\alpha_0)(w\,d/dw + 3)]\, F(w) = 0,$$

which can be factorized as

$$(d/dw + 2/w)(d/dw + w/2\alpha_0)\,F(w) = 0, \tag{50}$$

where $F(\mathbf{w})$ is written as $F(w)$ in view of the isotropy of the equations.

The solution of Eq.(50) entitled to be a probability distribution is obtained in the standard form as

$$F(w) = F_0(w) \equiv (4\pi\alpha_0)^{-3/2}\exp[-w^2/4\pi\alpha_0]. \tag{51}$$

The corresponding one-point velocity distribution is expressed in the original variables as

$$f(\mathbf{v},t) = f_0(\mathbf{v},t) \equiv (t/4\pi\alpha_0)^{3/2}\exp[-|\mathbf{v}|^2 t/4\pi\alpha_0], \tag{52}$$

and its one-dimensional component,

$$f(v,t) = f_0(v,t) \equiv (t/4\pi\alpha_0)^{1/2}\exp[-v^2 t/4\pi\alpha_0]. \tag{53}$$

Eqs.(52) and (53) represent the *inertial normal distribution*, N1 say, for the one-point velocity $\mathbf{v}$.

The distributions N1 change self-similarly in time. At the initial time $t = 0$, they represent a uniform distribution with zero probability density, grow up in time $t > 0$ as the normal distribution with decreasing variance, and eventually tend to the delta distribution around $|\mathbf{v}| = 0$ in the limit of time $t \to \infty$. The change of the one-dimensional distribution (53) of N1 in time is shown graphically in Fig.1.

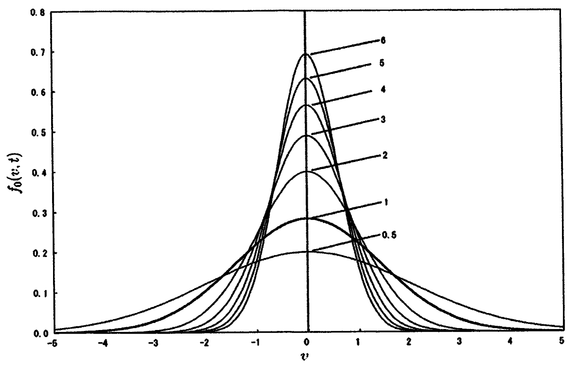

Fig.1. Inertial normal velocity distribution N1 (Eq.(53)). The numbers denote the time t/α_0.

The distributions N1 represented by Eqs.(52) and (53) have two remarkable features. One is the *inertial normality* of the velocity distribution. The *normality* itself is not surprising since, at small Reynolds numbers, the Navier-Stokes equation becomes essentially linear and the normality follows from the "*central limit theorem*" for the linear system (see Batchelor (1953), §8.2.). On the other hand, the *inertial normality* is concerned with large Reynolds numbers and hence located in the opposite limit to the *linear normality*.

Another point is that the *energy dissipation* $\alpha(t) = \varepsilon(t)/3$ is expressed as the integral of the *fluctuations* of small eddies as in Eq.(31). This is the manifestation of the "*fluctuation-dissipation theorem*" of statistical mechanics and indicates that the predominance of the scale-separation of turbulence into the large and small eddies at large Reynolds numbers.

4. Two-Point Velocity Distribution

4.1. Equation for two-point velocity distribution

The closed equation for the distribution $f^{(2)}$ is obtained by substituting the cross-independence hypothesis (25) and (26) into the $f^{(3)}$ terms of the Lundgren-Monin equation (13) for homogeneous isotropic turbulence, and following the similar manipulation to that for Eq.(28) as

$$[\partial/\partial t + (\mathbf{v}_2-\mathbf{v}_1)\cdot\partial/\partial\mathbf{r} + \alpha(t)(|\partial/\partial\mathbf{v}_1|^2+|\partial/\partial\mathbf{v}_2|^2) - \{\partial/\partial\mathbf{v}_1\cdot\partial/\partial\mathbf{x}_1\beta_1(\mathbf{v}_1,t) + \partial/\partial\mathbf{v}_2\cdot\partial/\partial\mathbf{x}_2\beta_2(\mathbf{v}_2,t)\}]f^{(2)}(\mathbf{v}_1,\mathbf{v}_2;\mathbf{r},t) = 0, \qquad (54)$$

where $\alpha(t)$ has been given by Eq.(11) and $\beta_1(\mathbf{v}_1,t)$ and $\beta_2(\mathbf{v}_1,t)$ are defined as

$$\beta_1(\mathbf{v}_1,t) = (1/4\pi)\iint|\mathbf{r}'|^{-1}((\mathbf{v}_1+2\mathbf{v}'_-)\cdot\partial/\partial\mathbf{r}')^2\, g_-(\mathbf{v}'_-,\mathbf{r}',t)d\mathbf{v}'_-d\mathbf{r}',$$

$$\beta_2(\mathbf{v}_2,t) = (1/4\pi)\iint|\mathbf{r}''|^{-1}((\mathbf{v}_2+2\mathbf{v}''_-)\cdot\partial/\partial\mathbf{r}'')^2\, g_-(\mathbf{v}''_-,\mathbf{r}'',t)d\mathbf{v}''_-d\mathbf{r}''. \qquad (55)$$

4.2. Equations for cross-velocity distributions

The two-point velocity distribution f is practically expressed in terms of the cross-velocity distributions g_+ and g_-. The closed equations for the distributions g_+ and g_- are obtained by substituting (25) and (26) into Eq.(54) and integrating the equation with respect to $\mathbf{v}_-$ and $\mathbf{v}_+$ respectively as

$$[\partial/\partial t + (1/2)\alpha(t)|\partial/\partial\mathbf{v}_+|^2]\, g_+(\mathbf{v}_+,\mathbf{r},t) = 0, \qquad (56)$$

$$[\partial/\partial t + (1/2)\alpha(t)|\partial/\partial\mathbf{v}_-|^2 + 2\mathbf{v}_-\cdot\partial/\partial\mathbf{r} + (1/2)\,\partial/\partial\mathbf{v}_-\cdot\{\partial/\partial\mathbf{x}_1\beta_1(\mathbf{v}_-,t) - \partial/\partial\mathbf{x}_2\beta_2(\mathbf{v}_-,t)\}]\, g_-(\mathbf{v}_-,\mathbf{r},t) = 0. \qquad (57)$$

Eq.(56) for the velocity-sum distribution g_+ is found to be independent of the distance $\mathbf{r}$ and identical with Eq.(32) for the one-point velocity distribution f except for that the parameter α of the latter has been replaced by $(1/2)\alpha$.

On the other hand, Eq.(57) for the velocity-difference distribution g_- includes the $\mathbf{r}$ dependent terms in addition to the $\mathbf{r}$- independent terms identical with those of Eq.(56). Since, however, it will be shown below that these $\mathbf{r}$- dependent terms give non-zero contribution only in the *local similarity range* of the order of Kolmogorov's length $\eta = (\nu^3/\varepsilon)^{1/4}$. Thus, under the inertial similarity $\nu\rightarrow 0$, Eq.(57) is expressed in the *outer similarity range* as

$$[\partial/\partial t + (1/2)\alpha(t)|\partial/\partial\mathbf{v}_-|^2]\, g_-(\mathbf{v}_-,\mathbf{r},t) = 0 \qquad \text{for} \quad |\mathbf{r}| > 0. \qquad (58)$$

4.3. Inertial normal distribution (N2)

According to the discussions in the previous subsection, the equations for the cross-velocity distributions g_+ and g_- are derived from Eqs.(56) and (58) as

$$[\partial/\partial t + (1/2)\alpha_0 t^{-2}|\partial/\partial \mathbf{v}_\pm|^2]g_\pm(\mathbf{v}_\pm,\mathbf{r},t) = 0, \tag{59}$$

where Eq.(44) has been taken into account.

Eq.(59) is solved in the same way as for Eq.(46) to give the three-dimensional normal distribution,

$$g_\pm(\mathbf{v}_\pm,\mathbf{r},t) = g_0(\mathbf{v}_\pm,t) \equiv (t/2\pi\alpha_0)^{3/2}\exp[-|\mathbf{v}_\pm|^2 t/2\pi\alpha_0], \tag{60}$$

and its one-dimensional component,

$$g_\pm(v_\pm,r,t) = g_0(v_\pm,t) \equiv (t/2\pi\alpha_0)^{3/2}\exp[-v_\pm^2 t/2\pi\alpha_0]. \tag{61}$$

Eqs.(60) and (61) give the same *inertial normal distribution*, N2 say, for the *cross-velocity distributions* $g_\pm$, indicating that the cross-velocities $\mathbf{v}_+$ and $\mathbf{v}_-$ are equivalent with each other in view of the symmetry of the one-point velocity distribution f. It may also be noted that the normal distribution N2 for the velocity-sum $\mathbf{v}_+$ is equivalent with the convolution of the two normal distributions N1 for the one-point velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, showing that the distributions $g_+(\mathbf{v}_+,\mathbf{r},t)$, $f(\mathbf{v}_1,t)$ and $f(\mathbf{v}_2,t)$ compose a self-consistent triplet of the *inertial normal distributions*.

4.4. Boundary conditions

Although the cross-velocity distributions $g_\pm$ have been obtained as the **r**- independent distributions (60) in the whole *outer range* $|\mathbf{r}| > 0$, they must satisfy the respective coincidence conditions (24) in the limit of $r \to 0$. This implies the discontinuous changes of the distributions $g_\pm$ at $r = 0$, but if we take into account the effect of the finite viscosity $\nu > 0$, such discontinuous changes of $g_\pm$ must be replaced by the continuous changes through the *local similarity range*.

5. Local Similarity of Velocity Distributions

5.1. Local variables

The continuous changes in the distributions g_+ and g_- in the local similarity range are described by means of the *local variables* based on Kolmogorov's length $\eta = (\nu^3/\varepsilon)^{1/4}$ and velocity $\upsilon = (\nu\varepsilon)^{1/4}$. In homogeneous turbulence, the energy dissipation rate $\varepsilon(t)$ changes in time, so that we have to take its value $\varepsilon(t_0) = \varepsilon_0 t_0^{-2}$ at a certain time $t = t_0$ for this purpose. Then, Kolmogorov's scales are expressed as

$$\text{Length} \quad \eta = (\nu^3/\varepsilon(t_0))^{1/4}, \qquad \text{Velocity} \quad \upsilon = (\nu\varepsilon(t_0))^{1/4}, \tag{62}$$

and the *local variables* based on these scales are defined as

$$\mathbf{x}^* = \mathbf{x}/\eta, \quad t^* = t/(\eta/\upsilon), \quad \mathbf{u}^*(\mathbf{x}^*,t^*) = \mathbf{u}(\mathbf{x},t)/\upsilon, \quad p^*(\mathbf{x}^*,t^*)/\rho = p(\mathbf{x},t)/\rho\upsilon^2, \tag{63}$$

where the suffix * denotes the non-dimensional local variables.

5.2. Equations in local range

The Navier-Stokes equation (1) and the incompressibility equation (2) are written in the local variables as

$$\partial \mathbf{u}^*/\partial t^* + (\mathbf{u}^* \cdot \partial/\partial \mathbf{x}^*)\mathbf{u}^* - |\partial \mathbf{x}^*|^2 \mathbf{u}^* = -(1/\rho)\partial p^*/\partial \mathbf{x}^*, \tag{64}$$

$$(\partial/\partial \mathbf{x}^*) \cdot \mathbf{u}^* = 0, \tag{65}$$

respectively, where the local viscosity ν^* becomes unity,

$$\nu^* = \nu/\eta\upsilon = 1/R^* = 1, \tag{66}$$

together with the Reynolds number R^* for the local range.

5.3. Equations in local variables

The closed equation for the distribution $f^{(2)}$ in the local range can be obtained from the Lundgren-Monin equation (13) written in the local variables and following the same process as that for Eq.(54). In this case, however, the distance $\mathbf{r}^*$ is so small that the cross-independence hypothesis has to be applied to the velocities $(\mathbf{v}^*_1, \mathbf{v}^*_2)$ as well.

The equation for $f^{(2)}$ thus obtained is expressed as

$$\begin{aligned}&[\partial/\partial t^* + (\mathbf{v}^*_2 - \mathbf{v}^*_1)\cdot\partial/\partial \mathbf{r}^* + \alpha^*_+(r^*,t^*)|\partial/\partial \mathbf{v}^*_+|^2 + \alpha^*_-(r^*,t^*)|\partial/\partial \mathbf{v}^*_-|^2 \\ &\quad - [\partial/\partial \mathbf{v}^*_1 \cdot \partial/\partial \mathbf{x}^*_1\{\beta^*_1(\mathbf{v}^*_1,t^*) + \gamma^*_1(\mathbf{v}^*_1,t^*)\} + \partial/\partial \mathbf{v}^*_2 \cdot \partial/\partial \mathbf{x}^*_2\{\beta^*_2(\mathbf{v}^*_2,t^*) + \gamma^*_2(\mathbf{v}^*_2,t^*)\}]\,]\times \\ &\qquad \times f^{(2)}(\mathbf{v}^*_1,\mathbf{v}^*_2;\mathbf{r}^*,t^*) = 0,\end{aligned} \tag{67}$$

with

$$\alpha^*_\pm(r^*,t^*) = (2/3)\lim_{|\mathbf{r}^{*\prime}|\to 0}|\partial/\partial \mathbf{r}^{*\prime}|^2 \int |\mathbf{v}^{*\prime}_\pm|^2 g_\pm(\mathbf{v}^{*\prime}_\pm;\mathbf{r}^*,\mathbf{r}^{*\prime};t^*)d\mathbf{v}^{*\prime}_\pm; \tag{68}$$

$$\begin{aligned}&\beta^*_1(\mathbf{v}^*_1,t^*) = (1/4\pi)\iint |\mathbf{r}^{*\prime}|^{-1}((\mathbf{v}^*_1+2\mathbf{v}^{*\prime}_-)\cdot\partial/\partial \mathbf{r}^{*\prime})^2 g_-(\mathbf{v}^{*\prime}_-,\mathbf{r}^{*\prime},t^*)d\mathbf{v}^{*\prime}_- d\mathbf{r}^{*\prime}, \\ &\beta^*_2(\mathbf{v}^*_2,t^*) = (1/4\pi)\iint |\mathbf{r}^{*\prime\prime}|^{-1}((\mathbf{v}^*_2+2\mathbf{v}^{*\prime\prime}_-)\cdot\partial/\partial \mathbf{r}^{*\prime\prime})^2 g_-(\mathbf{v}^{*\prime\prime}_-,\mathbf{r}^{*\prime\prime},t^*)d\mathbf{v}^{*\prime\prime}_- d\mathbf{r}^{*\prime\prime};\end{aligned} \tag{69}$$

$$\begin{aligned}&\gamma^*_1(\mathbf{v}^*_1,t^*) = (1/4\pi)\iint |\mathbf{r}^{*\prime}|^{-1}((\mathbf{v}^*_1+2\mathbf{v}^{*\prime}_-)\cdot\partial/\partial \mathbf{r}^{*\prime})^2(\mathbf{v}^{*\prime}_-\cdot\partial/\partial \mathbf{v}^*_1)g_-(\mathbf{v}^{*\prime}_-,\mathbf{r}^{*\prime},t^*)d\mathbf{v}^{*\prime}_- d\mathbf{r}^{*\prime}, \\ &\gamma^*_2(\mathbf{v}^*_2,t^*) = (1/4\pi)\iint |\mathbf{r}^{*\prime\prime}|^{-1}((\mathbf{v}^*_2+2\mathbf{v}^{*\prime\prime}_-)\cdot\partial/\partial \mathbf{r}^{*\prime\prime})^2(\mathbf{v}^{*\prime\prime}_-\cdot\partial/\partial \mathbf{v}^*_2)g_-(\mathbf{v}^{*\prime\prime}_-,\mathbf{r}^{*\prime\prime},t^*)d\mathbf{v}^{*\prime\prime}_- d\mathbf{r}^{*\prime\prime};\end{aligned} \tag{70}$$

where the following identity holds between $\alpha^*_\pm$ and α^* or the local expression of α defined by (31):

$$\begin{aligned}&\alpha^*_+(r^*,t^*) + \alpha^*_-(r^*,t^*) = \alpha^*(t^*), \\ &\alpha^*(t^*) = (2/3)\lim_{|\mathbf{r}^*|\to 0}|\partial/\partial \mathbf{r}^*|^2 \int |\mathbf{v}^*_-|^2 g_-(\mathbf{v}^*_-,\mathbf{r}^*,t^*)\, d\mathbf{v}^*_-.\end{aligned} \tag{71}$$

5.4. Equation for velocity-sum distribution

Equation for the velocity-sum distribution g_+ is obtained by substituting (25) and (26) into Eq.(67) and integrating the equation with respect to $\mathbf{v}^*_-$ as

$$[\partial/\partial t^* + \alpha^*_+(r^*,t^*)|\partial/\partial \mathbf{v}_+^*|^2 - (1/2)\partial/\partial \mathbf{v}^*_+\cdot [\partial/\partial \mathbf{x}^*_1\{\beta^*_1(\mathbf{v}^*_+,t^*)+(1/2)\gamma^*_1(\mathbf{v}^*_+,t^*)\}$$
$$+ \partial/\partial \mathbf{x}^*_2\{\beta^*_2(\mathbf{v}^*_+,t^*)+(1/2)\gamma^*_2(\mathbf{v}^*_+,t^*)\}]\,]\, g_+(\mathbf{v}^*_+,\mathbf{r}^*,t^*)=0.$$

Since, however, $\beta^*_1 = \beta^*_2$, $\gamma^*_1 = \gamma^*_2$ and $\partial/\partial \mathbf{x}^*_1 = -\partial/\partial \mathbf{x}^*_2 = -\partial/\partial \mathbf{r}^*$ in this context, the terms in [ ] identically vanish, so that the equation is simplified as

$$[\partial/\partial t^* + \alpha^*_+(r^*,t^*)|\partial/\partial \mathbf{v}_+|^2]\, g_+(\mathbf{v}^*_+,\mathbf{r}^*,t^*) = 0. \tag{72}$$

Eq.(72) gives the closed equation for the velocity-sum distribution g_+ for the local range, which has the same normal form as Eq.(56) for the distribution g_+ for the outer range but with the r^*-dependent parameter α^*_+ instead of the constant $\alpha/2$ for the latter.

5.5. Velocity-sum distribution (N3)

Eq.(72) is solved like Eq.(56) to give the three-dimensional normal distribution, N3 say,

$$g_+(\mathbf{v}^*_+,\mathbf{r}^*,t^*) = g_{+0}(\mathbf{v}^*_+,\mathbf{r}^*,t^*) \equiv (t^*/4\pi\alpha^*_{+0}(r^*))^{3/2}\exp[-|\mathbf{v}^*_+|^2 t^*/4\pi\alpha^*_{+0}(r^*)], \tag{73}$$

and its one-dimensional component,

$$g_+(\mathrm{v}^*_+,r^*,t^*) = g_{+0}(\mathrm{v}^*_+,r^*,t^*) \equiv (t^*/4\pi\alpha^*_{+0}(r^*))^{1/2}\exp[-\mathrm{v}^{*2}_+ t^*/4\pi\alpha^*_{+0}(r^*)], \tag{74}$$

with the parameter,

$$\alpha^*_+(r^*,t^*) = \alpha^*_{+0}(r^*)\, t^{*-2}. \tag{75}$$

Eq.(75) gives the decay laws of the *self-energy* $E^*_+(r^*,t^*)$ and the *self dissipation* $\varepsilon^*_+(r^*,t^*)$ associated with the velocity-sum distribution g_+ as follows:

$$E^*_+(r^*,t^*) = (1/2)\langle|\mathbf{u}^*_+(\mathbf{x}^*,\mathbf{r}^*,t^*)|^2\rangle = E^*_{+0}(r^*)\, t^{*-1},$$
$$\partial E^*_+(r^*,t^*)/\partial t^* = -\varepsilon^*_+(r^*,t^*),$$
$$\varepsilon^*_+(r^*,t^*) = 3\alpha^*_+(r^*,t^*) = \sum_{i,j=1}^3 \langle(\partial u^*_{+i}(\mathbf{x}^*,\mathbf{r}^*,t^*)/\partial x_j)^2\rangle = \varepsilon^*_{+0}(r^*)\, t^{*-2}. \tag{76}$$

The parameter $\alpha^*_+(r^*,t^*)$ must satisfy the asymptotic conditions for very large and small distances. In the limit of $r^*\to\infty$, the velocities $\mathbf{u}^*_1$ and $\mathbf{u}^*_2$ become independent with each other and it follows from (76) that

$$\alpha^*_+(r^*,t^*) = (1/3)\sum_{i,j=1}^3 \langle[(1/2)(\partial\{u^*_{1i}(\mathbf{x}^*_1,t^*) + u^*_{2i}(\mathbf{x}^*_1+\mathbf{r}^*,t^*)\}/\partial x^*_{1j})]^2\rangle$$
$$\to (1/12)\sum_{i,j=1}^3 \langle\{\partial u^*_{1i}(\mathbf{x}^*_1,t^*)/\partial x^*_{1j}\}^2 + \{\partial u^*_{2i}(\mathbf{x}^*_2,t^*)/\partial x^*_{2j}\}^2\rangle$$
$$= (1/6)\sum_{i,j=1}^3 \langle\{\partial u^*_{1i}(\mathbf{x}^*_1,t^*)/\partial x^*_{1j}\}^2\rangle$$
$$= (1/2)\ \alpha^*(t^*), \tag{77}$$

where the homogeneity of the distribution has been considered.

On the other hand, in the limit of $r^*\to 0$, the velocities $\mathbf{u}^*_1$ and $\mathbf{u}^*_2$ become identical with each other and it follows in the same way that

$$\alpha^*_+(r^*,t^*) = (1/3)\sum_{i,j=1}^{3} \langle[(1/2)(\partial\{u^*_{1i}(\mathbf{x}^*_1,t^*) + u^*_{2i}(\mathbf{x}^*_1+\mathbf{r}^*,t^*)\}/\partial x^*_{1j})]^2\rangle$$
$$\rightarrow (1/3)\sum_{i,j=1}^{3} \langle\{\partial u^*_{1i}(\mathbf{x}^*_1,t^*)/\partial x^*_{1j}\}^2\rangle$$
$$= \alpha^*(t^*). \qquad (78)$$

On substitution from (77) and (78) into Eqs.(73) and (74), we find that the distribution N3 tends to the distribution N2 represented by Eqs.(60) and (61) in the limit of $r^*\rightarrow\infty$ and to the distribution N1 represented by Eqs.(52) and (53) in the limit of $r^*\rightarrow 0$. This clearly shows that the discontinuous change of the velocity-sum distribution $g_+(\mathbf{v}_+,\mathbf{r},t)$ at $r = 0$ is actually represented by the continuous change of the distribution $g_+(\mathbf{v}^*_+,\mathbf{r}^*,t^*)$ throughout the local range $0 \le r^* < \infty$. The dependence of $\alpha^*_+(r^*,t^*)$ on the distance r^* is still indefinite at this stage, but it will be determined from the relation (71) after the parameter $\alpha^*_-(r^*,t^*)$ has been determined as a function of r^*. This clearly shows that the decay of the *self-energy* $E^*_+(r^*,t^*)$ and the *self-dissipation* $\varepsilon^*_+(r^*,t^*)$ of the velocity-sum $\mathbf{u}^*_+$ representing the energy-containing components of turbulence is obtained from Eq.(76) for arbitrary values of the distance r^*.

5.6. Equation for velocity-difference distribution

Equation for the velocity-difference distribution g_- is obtained by substituting (25) and (26) into Eq.(67) and integrating the equation with respect to $\mathbf{v}^*_+$ as

$$[\partial/\partial t^* + 2\mathbf{v}^*_-\cdot\partial/\partial\mathbf{r}^* + \alpha^*_-(r^*,t^*)|\partial/\partial\mathbf{v}_-^*|^2 + (1/2)\partial/\partial\mathbf{v}^*_-\cdot[\partial/\partial\mathbf{x}^*_1\{\beta^*_1(\mathbf{v}^*_-,t^*)+(1/2)\gamma^*_1(\mathbf{v}^*_-,t^*)\}$$
$$- \partial/\partial\mathbf{x}^*_2\{\beta^*_2(\mathbf{v}^*_-,t^*) + (1/2)\gamma^*_2(\mathbf{v}^*_-,t^*)\}]\,]\, g_-(\mathbf{v}^*_-,\mathbf{r}^*,t^*) = 0, \quad (79)$$

where the suffixes of β^*_1, γ^*_1 and β^*_2, γ^*_2 are retained according to their singularities at $\mathbf{x}^*_1$ and $\mathbf{x}^*_2$.

Eq.(79) gives the closed equation for the velocity-difference distribution g_- in the local range, but it has a serious difficulty that it does not satisfy the incompressibility condition (15). This is due to the fact that the compressibility of a viscous fluid appears as the finite thickness of the shock wave of $O(\nu)$, which is still smaller than the local length-scale based on Kolmogorov's length $\eta = O(\nu^{3/4})$. In order to avoid this difficulty, we employ the variables $(\mathbf{v}^*_1,\mathbf{v}^*_-)$ in the range of $O(\nu)$ in place of the variables $(\mathbf{v}^*_+,\mathbf{v}^*_-)$ in the range of $O(\nu^{3/4})$ (see §5.4. Tatsumi et al. (2007)).

Then, Eq.(79) for the velocity-difference distribution g_- is replaced by the equation,

$$[\partial/\partial t^* + 2\mathbf{v}^*_-\cdot\partial/\partial\mathbf{r}^* + \alpha^*_-(r^*,t^*)|\partial/\partial\mathbf{v}_-^*|^2$$
$$- (1/2)\partial/\partial\mathbf{v}^*_-\cdot\partial/\partial\mathbf{x}^*_2\{\beta^*_2(2\mathbf{v}^*_-,t^*) + (1/2)\gamma^*_2(2\mathbf{v}^*_-,t^*)\}]\, g_-(\mathbf{v}^*_-,\mathbf{r}^*,t^*) = 0, \qquad (80)$$

whereas Eq.(72) for the distribution g_+ remains unchanged only the argument $\mathbf{v}^*_+$ being replaced by $\mathbf{v}^*_1$. Eq.(80) gives the equation for the velocity-difference distribution g_- in the local range, which satisfies the incompressibility condition (15) identically.

5.7. Lateral and longitudinal velocity-difference distributions

In view of the axi-symmetry of Eq.(80) around the vector $\mathbf{r}^*$, the distribution g_- can be decomposed into the lateral and longitudinal components. Denoting the velocity arguments as $\mathbf{v}^*_- = (v^*_\parallel, v^*_\perp, v^*_{\perp\perp})$ with $\mathbf{r}^* = (r^*,0,0)$, we can define the one-dimensional components as

$$g_\parallel(v^*_\parallel, r^*, t^*) = \iint g_-(\mathbf{v}^*_-, \mathbf{r}^*, t^*)\, dv^*_\perp\, dv^*_{\perp\perp},$$
$$g_\perp(v^*_\perp, r^*, t^*) = \iint g_-(\mathbf{v}^*_-, \mathbf{r}^*, t^*)\, dv^*_\parallel dv^*_{\perp\perp},$$
$$g_\perp(v^*_{\perp\perp}, r^*, t^*) = \iint g_-(\mathbf{v}^*_-, \mathbf{r}^*, t^*)\, dv^*_\parallel dv^*_\perp. \qquad (81)$$

Eq.(80) for the distribution g_- is written in terms of its components (81) as

$$[\partial/\partial t^* + 2v^*_\parallel \partial/\partial r^* + \alpha^*_-(r^*,t^*)\, \{(\partial/\partial v^*_\parallel)^2 + (\partial/\partial v^*_\perp)^2 + (\partial/\partial v^*_{\perp\perp})^2\}$$
$$- (1/2)\, \partial/\partial v^*_\parallel \partial/\partial r^* \{\beta^*_2(2v^*_-, t^*) + (1/2)\, \gamma^*_2(2v^*_-, t^*)\}]\, g_\parallel(v^*_\parallel, r^*, t^*)\, g_\perp(v^*_\perp, r^*, t^*)\, g_\perp(v^*_{\perp\perp}, r^*, t^*) = 0. \qquad (82)$$

5.8. Lateral velocity-difference distribution (N4)

Integration of Eq.(82) with respect to $(v^*_\parallel, v^*_{\perp\perp})$ gives the equation for the lateral velocity-difference distribution $g_\perp$ in the local range as

$$[\partial/\partial t^* + \alpha^*_-(r^*,t^*)(\partial/\partial v^*_\parallel)^2]\, g_\perp(v^*_\perp, r^*, t^*) = 0. \qquad (83)$$

Since Eq.(83) is identical with the one-dimensional version of Eq.(72) for the distribution g_+, its solution is immediately derived from Eq.(74) as the one-dimensional inertial normal distribution, N4 say,

$$g_\perp(v^*_\perp, r^*, t^*) = g_{\perp 0}(v^*_\perp, r^*, t^*) \equiv (t^*/4\pi\alpha^*_{-0}(r^*))^{1/2} \exp[-v^{*2}_\perp t^*/4\pi\alpha^*_{-0}(r^*)], \qquad (84)$$

with the parameter,

$$\alpha^*_-(r^*,t^*) = \alpha^*_{-0}(r^*)\, t^{*-2}. \qquad (85)$$

It may be interesting to note that the velocity-difference distribution g_-, which should be singular in the limit of $|\mathbf{r}| \to 0$ according to the coincidence condition (24), can be regular as far as its lateral component $g_\perp$ in the local coordinate is concerned. This, however, does not exclude the possibility that the distribution $g_\perp$ is singular at still smaller length-scale of $O(\nu)$ as mentioned in the section **5.5** .

6. Longitudinal Velocity-Difference Distribution

Unlike the lateral velocity-difference distribution $g_\perp$, its longitudinal counterpart $g_\parallel$ is governed by the full equation of Eq.(82) including the r^*-dependent terms, so that it seems to require considerable amount of work to clarify the behavior of this distribution in the local range.

6.1. Equation for longitudinal velocity-difference distribution

Integration of Eq.(82) with respect to $(v^*_\perp, v^*_{\perp\perp})$ gives the equation for the longitudinal velocity-difference distribution $g_\parallel$ in the local range as

$$[\partial/\partial t^* + \alpha^*_-(r^*,t^*)(\partial/\partial v^*_\parallel)^2 + 2v^*_\parallel \partial/\partial r^*$$
$$-(1/2)\,\partial/\partial v^*_\parallel \partial/\partial r^* M_{v^*\perp,v^*\perp\perp}\{\beta^*_2(2v^*_-,t^*) + (1/2)\,\gamma^*_2(2v^*_-,t^*)\}]\, g_\parallel(v^*_\parallel,r^*,t^*) = 0, \quad (86)$$

where $M_{v^*\perp,v^*\perp\perp}$ denotes the mean with respect to the arguments $(v^*_\perp, v^*_{\perp\perp})$. Since Eq.(86) includes the r^*-dependent terms, its solution is generally not normal.

Eq.(85) gives the decay laws of the *self-energy* $E^*_-(r^*,t^*)$ and the *self-dissipation* $\varepsilon^*_-(r^*,t^*)$ associated with the velocity-difference distribution g_- as follows:

$$E^*_-(r^*,t^*) = (1/2)\,\langle|\mathbf{u}^*_-(\mathbf{x}^*,\mathbf{r}^*,t^*)|^2\rangle = E^*_{-0}(r^*)\,t^{*-1},$$
$$\varepsilon^*_-(r^*,t^*) = 3\alpha^*_-(r^*,t^*) = \sum_{i,j=1}^3 \langle(\partial u^*_{-i}(\mathbf{x}^*,\mathbf{r}^*,t^*)/\partial x_j)^2\rangle = \varepsilon^*_{-0}(r^*)\,t^{*-2}. \quad (87)$$

The parameter $\alpha^*_-(r^*,t^*)$ must satisfy the asymptotic conditions for very large and small distances. In the limit of $r^*\to\infty$, the velocities $\mathbf{u}^*_1$ and $\mathbf{u}^*_2$ become independent with each other and it follows from (86) that

$$\alpha^*_-(r^*,t^*) = (1/3)\sum_{i,j=1}^3 \langle[(1/2)(\partial\{u^*_{2i}(\mathbf{x}^*_1+\mathbf{r}^*,t^*) - u^*_{1i}(\mathbf{x}^*_1,t^*)\}/\partial x^*_{1j})]^2\rangle$$
$$\to (1/12)\sum_{i,j=1}^3 \langle\{\partial u^*_{2i}(\mathbf{x}^*_2,t^*)/\partial x^*_{2j}\}^2 + \{\partial u^*_{1i}(\mathbf{x}^*_1,t^*)/\partial x^*_{1j}\}^2\rangle$$
$$= (1/6)\sum_{i,j=1}^3 \langle\{\partial u^*_{1i}(\mathbf{x}^*_1,t^*)/\partial x^*_{1j}\}^2\rangle$$
$$= (1/2)\ \alpha^*(t^*), \quad (88)$$

On the other hand, in the limit of $r^*\to 0$, the velocities $\mathbf{u}^*_1$ and $\mathbf{u}^*_2$ become identical with each other and it immediately follows

$$\alpha^*_-(r^*,t^*) = (1/3)\sum_{i,j=1}^3 \langle[(1/2)(\partial\{u^*_{2i}(\mathbf{x}^*_1+\mathbf{r}^*,t^*) - u^*_{1i}(\mathbf{x}^*_1,t^*)\}/\partial x^*_{1j})]^2\rangle$$
$$\to 0. \quad (89)$$

On substitution from (88) and (89) into Eq.(84), we find that the distribution N4 tends to the distribution N2 represented by Eq.(61) in the limit of $r^*\to\infty$ and to the delta distribution in the limit of $r^*\to 0$. This clearly shows that the discontinuous change of the velocity-difference distribution $g_-(\mathbf{v}_-,\mathbf{r},t)$ at $r = 0$ is actually represented by the continuous change of the distribution $g_-(\mathbf{v}^*_-,\mathbf{r}^*,t^*)$ in the local range $0 \le r^* < \infty$. The dependence of $\alpha^*_-(r^*,t^*)$ on the distance r^* is still indefinite at this stage, but it will be determined by working out the longitudinal component $g_\parallel$ numerically.

6.2. Simplified equation

In order to deal with Eq.(86) for the distribution $g_\parallel$, it may be practical to simplify the non-local β^*_2

and γ^*_2 terms. These terms defined (69) and (70) are rewritten assuming the isotropic distribution of $\mathbf{v}^{*\prime\prime}_{-}$ around the vector $\mathbf{r}^{*\prime\prime}$ as follows:

$$\beta^*_2(2\mathbf{v}^*_{-},t^*) + (1/2)\gamma^*_2(2\mathbf{v}^*_{-},t^*)$$
$$=(1/4\pi)\iint |\mathbf{r}^{*\prime\prime}|^{-1}((2\mathbf{v}^*_{-}+2\mathbf{v}^{*\prime\prime}_{-})\cdot\partial/\partial\mathbf{r}^{*\prime\prime})^2\{1+(1/4)(\mathbf{v}^{*\prime\prime}_{-}\cdot\partial/\partial\mathbf{v}^*_{-})\}\, g_{-}(\mathbf{v}^{*\prime\prime}_{-},\mathbf{r}^{*\prime\prime},t^*)d\mathbf{v}^{*\prime\prime}_{-}d\mathbf{r}^{*\prime\prime}$$
$$=(1/4\pi)\iint |\mathbf{r}^{*\prime\prime}|^{-1}(1/3)|2\mathbf{v}^*_{-}+2\mathbf{v}^{*\prime\prime}_{-}|^2|\partial/\partial\mathbf{r}^{*\prime\prime}|^2\{1+(1/4)(\mathbf{v}^{*\prime\prime}_{-}\cdot\partial/\partial\mathbf{v}^*_{-})\}\, g_{-}(\mathbf{v}^{*\prime\prime}_{-},\mathbf{r}^{*\prime\prime},t^*)d\mathbf{v}^{*\prime\prime}_{-}d\mathbf{r}^{*\prime\prime}$$
$$=-(1/3)\iint \delta(\mathbf{r}^{*\prime\prime})|2\mathbf{v}^*_{-}+2\mathbf{v}^{*\prime\prime}_{-}|^2\{1+(1/4)(\mathbf{v}^{*\prime\prime}_{-}\cdot\partial/\partial\mathbf{v}^*_{-})\}\, g_{-}(\mathbf{v}^{*\prime\prime}_{-},\mathbf{r}^{*\prime\prime},t^*)d\mathbf{v}^{*\prime\prime}_{-}d\mathbf{r}^{*\prime\prime}$$
$$=-(1/3)\int |2\mathbf{v}^*_{-}+2\mathbf{v}^{*\prime\prime}_{-}|^2\{1+(1/4)(\mathbf{v}^{*\prime\prime}_{-}\cdot\partial/\partial\mathbf{v}^*_{-})\}\delta(\mathbf{v}^{*\prime\prime}_{-})d\mathbf{v}^{*\prime\prime}_{-}$$
$$=-(4/3)|\mathbf{v}^*_{-}|^2$$
$$=-(4/3)(\mathrm{v}^{*2}_{\parallel}+\mathrm{v}^{*}_{\perp}{}^2+\mathrm{v}^{*}_{\perp\perp}{}^2), \quad (90)$$

where the identity for the Laplacian operator,

$$(1/4\pi)\,|\partial/\partial\mathbf{x}^*_2|^2\,|\mathbf{x}^*_3-\mathbf{x}^*_2|^{-1} = \delta(\mathbf{x}^*_3-\mathbf{x}^*_2), \quad (91)$$

has been used for the integration with respect to $\mathbf{r}^{*\prime\prime}=\mathbf{x}^*_3-\mathbf{x}^*_2$.

On substitution from (90), Eq.(86) is written as

$$[\partial/\partial t^* + \alpha^*_{-}(r^*,t^*)(\partial/\partial\mathrm{v}^*_{\parallel})^2 + 2\mathrm{v}^*_{\parallel}\partial/\partial r^* + (2/3)\,\partial/\partial\mathrm{v}^*_{\parallel}\partial/\partial r^*\,\{\mathrm{v}^{*2}_{\parallel} + \mathrm{M}_{\mathrm{v}^*\perp,\mathrm{v}^*\perp\perp}(\mathrm{v}^*_{\perp}{}^2+\mathrm{v}^*_{\perp\perp}{}^2)\}]\,g_{\parallel}(\mathrm{v}^*_{\parallel},r^*,t^*)=0,$$

which is rewritten as

$$[\partial/\partial t^* + \alpha^*_{-}(r^*,t^*)(\partial/\partial\mathrm{v}^*_{\parallel})^2 + 2\mathrm{v}^*_{\parallel}\partial/\partial r^* + (2/3)\,\partial/\partial\mathrm{v}^*_{\parallel}\partial/\partial r^*\,\{\mathrm{v}^{*2}_{\parallel} + 2\langle\mathrm{u}^*_{\perp}{}^2\rangle\}]\,g_{\parallel}(\mathrm{v}^*_{\parallel},r^*,t^*) = 0, \quad (92)$$

where $\mathrm{u}^*_{\perp}$ denotes the real velocity corresponding to the variable $\mathrm{v}^*_{\perp}$. Eq.(92) gives the simplified version of Eq.(86) for the longitudinal velocity-difference distribution $g_{\parallel}$ in the local range.

6.3. Self-similar solution

Like other velocity distributions, we look for the self-similar solutions of Eq.(92) in time, using the same similarity variables as (47) and (48),

$$w=\mathrm{v}^*_{\parallel}t^{*1/2}, \quad s=r^*t^{*-1/2}, \quad G(w,s)=t^{-3/2}\,g_{\parallel}(\mathrm{v}^*_{\parallel},r^*,t^*). \quad (93)$$

On substitution from (93) and (85), Eq.(92) is written in the self-similar form as

$$[(1/2)(1+w\partial/\partial w-s\partial/\partial s)+\alpha^*_{-0}(r^*)(\partial/\partial w)^2+(2/3)\,\partial/\partial s\{5w+(w^2+4\alpha^*_{-0}(r^*))\partial/\partial w\}]\,G(w,s)=0, \quad (94)$$

which gives the equation for the longitudinal velocity-difference distribution $g_{\parallel}$ in the local range.

The solution of the s-independent part of Eq.(94), that is,

$$[(1/2)(1+w\partial/\partial w) + \alpha^*_{-0}(r^*)(\partial/\partial w)^2]\,G(w,s) = 0, \quad (95)$$

is immediately obtained as

$$G(w,s) = G_0(w,s) \equiv (4\pi\alpha^*_{-0}(r^*))^{-1/2}\exp[-w^2/4\pi\alpha^*_{-0}(r^*)], \quad (96)$$

which is nothing but the one-dimensional inertial normal distribution N4 represented by Eq.(84).

6.4. Spatially similar solutions

Although Eq.(94) for the distribution $G(w,s)$ is only a linear partial differential equation, it is still difficult to solve analytically. This is because, at large Reynolds numbers, the local range in the co-ordinate s is divided into still smaller subranges having their own spatial similarity. In order to deal with such an equation, it is convenient to express the solution $G(w,s)$ in the spatially similar form as

$$G(w,s)=s^{-\theta}H(\xi), \qquad \xi = w/s^{\theta}, \tag{97}$$

where θ denotes the exponent of each similarity subrange.

On substitution from (97), Eq.(94) is written for the distribution $H(\xi)$ as

$$(1/2)(1+\theta)(H+\xi H') + \alpha^{*}_{-0}(r^{*})s^{-2\theta}H'' - (10/3)\theta\,\xi\,(H+\xi H')\,s^{\theta-1}$$
$$- (2/3)\theta\{\xi^{2}+4\alpha^{*}_{-0}(r^{*})s^{-2\theta}\}(2H+\xi H'')\,s^{\theta-1} + (8/3)(\mathrm{d}\alpha^{*}_{-0}(r^{*})/\mathrm{d}s)s^{-\theta}H' = 0, \tag{98}$$

where the symbol $'$ denotes $\mathrm{d}/\mathrm{d}\xi$.

The parameter $\alpha^{*}_{-0}(r^{*})$ in the equation can be expressed in terms of s by taking the integral moments,

$$H_n = \int \xi^{n}H(\xi)\,\mathrm{d}\xi, \quad n\text{: positive integers}, \tag{99}$$

as .

$$\alpha^{*}_{-0}(r^{*}) = a_{\theta}s^{2\theta}, \qquad a_{\theta} = (1/2)(1+\theta)H_2, \tag{100}$$

where use has been made of the conditions $H_0=1, H_1=H_3=0$.

On substitution from (100), Eq.(98) is written as

$$[(1/2)(1+\theta)(H+\xi H') + a_{\theta}H''] - (2/3)\theta\xi\,[5H + 7\xi H' + (\xi^{2}+4a_{\theta})H'']\,s^{\theta-1} = 0, \tag{101}$$

which gives the equation for the spatially similar distribution $H(\xi)$ in each similarity subrange.

6.5. Viscous and inertial subranges

The exponent θ in Eq.(101) can be determined for each similarity subranges by considering the balance of the dominant terms of Eq.(94) in the respective ranges.

Viscous subrange:

For very small s, the distribution G tends to the delta-like distribution according to the coincidence condition (24). In such a case, the second derivative terms dominate in Eq.(94) under the condition, $w^2 \ll 4\alpha^{*}_{-0}(r^{*})$. Then, Eq.(94) is approximated by the equation,

$$[(\partial/\partial w)^{2} + (8/3)\,\partial/\partial s\partial/\partial w]\,G(w,s) = 0, \tag{102}$$

with an obvious solution,

$$G(w,s) = \delta_s(w-(3/8)s), \tag{103}$$

where δ_s denotes a delta-like distribution. Eq.(103) clearly shows that $w \propto s$, or the exponent $\theta=1$ for the viscous subrange at very small s.

Inertial subrange:

For much larger s than those in the viscous subrange, the second derivative terms still dominate in Eq.(94) but the opposite condition $w^2 \gg 4\alpha^*_{-0}(r^*)$ is valid there. Then, Eq.(94) is approximated by the equation,

$$[\alpha^*_{-0}(r^*)(\partial/\partial w)^2 + (2/3)\,\partial/\partial s\,\{5w + w^2\,\partial/\partial w\}]\,G(w,s) = 0. \qquad (104)$$

Eq.(104) is clearly satisfied by the function of the variable $w/s^{1/3}$ so far as $\alpha^*_{-0}(r^*)$ is taken to be constant there, showing that the exponent $\theta = 1/3$ applies to the inertial subrange associated with much larger s than in the viscous range.

6.6. Distributions in intermediate and inertial subranges

Eq.(101) is generally composed of two parts with different dependence on s except for $\theta = 1$ corresponding to the viscous subrange. Therefore, in the inertial subrange, in order that Eq.(101) be satisfied for arbitrary s, the two parts must vanish identically. Thus, we obtain the following equations for $H(\xi)$ in different domains of s in the inertial subrange:

$$(1/2)(1+\theta)(H + \xi H') + a_\theta H'' = 0, \qquad (105)$$

for relatively large s, and

$$5H + 7\xi H' + (\xi^2 + 4a_\theta)H'' = 0, \qquad (106)$$

for relatively small s.

It may be strange that the inertial subrange should be divided into two regions governed by different equations (105) and (106). However, if look back the origin of these equations to Eq.(94) for G and further to Eq.(86) for $g_\parallel$, this division may be understandable. It will be found that Eq.(105) represents the linear part of the equation of motion Eq.(1) composed of the time-derivative and the viscous terms, while Eq.(106) represents the nonlinear part composed of the transfer and the pressure terms. If we consider that the linear part is mostly contributed by large-scale components of turbulence while the nonlinear part by small-scale components, the independence of these two parts must be the natural consequence at large Reynolds numbers. In this sense, the region governed by Eq.(106) corresponds to the *inertial range* in the proper sense and that of Eq.(105) to the *intermediate range* between the local and the outer ranges.

6.6.1. Distribution in intermediate subrange

In the intermediate subrange, Eq.(105) gives a solution representing a normal distribution, N5 say,

$$H(\xi)=H_0(\xi) \equiv (4\pi\alpha_\theta/(1+\theta))^{-1/2}\exp[-((1+\theta)/4\alpha_\theta)\xi^2], \quad (107)$$

with the variable parameter α_θ. The exponent θ in this subrange is thought to change with s from 1/3 at the lower end of the subrange to 0 at the upper end.

The corresponding distribution $G(w,s)$ is expressed as

$$G(w,s)=G_0(w,s) \equiv (4\pi\alpha_\theta/(1+\theta))^{-1/2}\, s^{-\theta} \exp[-((1+\theta)/4\alpha_\theta)(w/s^\theta)^2], \quad (108)$$

which gives the *inertial normal distribution* N5,

$$g_\|(v^*_\|,r^*,t^*)=g_{\|0}(v^*_\|,r^*,t^*) \equiv (4\pi\alpha_\theta/(1+\theta))^{-1/2}(r^*/t^{*2})^{-\theta}\exp[-((1+\theta)/4\alpha_\theta)\{v^*_\| /(r^*/t^{*2})^\theta\}^2], \quad (109)$$

for the longitudinal velocity-difference in the intermediate subrange.

The normal similarity of the distributions (107)~(109) and the connection of Eq.(105) with the large-scale components of turbulence clearly indicate that the distribution N5 has the intermediate nature between those of the inertial subrange and the outer range. The distribution N5 represented by Eq.(107) is shown graphically in Fig.2 in the standard form with unit variance.

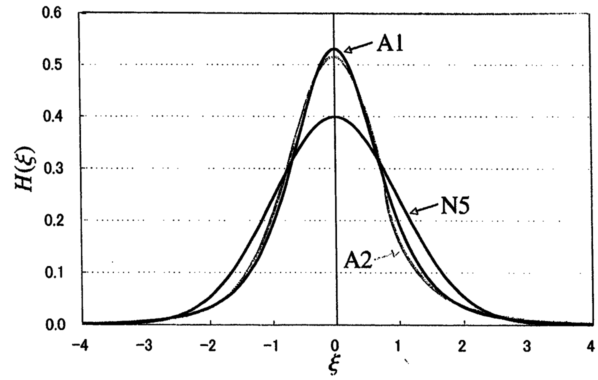

Fig.2. Longitudinal velocity-difference distributions in the local range. N5: Inertial normal distribution Eq.(107). A1: Inertial algebraic distribution Eq.(110). A2: Viscous algebraic distribution obtained from Eq.(114) numerically

6.6.2. Distribution in inertial subrange

In the inertial subrange, Eq.(106) has a solution representing an algebraic distribution, A1 say,

$$H(\xi)=H_c(\xi)\equiv 12\pi\alpha_{1/3}^2(\xi^2+4\alpha_{1/3})^{-5/2}, \quad (110)$$

with the exponent θ=1/3.

The corresponding distribution $G(w,s)$ is expressed as

$$G(w,s) = G_c(w,s) \equiv 12\pi\alpha_{1/3}^2\{(w/s^{1/3})^2 + 4\alpha_{1/3}\}^{-5/2}, \quad (111)$$

which gives the *inertial algebraic distribution* A1,

$$g_\|(v^*_\|,r^*,t^*)=g_{\|c}(v^*_\|,r^*,t^*) \equiv 12\pi\alpha_{1/3}^2[\{v^*_\|/(r^*/t^{*2})^{1/3}\}^2+4\alpha_{1/3}]^{-5/2}, \quad (112)$$

for the longitudinal velocity-difference in the inertial subrange.

The independence of the distributions (110)~(112) from those of (107)~(109) and the connection of

Eq.(106) with the small-scale components of turbulence clearly indicate the canonical character of the distribution A1 in the inertial subrange.

The distribution A1 represented by Eq.(110) is shown graphically in Fig.2 in the standard form with unit variance. It may be seen that the distribution A1 has algebraic tails,

$$H_c(\xi) \approx |\xi|^{-5} \qquad \text{for} \quad |\xi| \to \infty, \tag{113}$$

so that A1 has no finite integral moments H_n for $n \geq 4$.

6.7. Distribution in viscous subrange

For the viscous subrange, Eq.(101) gives the single equation with the exponent $\theta = 1$,

$$\{1-(10/3)\xi\}H + \{1-(14/3)\xi\}\xi H' + \{a_1(1-(8/3)\xi) - (2/3)\xi^3\}H'' = 0, \tag{114}$$

where the parameter a_1 is determined using the moment relations as $a_1 = 9/128$. Eq.(114) has a singularity at the zero-point of the H'' term, $\xi = \xi_s = 0.28909$, and two asymptotic solutions,

$$H(\xi) \approx |\xi|^{-5} \text{ and } |\xi|^{-1} \qquad \text{for} \quad |\xi| \to \infty. \tag{115}$$

Among the two asymptotic solutions (115), only the first is qualified for a distribution and identical to that of the distribution A1 expressed by (113).

Eq.(114) is solved numerically by integrating the equation from certain points around ξ_s in both directions, adjusting the initial value of H' at the points to attain the asymptotic solutions $H(\xi) \approx |\xi|^{-5}$ for $|\xi| \to \infty$. The distribution thus obtained may be called the *viscous algebraic distribution* A2. The distribution A2, which is shown graphically in Fig.2 in the standard form, is generally similar to A1 but a little asymmetric with respect to ξ and also has the divergent moments H_n for $n \geq 4$.

7. Overview of Two-point Velocity Distributions

The various two-point velocity distributions obtained here may be clearly overviewed in the diagram representing the dependence of their variance upon the distance $r^* = r/\eta$. Fig.3 shows graphically the experimental results of the longitudinal cross-velocity correlations, $\langle u^{*2}_+\rangle$, $\langle u^{*2}_-\rangle$ and $\langle u^*_+ u^*_-\rangle$, measured by Makita et al. (2005) using their intensive turbulence wind-tunnel. It may be noted that the curve of $\langle u^{*2}_+\rangle$ indicated by □ makes a clear mirror image of that of $\langle u^{*2}_-\rangle$ indicated by Δ with respect to a horizontal line between them. Actually this reflects the relation Eq.(71) between the self-dissipations $\alpha^*_+(r^*,t^*)$ $(=\langle u^{*2}_+\rangle/2t^*)$ and $\alpha^*_-(r^*,t^*)$ $(=\langle u^{*2}_-\rangle/2t^*)$. The curve of $\langle u^{*2}_+\rangle$ for the velocity-sum distribution N3 coincides with $\langle u^{*2}_1\rangle$ for the one-point velocity distribution N1 at $r^* = 0$ and decreases monotonically with increasing r^* to coincide with $\langle u^{*2}_1\rangle/2$ for the velocity-sum distribution N2 in the outer range. On the other hand, the curve of $\langle u^{*2}_{-}\rangle$ representing the velocity-difference distribution starts from zero of the delta distribution at $r^* = 0$ and increases monotonically with r^* to coincide with $\langle u^{*2}_{1}\rangle/2$ for the velocity-sum distribution N2. During this change, the longitudinal velocity-difference distribution begins as the viscous algebraic distribution A2 and changes successively through the inertial algebraic distribution A1 and the intermediate normal distribution N5 to the distribution N2.

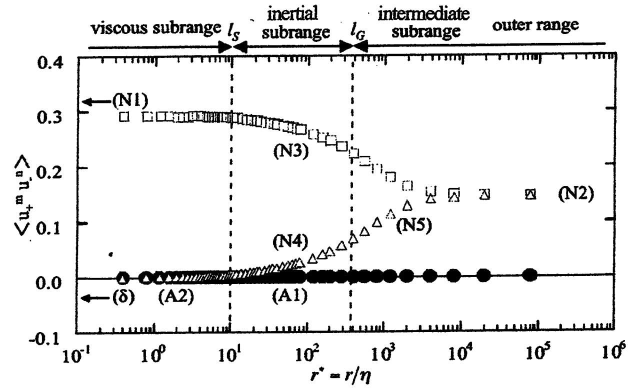

Fig.3. Overview of the two-point velocity distributions on the figure of the longitudinal cross-velocity correlations measured by Makita et al.(2005) at R_λ=350. □ : $\langle u^{*2}_{+}\rangle$, Δ : $\langle u^{*2}_{-}\rangle$, • : $\langle u^*_+ u^*_-\rangle$.

8. Summary and Discussions

8.1. Statistical mechanics of turbulence

The present theory of the Navier-Stokes turbulence can be summarized as follows.

1) Turbulence at large Reynolds numbers is in the state of *dynamical equilibrium* governed by two parameters: the mean *energy dissipation rate* ε and the *viscosity* ν. (Kolmogorov's first hypothesis)

2) The two parameters are subject to the *inviscid catastrophe*, or the finite *energy dissipation rate* $\varepsilon > 0$ in the limit of vanishing *viscosity* $\nu \rightarrow 0$. (Kolmogorov's premice)

3) In homogeneous turbulence, it follows from 1) and 2) that the *energy dissipation rate* decays in time as $\varepsilon \propto t^{-2}$, and thus the *kinetic energy* decays as $E \propto t^{-1}$.

4) Then, the one-point velocity distribution f, the velocity-sum distribution g_+, and the velocity-difference distribution g_- are expressed as the *inertial normal distributions* including only parameter $\varepsilon(t_0)$ for f, and $\varepsilon(t_0)/2$ for g_+ and g_- at all finite distance $r > 0$, where t_0 denotes a certain initial time.

5) The distributions g_+ and g_- must change discontinuously at $r = 0$ in order to coincide with f and the delta distribution respectively.

6) If we use the local coordinate $r^* = r/\eta$, $\eta = O((\nu^3/\varepsilon)^{1/4})$ being Kolmogorov's length, g_+ and the lateral

component of g_- are expressed as the *inertial normal distributions* in the local variables, which satisfy the coincidence conditions at the boundaries of the local range.

7) The longitudinal component of g_- takes three different similarity forms in the local range, the *inertial normal distribution* in the intermediate subrange and the *algebraic distributions* in the inertial and the viscous subranges.

8.2. Mathematical physics of turbulence

It should be noted that the statistical information on homogeneous isotropic turbulence described above has been obtained only through mathematical analyses of the equations for the velocity distributions, which have been derived from the Navier-Stokes equation. Otherwise, we have to be satisfied by physical conjectures based on dimensional arguments.

Historically speaking, fluid mechanics has been the representative field of mathematical physics, but such a good tradition seems to have been terminated by the appearance of Kolmogorov's revolutionary theory of turbulence which uses only dimensional analysis. It is eagerly hoped that this small piece of work would be able to reopen this good tradition by showing how nicely Kolmogorov's idea can be incorporated with modern mathematical physics.

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