Torsion representations arising from $(\varphi, \hat{G})$-modules

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Abstract

The notion of a $(\varphi, \hat{G})$-module is defined by Tong Liu in 2010 to classify lattices in semi-stable representations. In this paper, we study torsion $(\varphi, \hat{G})$-modules, and torsion p-adic representations associated with them, including the case where $p = 2$. First we prove that the category of torsion p-adic representations arising from torsion $(\varphi, \hat{G})$-modules is an abelian category. Secondly, we construct a maximal (minimal) theory for $(\varphi, \hat{G})$-modules by using the theory of étale $(\varphi, \hat{G})$-modules, essentially proved by Xavier Caruso, which is an analogue of Fontaine's theory of étale (φ, Γ)-modules. Non-isomorphic two maximal (minimal) objects give non-isomorphic two torsion p-adic representations.

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1 Introduction

The notion of a $(\varphi, \hat{G})$-module was introduced by T. Liu in [Li3] to classify lattices in semi-stable representations. In this paper, we give various properties of torsion $(\varphi, \hat{G})$-modules such as the Cartier duality theorem. Furthermore, we study the category of torsion representations arising from torsion $(\varphi, \hat{G})$-modules. Let G be the absolute Galois group of a complete discrete valuation field K of mixed characteristic $(0, p)$ with perfect residue field. Fix $r \in \{0, 1, 2, \dots, \infty\}$. Our study is motivated by the following question:

> Is any torsion $\mathbb{Z}_p$-representation of G a torsion semi-stable representation with Hodge-Tate weights in $[0, r]$?

Here, a torsion $\mathbb{Z}_p$-representation of G is said to be *torsion semi-stable with Hodge-Tate weights in* $[0, r]$ if it can be written as the quotient of two lattices in a semi-stable p-adic representation of G with Hodge-Tate weights in $[0, r]$. It is known that the above question does not have an affirmative answer if $r < \infty$ and thus it makes sense only if $r = \infty$. We propose an approach to this question by using $(\varphi, \hat{G})$-modules which give descriptions of (torsion) semi-stable p-adic representations with Hodge-Tate weights in $[0, r]$. The theory of Breuil modules also gives descriptions of these representations in terms of linear algebra (cf. [Li2]), however, for technical reasons, we have to assume $r < p - 1$ when we use this theory for integral or torsion representations. On the other hand, there is no restriction on r in the theory of $(\varphi, \hat{G})$-modules. This is the main reason why we focus on $(\varphi, \hat{G})$-modules.

Let $\mathrm{Rep}_{\mathrm{tor}}(G)$ be the category of torsion $\mathbb{Z}_p$-representations. Let $\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G)$ be the category of torsion semi-stable representations. We denote by $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ the category of torsion $(\varphi, \hat{G})$-modules of height r and $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ the associated functor (see Section 2). Let $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ be the category of torsion representations arising from torsion $(\varphi, \hat{G})$-modules, that is, the essential image of $\hat{T}$ on $\mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty}$. Then inclusions

$$\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}_{\mathrm{tor}}(G)$$

are known (cf. [CL2, Theorem 3.1.3]). Since our interest is related with the equality of categories $\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G)$ and $\mathrm{Rep}_{\mathrm{tor}}(G)$, we want to know differences between above three categories. The following is the first main result of this paper:

Theorem 1.1. *The category* $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ *is an abelian full subcategory of* $\mathrm{Rep}_{\mathrm{tor}}(G)$ *which is stable under subquotients,* $\oplus$*,* $\otimes$ *and the dual.*

To show the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is abelian, we give two different proofs. The first one uses a deep relation, proved by T. Liu, between $(\varphi, \hat{G})$-modules and representations associated with them (cf. Lemma 4.2). The second proof is based on a result on *maximal* (*minimal*) *objects* of $(\varphi, \hat{G})$-modules. In general, the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is not abelian and $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is not fully faithful. The theory of maximal (minimal) objects allows us to avoid this problem. Denote by $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ the full subcategory of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ whose objects are maximal. Then we obtain the functor $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ which is a retraction of the natural inclusion $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \hookrightarrow \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ and commutes with $\hat{T}$. We prove

Theorem 1.2. *The category* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is abelian and, if* $r < \infty$*, it is Artinian. Furthermore, the restriction of* $\hat{T}$ *on* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is exact and fully faithful, and its essential image is stable under subquotients.*

In particular, we immediately find that the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is abelian. If $r < \infty$, we can also define the full subcategory $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ whose objects are minimal and the functor $\mathrm{Min}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$; they satisfy analogous properties as those stated in Theorem 1.2. Furthermore, the Cartier duality theorem gives a connection between maximal objects and minimal objects (cf. Proposition 5.28). Maximal (minimal) objects are first defined for finite flat group schemes by M. Raynaud [Ra]. X. Caruso and T. Liu generalized Raynaud's theory, with respect to finite flat group schemes killed by a power of p, to torsion Kisin modules [CL1], whose representations are defined on G_∞. Here $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$ and $K_\infty = \cup_{n\geq 0} K(\pi_n)$, $\pi_0 = \pi$ a uniformizer of K, $\pi^p_{n+1} = \pi_n$. Furthermore, a categorical interpretation of maximal (minimal) objects is given in [Ca3]. Our theorem described above is an extended result of [CL1] in a certain sense. In the case where $r = \infty$, we obtain the following:

Corollary 1.3. *The functor $\hat{T}\colon \mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ induces an equivalence of abelian categories between the category $\mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty}$ of maximal torsion $(\varphi,\hat{G})$-modules of finite height and the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ of torsion $\mathbb{Z}_p$-representations of G arising from $(\varphi,\hat{G})$-modules.*

To define maximal (minimal) objects of torsion $(\varphi,\hat{G})$-modules, we introduce an *étale $(\varphi,\hat{G})$-module*, which is an étale φ-module (in the sense of J.-M. Fontaine [Fo]) equipped with certain Galois action. Arguments in the theory of (φ,τ)-modules of [Ca4] give us the fact that the category of torsion étale $(\varphi,\hat{G})$-modules is equivalent to $\mathrm{Rep}_{\mathrm{tor}}(G)$.

Now denote by e the absolute ramification index of K. If $er < p-1$, then all torsion $(\varphi,\hat{G})$-modules of height r are automatically maximal and minimal. Therefore, we have

Corollary 1.4 (= Corollary 5.34)**.** *Suppose $er < p-1$. Then the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is abelian and Artinian. Furthermore, the functor $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is exact and fully faithful, and its essential image is stable under subquotients.*

The corresponding result on torsion Breuil modules has been proven by X. Caruso (cf. [Ca2, Théorème 1.0.4]).
We hope our study will be useful to solve the question described in the beginning of this paper (cf. Section 5.7).

Now we describe an organization of this paper. In Section 2, we recall some results on Kisin modules and $(\varphi,\hat{G})$-modules, and prove some fundamental properties of them which are often used in this paper. In Section 3, we prove the Cartier duality theorem for $(\varphi,\hat{G})$-modules. In Section 4, we prove Theorem 1.1. Finally in Section 5, we give a theory of étale $(\varphi,\hat{G})$-modules, define maximal (minimal) objects for $(\varphi,\hat{G})$-modules, and prove Theorem 1.2.

Convention. For any $\mathbb{Z}$-module M, we always use M_n to denote M/p^nM for a positive integer n and $M_\infty = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. We reserve φ to represent various Frobenius structures and φ_M will denote the Frobenius on M. However, we often drop the subscript if no confusion arises. All representations and actions are assumed to be continuous.

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2 Preliminaries

In this section, we recall some notions and results which will be used throughout this paper.

2.1 Notation

Let k be a perfect field of characteristic $p \geq 2$, $W(k)$ its ring of Witt vectors, $K_0 = W(k)[1/p]$, K a finite totally ramified extension of K_0, $\bar{K}$ a fixed algebraic closure of K and $G = \mathrm{Gal}(\bar{K}/K)$. Throughout this paper, we fix a uniformizer $\pi \in K$ and denote by $E(u)$ its Eisenstein polynomial over K_0. Put $\mathfrak{S} = W(k)[\![u]\!]$. We define a Frobenius endomorphism φ of $\mathfrak{S}$ by $u \mapsto u^p$, extending the Frobenius on $W(k)$.

Let $R = \varprojlim \mathcal{O}_{\bar{K}}/p$ where $\mathcal{O}_{\bar{K}}$ is the integer ring of $\bar{K}$ and the transition maps are given by the p-th power map. By the universal property of the ring of Witt vectors $W(R)$ of R, there exists a unique surjective projection map $\theta\colon W(R) \to \widehat{\mathcal{O}}_{\bar{K}}$ which lifts the projection $R \to \mathcal{O}_{\bar{K}}/p$ onto the first factor in the inverse limit, where $\widehat{\mathcal{O}}_{\bar{K}}$ is the p-adic completion of $\mathcal{O}_{\bar{K}}$. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to the kernel of θ. We put $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$. For any integer $n \geq 0$, let $\pi_n \in \bar{K}$ be a p^n-th root of π such that $\pi^p_{n+1} = \pi_n$ and write $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$. Let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative of $\underline{\pi}$. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ by the map $u \mapsto [\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$, which is compatible with Frobenius endomorphisms.

Let $\mathcal{O}$ be the p-adic completion of $\mathfrak{S}[1/u]$, which is a discrete valuation ring with uniformizer p and residue field $k(\!(u)\!)$. Denote by $\mathcal{E}$ the field of fractions of $\mathcal{O}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to inclusions $\mathcal{O} \hookrightarrow W(\mathrm{Fr}R)$ and $\mathcal{E} \hookrightarrow W(\mathrm{Fr}R)[1/p]$. Here $\mathrm{Fr}R$ is the field of fractions of R. It is not difficult to see that $\mathrm{Fr}R$ is algebraically closed. We denote by $\mathcal{E}^{\mathrm{ur}}$ the maximal unramified field extension of $\mathcal{E}$ in $W(\mathrm{Fr}R)[1/p]$ and $\mathcal{O}^{\mathrm{ur}}$ its integer ring. Let $\widehat{\mathcal{E}^{\mathrm{ur}}}$ be the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$ and $\widehat{\mathcal{O}^{\mathrm{ur}}}$ its integer ring. The ring $\widehat{\mathcal{E}^{\mathrm{ur}}}$ (resp. $\widehat{\mathcal{O}^{\mathrm{ur}}}$) is equal to the closure of $\mathcal{E}^{\mathrm{ur}}$ in $W(\mathrm{Fr}R)[1/p]$ (resp. the closure of $\mathcal{O}^{\mathrm{ur}}$ in $W(\mathrm{Fr}R)$). Put $\mathfrak{S}^{\mathrm{ur}} = \widehat{\mathcal{O}^{\mathrm{ur}}} \cap W(R)$. We regard all these rings as subrings of $W(\mathrm{Fr}R)[1/p]$.

Let $K_\infty = \cup_{n\geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\bar{K}/K_\infty)$. Then G_∞ acts on $\mathfrak{S}^{\mathrm{ur}}$ and $\mathcal{E}^{\mathrm{ur}}$ continuously and fixes the subring $\mathfrak{S} \subset W(R)$. We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$) the category of continuous $\mathbb{Z}_p$-representations of G_∞ on $\mathbb{Z}_p$-modules of finite type (resp. the category of continuous $\mathbb{Q}_p$-representations of G_∞ on finite dimensional $\mathbb{Q}_p$-vector spaces). We denote by $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathrm{fr}}(G_\infty)$) the full subcategory of $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ consisting of $\mathbb{Z}_p$-modules killed by some power of p (resp. free $\mathbb{Z}_p$-modules). Similarly, we define categories $\mathrm{Rep}_{\mathbb{Z}_p}(G), \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty), \mathrm{Rep}_{\mathrm{tor}}(G)$ and $\mathrm{Rep}_{\mathrm{fr}}(G)$ by replacing G_∞ with G.

2.2 Étale φ-modules

In this subsection, We recall the theory of Fontaine's étale φ-modules. For more precise information, see [Fo, A 1.2].

An *étale φ-module over $\mathcal{O}$* is an $\mathcal{O}$-module M of finite type, equipped with a φ-semi-linear map $\varphi_M\colon M \to M$ such that φ^*_M is an isomorphism. Here, φ^*_M stands for the $\mathcal{O}$-linearization $1 \otimes \varphi_M\colon \mathcal{O} \otimes_{\varphi,\mathcal{O}} M \to M$ of φ_M. An *étale φ-module over $\mathcal{E}$* is a finite dimensional $\mathcal{E}$-vector space M, equipped with a φ-semi-linear map $\varphi_M\colon M \to M$ such that there exists a φ-stable $\mathcal{O}$-lattice L of M and that L is an étale φ-module over $\mathcal{O}$. We denote by $'\mathbf{\Phi M}_{/\mathcal{O}}$ (resp. $\mathbf{\Phi M}_{/\mathcal{E}}$) the category of étale φ-modules over $\mathcal{O}$ (resp. the category of étale φ-modules over $\mathcal{E}$) with the obvious morphisms. Note that the extension K_∞/K is a strictly APF extension in the sense of [Wi] and thus G_∞ is naturally isomorphic to the absolute Galois group of $k(\!(u)\!)$ by the theory of norm fields. Combining this fact and Fontaine's theory in [Fo, A 1.2.6], we have that functors

$$\mathcal{T}_*\colon {'\mathbf{\Phi M}_{/\mathcal{O}}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty), \quad M \mapsto (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}} M)^{\varphi=1}$$

and

$$\mathcal{T}_*\colon \mathbf{\Phi M}_{/\mathcal{E}} \to \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty), \quad M \mapsto (\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} M)^{\varphi=1}$$

are equivalences of abelian categories and there exist natural $\widehat{\mathcal{O}^{\mathrm{ur}}}$-linear isomorphisms which are compatible with φ-structures and G_∞-actions:

$$\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathbb{Z}_p} \mathcal{T}_*(M) \xrightarrow{\sim} \widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}} M \quad \text{for } M \in {'\mathbf{\Phi M}_{/\mathcal{O}}} \tag{2.2.1}$$

and

$$\widehat{\mathcal{E}^{\mathrm{ur}}}\otimes_{\mathbb{Q}_p}\mathcal{T}_*(M)\xrightarrow{\sim}\widehat{\mathcal{E}^{\mathrm{ur}}}\otimes_{\mathcal{E}}M\quad\text{for } M\in\mathbf{\Phi M}_{/\mathcal{E}}.\tag{2.2.2}$$

On the other hand, define functors

$$\mathcal{M}_*\colon \mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)\to {}'\mathbf{\Phi M}_{/\mathcal{O}},\quad T\mapsto(\widehat{\mathcal{O}^{\mathrm{ur}}}\otimes_{\mathbb{Z}_p}T)^{G_\infty}$$

and

$$\mathcal{M}_*\colon \mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)\to \mathbf{\Phi M}_{/\mathcal{E}},\quad T\mapsto(\widehat{\mathcal{E}^{\mathrm{ur}}}\otimes_{\mathbb{Q}_p}T)^{G_\infty}.$$

There exist natural $\widehat{\mathcal{O}^{\mathrm{ur}}}$-linear isomorphisms which are compatible with φ-structures and G_∞-actions:

$$\widehat{\mathcal{O}^{\mathrm{ur}}}\otimes_{\mathcal{O}}\mathcal{M}_*(T)\xrightarrow{\sim}\widehat{\mathcal{O}^{\mathrm{ur}}}\otimes_{\mathbb{Z}_p}T\quad\text{for } T\in\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)\tag{2.2.3}$$

and

$$\widehat{\mathcal{E}^{\mathrm{ur}}}\otimes_{\mathcal{E}}\mathcal{M}_*(T)\xrightarrow{\sim}\widehat{\mathcal{E}^{\mathrm{ur}}}\otimes_{\mathbb{Q}_p}T\quad\text{for } T\in\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty).\tag{2.2.4}$$

We denote by $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$ (resp. $\mathbf{\Phi M}_{/\mathcal{O}}$) the category of étale φ-modules over $\mathcal{O}$ which are killed by some power of p (resp. the category of étale φ-modules over $\mathcal{O}$ which are p-torsion free). We call objects of $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$ (resp. $\mathbf{\Phi M}_{/\mathcal{O}}$) *torsion étale φ-modules over $\mathcal{O}$* (resp. *free étale φ-modules over $\mathcal{O}$*).

Proposition 2.1. *The functor $\mathcal{T}_*$ induces equivalences of categories between $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$ (resp. $\mathbf{\Phi M}_{/\mathcal{O}}$, resp. $\mathbf{\Phi M}_{/\mathcal{E}}$) and $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ (resp. $\mathrm{Rep}_{\mathrm{fr}}(G_\infty)$, resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$). Furthermore, $\mathcal{M}_*$ is a quasi-inverse of $\mathcal{T}_*$.*

The contravariant version of the functor $\mathcal{T}_*$ is useful for integral theory. For any $T\in\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, put

$$\mathcal{M}(T)=\mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T,\mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}})\quad\text{if } T \text{ is killed by some power of } p,$$

$$\mathcal{M}(T)=\mathrm{Hom}_{\mathbb{Z}_p[G_\infty]}(T,\widehat{\mathcal{O}^{\mathrm{ur}}})\quad\text{if } T \text{ is free,}$$

and for any $T\in\mathrm{Rep}_{\mathbb{Q}_p}(G_\infty)$, put

$$\mathcal{M}(T)=\mathrm{Hom}_{\mathbb{Q}_p[G_\infty]}(T,\widehat{\mathcal{E}^{\mathrm{ur}}}).$$

Then we can check that $\mathcal{T}(M)$ is the dual representation of $\mathcal{T}_*(M)$. For any $M\in\mathbf{\Phi M}_{/\mathcal{O}}$, put

$$\mathcal{T}(M)=\mathrm{Hom}_{\mathcal{O},\varphi}(M,\mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}})\quad\text{if } M \text{ is killed by some power of } p,$$

$$\mathcal{T}(M)=\mathrm{Hom}_{\mathcal{O},\varphi}(M,\widehat{\mathcal{O}^{\mathrm{ur}}})\quad\text{if } M \text{ is } p\text{-torsion free,}$$

and for any $M\in\mathbf{\Phi M}_{/\mathcal{E}}$, put

$$\mathcal{T}(M)=\mathrm{Hom}_{\mathcal{E},\varphi}(M,\widehat{\mathcal{E}^{\mathrm{ur}}}).$$

These formulations give us contravariant functors $\mathcal{T}$ and $\mathcal{M}$ (on appropriate categories) such that $\mathcal{M}\circ\mathcal{T}\simeq\mathrm{Id},\mathcal{T}\circ\mathcal{M}\simeq\mathrm{Id}$.

2.3 Kisin modules

A *φ-module* (*over* $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi\colon\mathfrak{M}\to\mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1,\varphi_1)$ and $(\mathfrak{M}_2,\varphi_2)$ is an $\mathfrak{S}$-linear morphism $\mathfrak{M}_1\to\mathfrak{M}_2$ compatible with φ_1 and φ_2. Denote by ${}'\mathrm{Mod}^r_{/\mathfrak{S}}$ the category of φ-modules $\mathfrak{M}$ *of height* r in the following sense;

- if $r<\infty$, then $\mathfrak{M}$ is of finite type over $\mathfrak{S}$ and the cokernel of φ^* is killed by $E(u)^r$, where φ^* is the $\mathfrak{S}$-linearization $1\otimes\varphi\colon\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to\mathfrak{M}$ of φ,

- if $r=\infty$, then $\mathfrak{M}$ is of height r' for some integer $0\le r'<\infty$. In this case, $\mathfrak{M}$ is called *of finite height.*

Let $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ be the full subcategory of $'\mathrm{Mod}^r_{/\mathfrak{S}}$ consisting of $\mathfrak{S}$-modules $\mathfrak{M}$ of finite type which satisfy the following:

- $\mathfrak{M}$ is killed by some power of p,
- $\mathfrak{M}$ has a two term resolution by finite free $\mathfrak{S}$-modules, that is, there exists an exact sequence

$$0\to\mathfrak{N}_1\to\mathfrak{N}_2\to\mathfrak{M}\to 0$$

  of $\mathfrak{S}$-modules where $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are finite free $\mathfrak{S}$-modules.

Let $\mathrm{Mod}^r_{/\mathfrak{S}}$ be the full subcategory of $'\mathrm{Mod}^r_{/\mathfrak{S}}$ consisting of finite free $\mathfrak{S}$-modules. An object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Mod}^r_{/\mathfrak{S}}$) is called a *torsion Kisin module* (resp. a *free Kisin module*) *of height* r.

A φ-modules $\mathfrak{M}$ is called *p'-torsion free* if for any non-zero element $x\in\mathfrak{M}$, $\mathrm{Ann}_{\mathfrak{S}}(x)=0$ or $\mathrm{Ann}_{\mathfrak{S}}(x)=p^n\mathfrak{S}$ for some integer n. This is equivalent to the natural map $\mathfrak{M}\to\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M}$ being injective. If $\mathfrak{M}$ is killed by some power of p, then $\mathfrak{M}$ is p'-torsion free if and only if $\mathfrak{M}$ is u-torsion free. Therefore, if $\mathfrak{M}\in{}'\mathrm{Mod}^r_{/\mathfrak{S}}$ is killed by p and p'-torsion free, then $\mathfrak{M}$ is finite free as a $k[\![u]\!]$-module. A φ-module $\mathfrak{M}$ is called *étale* if $\mathfrak{M}$ is p'-torsion free and $\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M}$ is an étale φ-module over $\mathcal{O}$. Since $E(u)$ is a unit of $\mathcal{O}$, we see that $\mathfrak{M}$ is étale if and only if $\mathfrak{M}$ is p'-torsion free for any $\mathfrak{M}\in{}'\mathrm{Mod}^r_{/\mathfrak{S}}$. Any object of $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$ is clearly étale.

For any $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, we define a $\mathbb{Z}_p[G_\infty]$-module by

$$T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_\infty),$$

where a G_∞-action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(\sigma.g)(x)=\sigma(g(x))$ for $\sigma\in G_\infty, g\in T_{\mathfrak{S}}(\mathfrak{M}), x\in\mathfrak{M}$. The representation $T_{\mathfrak{S}}(\mathfrak{M})$ is an object of $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$.

Proposition 2.2 ([CL1, Corollary 2.1.6])**.** *The functor* $T_{\mathfrak{S}}\colon\mathrm{Mod}^r_{/\mathfrak{S}_\infty}\to\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *is exact and faithful.*

Proof. The exactness follows from Proposition 2.4 below and the fact that the functor $(\mathfrak{M}\mapsto\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M})$ from $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ to $\mathbf{\Phi M}_{/\mathcal{O}}$ is exact (since $\mathcal{O}$ is flat over $\mathfrak{S}$). $\square$

Similarly, for any $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$, we define a $\mathbb{Z}_p[G_\infty]$-module by

$$T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}).$$

The representation $T_{\mathfrak{S}}(\mathfrak{M})$ is an object of $\mathrm{Rep}_{\mathrm{fr}}(G_\infty)$ and $\mathrm{rank}_{\mathbb{Z}_p}T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{rank}_{\mathfrak{S}}\mathfrak{M}$.

Proposition 2.3 ([Ki, Corollary 2.1.4, Proposition 2.1.12])**.** *The functor* $T_{\mathfrak{S}}\colon\mathrm{Mod}^r_{/\mathfrak{S}}\to\mathrm{Rep}_{\mathrm{fr}}(G_\infty)$ *is exact and fully faithful.*

Let $\mathfrak{M}$ be a torsion Kisin module (resp. a free Kisin module). Since $E(u)$ is a unit in $\mathcal{O}$, we see that $M=\mathfrak{M}[1/u]:=\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M}$ is a torsion étale φ-module over $\mathcal{O}$ (resp. a free étale φ-module over $\mathcal{O}$). Here a Frobenius φ_M on M is given by $\varphi_M=\varphi_{\mathcal{O}}\otimes\varphi_{\mathfrak{M}}$.

Proposition 2.4 ([Br2, Lemma 2.3.3], [Li1, Corollary 2.2.2])**.** *Suppose that $\mathfrak{M}$ is an object of* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *or* $\mathrm{Mod}^r_{/\mathfrak{S}}$*. Then the natural map*

$$T_{\mathfrak{S}}(\mathfrak{M})\to\mathcal{T}(\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M})$$

is an isomorphism of $\mathbb{Z}_p$-representations of G_∞.

2.4 $(\varphi, \hat{G})$-modules

Let S be the p-adic completion of $W(k)[u, \frac{E(u)^i}{i!}]_{i\geq 0}$ and endow S with the following structures:

- a continuous φ-semi-linear Frobenius $\varphi\colon S \to S$ defined by $\varphi(u) = u^p$.
- a continuous linear derivation $N\colon S \to S$ defined by $N(u) = -u$.
- a decreasing filtration $(\mathrm{Fil}^i S)_{i\geq 0}$ in S. Here $\mathrm{Fil}^i S$ is the p-adic closure of the ideal generated by the divided powers $\gamma_j(E(u)) = \frac{E(u)^j}{j!}$ for all $j \geq i$.

Put $S_{K_0} = S[1/p] = K_0 \otimes_{W(k)} S$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ induces inclusions $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ and $S_{K_0} \hookrightarrow B^+_{\mathrm{cris}}$. (See subsection 2.1 for definitions of rings A_{cris} and B^+_{cris}.) We regard all these rings as subrings of B^+_{cris}.

Fix a choice of primitive p^i-th root of unity ζ_{p^i} for $i \geq 0$ such that $\zeta^p_{p^{i+1}} = \zeta_{p^i}$. Put $\underline{\varepsilon} = (\zeta_{p^i})_{i\geq 0} \in R^\times$ and $t = \log([\underline{\varepsilon}]) \in A_{\mathrm{cris}}$. Denote by $\nu\colon W(R) \to W(\bar{k})$ the unique lift of the projection $R \to \bar{k}$. Since $\nu(\mathrm{Ker}(\theta))$ is contained in the set $pW(\bar{k})$, ν extends to maps $\nu\colon A_{\mathrm{cris}} \to W(\bar{k})$ and $\nu\colon B^+_{\mathrm{cris}} \to W(\bar{k})[1/p]$. For any subring $A \subset B^+_{\mathrm{cris}}$, we put $I_+A = \mathrm{Ker}(\nu \text{ on } B^+_{\mathrm{cris}}) \cap A$. For any integer $n \geq 0$, put $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power.

We define a subring $\mathcal{R}_{K_0}$ of B^+_{cris} as below:

$$\mathcal{R}_{K_0} = \{\sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to \infty\}.$$

Put $\widehat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$ and $I_+ = I_+\widehat{\mathcal{R}}$.

For any field F over $\mathbb{Q}_p$, set $F_{p^\infty} = \cup^\infty_{n\geq 0} F(\zeta_{p^n})$. Recall $K_\infty = \cup_{n\geq 0} K(\pi_n)$ and note that $K_{\infty,p^\infty} = \cup_{n\geq 0} K(\pi_n, \zeta_{p^\infty})$ is the Galois closure of K_∞ over K. Put $H_K = \mathrm{Gal}(K_{\infty,p^\infty}/K_\infty)$, $H_\infty = \mathrm{Gal}(\bar{K}/K_{\infty,p^\infty})$, $G_{p^\infty} = \mathrm{Gal}(K_{\infty,p^\infty}/K_{p^\infty})$ and $\hat{G} = \mathrm{Gal}(K_{\infty,p^\infty}/K)$.

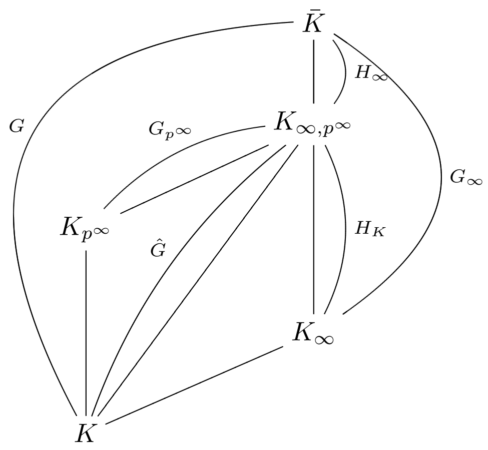

Figure 1: Galois groups of field extensions

Proposition 2.5 ([Li3, Lemma 2.2.1])**.** (1) *$\widehat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a φ-stable $\mathfrak{S}$-algebra as a subring in $W(R)$ (resp. B^+_{cris}).*
(2) *$\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) are G-stable. The G-action on $\widehat{\mathcal{R}}$ and I_+ (resp. $\mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$) factors through $\hat{G}$.*
(3) *There exist natural isomorphisms $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/I_+\mathfrak{S} \simeq W(k)$.*

For any Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of height r, we equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with a Frobenius by $\phi_{\widehat{\mathcal{R}}} \otimes \varphi_{\hat{\mathfrak{M}}}$. It is known that the natural map

$$\mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

is an injection (cf. [CL2, Section 3.1]). By this injection, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-stable submodule in $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Definition 2.6. A *weak* $(\varphi, \hat{G})$*-module* (*of height* r) is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module (of height r),
(2) $\hat{G}$ is an $\widehat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ which induces a continuous G-action on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ for the weak topology[1],
(3) the $\hat{G}$-action commutes with $\phi_{\widehat{\mathcal{R}}} \otimes \varphi_{\hat{\mathfrak{M}}}$,
(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^{H_K}$.

If $\mathfrak{M}$ is a torsion (resp. free) Kisin module of (height r), we call $\hat{\mathfrak{M}}$ a *torsion* (resp. *free*) *weak* $(\varphi, \hat{G})$*-module* (*of* height r). A weak $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$ is called a $(\varphi, \hat{G})$*-module* if it satisfies the additional condition

(5) $\hat{G}$ acts on the $W(k)$-module $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ trivially.

If $\mathfrak{M}$ is a torsion (resp. free) Kisin module of (height r), we call $\hat{\mathfrak{M}}$ a *torsion* (resp. *free*) $(\varphi, \hat{G})$*-module* (*of* height r). If $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is a weak $(\varphi, \hat{G})$-module, we often abuse notations by writing $\hat{\mathfrak{M}}$ for the underlying module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

A morphism $f\colon (\mathfrak{M}, \varphi, \hat{G}) \to (\mathfrak{M}', \varphi', \hat{G})$ between two weak $(\varphi, \hat{G})$-modules is a morphism $f\colon (\mathfrak{M}, \varphi) \to (\mathfrak{M}', \varphi')$ of Kisin-modules such that $\widehat{\mathcal{R}} \otimes f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ is $\hat{G}$-equivariant. We denote by ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$, resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$) the category of torsion weak $(\varphi, \hat{G})$-modules (resp. free weak $(\varphi, \hat{G})$-modules, resp. torsion $(\varphi, \hat{G})$-modules, resp. free $(\varphi, \hat{G})$-modules). We regard $\hat{\mathfrak{M}}$ as a G-module via the projection $G \twoheadrightarrow \hat{G}$. A sequence $0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$ of (weak) $(\varphi, \hat{G})$-modules is *exact* if it is exact as $\mathfrak{S}$-modules and all morphisms are morphisms of (weak) $(\varphi, \hat{G})$-modules.

For a weak $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$, we define a $\mathbb{Z}_p[G]$-module as below:

$$\hat{T}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R)_\infty) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\hat{T}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R)) \quad \text{if } \mathfrak{M} \text{ is free.}$$

Here, G acts on $\hat{T}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G,\ f \in \hat{T}(\hat{\mathfrak{M}}),\ x \in \hat{\mathfrak{M}}$.

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a weak $(\varphi, \hat{G})$-module. There exists a natural map

$$\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$$

defined by

$$\theta(f)(a \otimes m) = a\varphi(f(m)) \quad \text{for } f \in T_{\mathfrak{S}}(\mathfrak{M}),\ a \in \widehat{\mathcal{R}}, m \in \mathfrak{M},$$

which is G_∞-equivariant.

Let denote by $\mathrm{Rep}^r_{\mathbb{Z}_p}(G)$ the category of G-stable $\mathbb{Z}_p$-lattices in semi-stable p-adic representations of G with Hodge-Tate weights in $[0, r]$.

[1] Suppose that $\mathfrak{M}$ is free as an $\mathfrak{S}$-module. We equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ (resp. $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$) with the weak topology using any $\widehat{\mathcal{R}}$-basis (resp. $W(\mathrm{Fr}R)$-basis), which is independent of the choice of basis. Then we may replace the condition (2) with the following condition (2)':
(2)' $\hat{G}$ is a continuous $\widehat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ for the weak topology.
In fact, if G acts on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ continuously, then the G-action on $\hat{T}(\hat{\mathfrak{M}})$ is continuous for the p-adic topology (the definition for $\hat{T}(\hat{\mathfrak{M}})$ is given before Theorem 2.7). Since the map $\hat{\iota}$ in Lemma 4.2 (4) is a topological isomorphism for weak topologies on both sides, we see that the G-action on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is automatically continuous.

Theorem 2.7 ([CL2, Theorem 3.1.3], [Li3, Theorem 2.3.1])**.** (1) *For a weak* $(\varphi, \hat{G})$*-module* $\hat{\mathfrak{M}}$*, the map* $\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ *is an isomorphism of* $\mathbb{Z}_p[G_\infty]$*-modules.*
(2) *The functor* $\hat{T}$ *induces an anti-equivalence between* $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G}}$ *and* $\mathrm{Rep}_{\mathbb{Z}_p}^r(G)$*.*

Corollary 2.8. *The functor* $\hat{T}\colon {}_{\mathrm{w}}\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\hat{G}} \to \mathrm{Rep}_{\mathrm{tor}}^r(G)$ *is exact and faithful.*

Proof. The exactness of the functor $\hat{T}$ follows from Proposition 2.2 and Theorem 2.7 (1). Since $T_{\mathfrak{S}}\colon \mathrm{Mod}_{/\mathfrak{S}_\infty}^r \to \mathrm{Rep}_{\mathrm{tor}}^r(G_\infty)$ is faithful, the faithfulness of $\hat{T}$ follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{{}_{\mathrm{w}}\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\hat{G}}}(\hat{\mathfrak{M}}, \hat{\mathfrak{M}}') \arrow[r, hook] \arrow[d, "\hat{T}"'] & \mathrm{Hom}_{\mathrm{Mod}_{/\mathfrak{S}_\infty}^r}(\mathfrak{M}, \mathfrak{M}') \arrow[r, hook, "T_{\mathfrak{S}}"] & \mathrm{Hom}_{G_\infty}(T_{\mathfrak{S}}(\mathfrak{M}'), T_{\mathfrak{S}}(\mathfrak{M})) \arrow[d, "\wr"] \\
\mathrm{Hom}_G(\hat{T}(\hat{\mathfrak{M}}'), \hat{T}(\hat{\mathfrak{M}})) \arrow[rr, hook] & & \mathrm{Hom}_{G_\infty}(\hat{T}(\hat{\mathfrak{M}}'), \hat{T}(\hat{\mathfrak{M}})).
\end{tikzcd}$$

□

2.5 Some fundamental properties

In this subsection, we give some fundamental, but important, results on Kisin modules and $(\varphi, \hat{G})$-modules. We start with the following proposition which plays an important role throughout this paper.

Proposition 2.9 ([Li1, Proposition 2.3.2])**.** *Let* $\mathfrak{M}$ *be an object of* $'\mathrm{Mod}_{/\mathfrak{S}}^r$ *which is killed by* p^n*. The following statements are equivalent:*
(1) $\mathfrak{M}$ *is an object of* $\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$*.*
(2) $\mathfrak{M}$ *is u-torsion free.*
(3) $\mathfrak{M}$ *is étale.*
(4) $\mathfrak{M}$ *is a successive extension of finite free* $k[\![u]\!]$*-modules in* $'\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$*, that is, there exists a sequence of extensions*

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_k = \mathfrak{M}$$

in $'\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$ *such that* $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *is an object of* $'\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$*, and* $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *is a finite free* $k[\![u]\!]$*-module.*
(5) $\mathfrak{M}$ *is the quotient of two finite free* $\mathfrak{S}$*-modules* $\mathfrak{N}'$ *and* $\mathfrak{N}''$ *with* $\mathfrak{N}', \mathfrak{N}'' \in \mathrm{Mod}_{/\mathfrak{S}}^r$*.*

Remark 2.10. By Lemma 2.3.1 of [Li1], it is easy to see that, for any i, $\mathfrak{M}_i$ and $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ appeared in Proposition 2.9 (4) are in fact objects of $\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$.

Corollary 2.11. *Let* A *be an* $\mathfrak{S}$*-algebra without* p*-torsion. Then* $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, A) = 0$ *for any Kisin module* $\mathfrak{M}$*. In particular, the functor* $\mathfrak{M} \mapsto A \otimes_{\mathfrak{S}} \mathfrak{M}$ *is an exact functor from the category of Kisin modules to the category of* A*-modules.*

Proof. If $\mathfrak{M}$ is a free Kisin module, then the fact $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, A) = 0$ is clear. Let $\mathfrak{M}$ be a torsion Kisin module and let show $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, A) = 0$. For this proof, we use Proposition 2.9 (4) and *dévissage* to reduce the proof to the case where $\mathfrak{M}$ is killed by p. Then it suffices to show $\mathrm{Tor}_1^{\mathfrak{S}}(k[\![u]\!], A) = 0$. The exact sequence $0 \to \mathfrak{S} \overset{p}{\to} \mathfrak{S} \to k[\![u]\!] \to 0$ induces an exact sequence $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{S}, A) \to \mathrm{Tor}_1^{\mathfrak{S}}(k[\![u]\!], A) \to A \overset{p}{\to} A$. Since $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{S}, A) = 0$ and A has no p-torsion, we obtain $\mathrm{Tor}_1^{\mathfrak{S}}(k[\![u]\!], A) = 0$. □

Recall that, for any $\mathbb{Z}$-module M and any positive integer n, we always use M_n to denote $M/p^n M$.

Corollary 2.12. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ or $\mathrm{Mod}^r_{/\mathfrak{S}}$. Let $A \subset B$ be a ring extension of p-torsion free $\mathfrak{S}$-algebras. Suppose that the natural map $A_1 \to B_1$ is injective. Then the natural map $A \otimes_{\mathfrak{S}} \mathfrak{M} \to B \otimes_{\mathfrak{S}} \mathfrak{M}$ is injective.*

In this paper, we often regard $A \otimes_{\mathfrak{S}} \mathfrak{M}$ (resp. $A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$) as a submodule of $B \otimes_{\mathfrak{S}} \mathfrak{M}$ (resp. $B \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$).

Proof. The statement is clear if $\mathfrak{M}$ is free over $\mathfrak{S}$ or killed by p (since $A_1 \subset B_1$). Suppose that $\mathfrak{M}$ is killed by some power of p. Take a sequence of extensions $0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_k = \mathfrak{M}$ as in Proposition 2.9 (4). Note that $\mathfrak{M}_i$ and $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ are objects of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (cf. Remark 2.10). Since two horizontal sequences in the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \otimes_{\mathfrak{S}} \mathfrak{M}_{i-1} \arrow[r] \arrow[d] & A \otimes_{\mathfrak{S}} \mathfrak{M}_i \arrow[r] \arrow[d] & A \otimes_{\mathfrak{S}} \mathfrak{M}_{i-1} \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & B \otimes_{\mathfrak{S}} \mathfrak{M}_{i-1} \arrow[r] & B \otimes_{\mathfrak{S}} \mathfrak{M}_i \arrow[r] & B \otimes_{\mathfrak{S}} \mathfrak{M}_{i-1} \arrow[r] & 0 \end{tikzcd}$$

are exact (see Corollary 2.11), induction on i gives the desired result. □

Corollary 2.13. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $\mathfrak{N}$ a φ-module over $\mathfrak{S}$ with $\mathfrak{M} \subset \mathfrak{N}$. Let $\mathfrak{S} \subset A \subset W(\mathrm{Fr}R)$ be ring extensions such that $A_1 \to \mathrm{Fr}R$ is injective.*
(1) *The natural map $A \otimes_{\mathfrak{S}} \mathfrak{M} \to A \otimes_{\mathfrak{S}} \mathfrak{N}$ is injective.*
(2) *If A is φ-stable, then the natural map $A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to A \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ is injective.*

Proof. We only prove (2) (a proof for (1) is similar). See the following commutative diagram:

$$\begin{tikzcd} A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[d] & A \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} \arrow[d] \\ W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r] & W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}. \end{tikzcd}$$

The left vertical map is injective by Corollary 2.12 and the bottom horizontal map is also injective since $\varphi\colon \mathfrak{S} \to W(\mathrm{Fr}R)$ is flat. Hence we obtain the desired result. □

Remark 2.14. Let $n > 0$ be an integer.
(1) Let $\mathfrak{S} \subset A \subset B \subset W(\mathrm{Fr}R)$ be ring extensions. If the natural map $A_n \to W_n(\mathrm{Fr}R)$ is injective, then the map $A_n \to B_n$ is also injective.
(2) (cf. [CL2, Lemma 3.1.1], [Fo, Proposition 1.8.3]) We have the following inclusions:

$$\begin{tikzcd} \widehat{\mathcal{R}}_n \arrow[r, hook] & W_n(R) \arrow[r, hook] & W_n(\mathrm{Fr}R) \\ \mathfrak{S}_n \arrow[r, hook] \arrow[u, hook] & \mathfrak{S}_n^{\mathrm{ur}} \arrow[r, hook] \arrow[u, hook] & \mathcal{O}_n^{\mathrm{ur}}. \arrow[u, hook] \end{tikzcd}$$

Corollary 2.15. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $n \geq 0$ an integer. Then $p^n T_{\mathfrak{S}}(\mathfrak{M}) = 0$ if and only if $p^n \mathfrak{M} = 0$.*

Proof. The sufficiency is clear from the definition of $T_{\mathfrak{S}}$. Suppose $p^n T_{\mathfrak{S}}(\mathfrak{M}) = 0$. First we prove the case where $n = 0$. By Proposition 2.9 and Remark 2.10, there exists a sequence of extensions

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_k = \mathfrak{M}$$

in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ such that $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and is a finite free $k[\![u]\!]$-module. Applying $T_{\mathfrak{S}}$ to the exact sequence $0 \to \mathfrak{M}_i \to \mathfrak{M}_{i+1} \to \mathfrak{M}_{i+1}/\mathfrak{M}_i \to 0$, we obtain an exact sequence $0 \to T_{\mathfrak{S}}(\mathfrak{M}_{i+1}/\mathfrak{M}_i) \to T_{\mathfrak{S}}(\mathfrak{M}_{i+1}) \to T_{\mathfrak{S}}(\mathfrak{M}_i) \to 0$ of $\mathbb{Z}_p[G_\infty]$-modules. Since $T_{\mathfrak{S}}(\mathfrak{M}_k) = T_{\mathfrak{S}}(\mathfrak{M}) = 0$, we obtain $T_{\mathfrak{S}}(\mathfrak{M}_k/\mathfrak{M}_{k-1}) = 0$. By Lemma 2.1.2 of [Ki], this implies $\mathfrak{M}_k = \mathfrak{M}_{k-1}$ and in particular,

$T_{\mathfrak{S}}(\mathfrak{M}_{k-1}) = 0$. Inductively, we obtain $\mathfrak{M}_k = \mathfrak{M}_{k-1} = \cdots = \mathfrak{M}_0 = 0$. For general $n \geq 0$, we consider the exact sequence $0 \to \ker(p^n) \to \mathfrak{M} \xrightarrow{p^n} \mathfrak{M}$ in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Since $p^n T_{\mathfrak{S}}(\mathfrak{M}) = 0$, we obtain $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\ker(p^n))$. Therefore, applying $T_{\mathfrak{S}}$ to the exact sequence $0 \to \ker(p^n) \to \mathfrak{M} \to \mathfrak{M}/\ker(p^n) \to 0$ in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, we obtain $T_{\mathfrak{S}}(\mathfrak{M}/\ker(p^n)) = 0$ and then $\mathfrak{M}/\ker(p^n) = 0$. □

Lemma 2.16. *Let $\mathfrak{M}$ be an $\mathfrak{S}$-module of finite type. If $\mathfrak{M}$ is p'-torsion free, then so is $\mathfrak{M}/p\mathfrak{M}$.*

Proof. We may suppose that $\mathfrak{M} \neq 0$. By an elementary ring theory, we obtain $\sqrt{\mathrm{Ann}_{\mathfrak{S}}(\mathfrak{M}/p\mathfrak{M})} = \sqrt{\mathrm{Ann}_{\mathfrak{S}}(\mathfrak{M}) + p\mathfrak{S}} = p\mathfrak{S}$ and thus $\mathrm{Ann}_{\mathfrak{S}}(\mathfrak{M}/p\mathfrak{M}) = p\mathfrak{S}$. □

Proposition 2.17. *Let $\mathfrak{M}$ (resp. $\mathfrak{M}'$) be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Mod}^{r'}_{/\mathfrak{S}_\infty}$) for some $r \in \{0, 1, \ldots, \infty\}$ (resp. $r' \in \{0, 1, \ldots, \infty\}$). Then $\frac{\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{M}'}{u\text{-tor}}$ is an object of $\mathrm{Mod}^{r+r'}_{/\mathfrak{S}_\infty}$. If we put $\mathfrak{M}\otimes\mathfrak{M}' = \frac{\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{M}'}{u\text{-tor}}$, then there exists a canonical isomorphism $T_{\mathfrak{S}}(\mathfrak{M}\otimes\mathfrak{M}') \simeq T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M}')$ of $\mathbb{Z}_p[G_\infty]$-modules. Furthermore, if $\mathfrak{M}$ or $\mathfrak{M}'$ is killed by p, then $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'$ is u-torsion free.*

Proof. To check $\frac{\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{M}'}{u\text{-tor}} \in \mathrm{Mod}^{r+r'}_{/\mathfrak{S}_\infty}$ is not difficult. Putting $M = \mathfrak{M}[1/u]$ and $M' = \mathfrak{M}'[1/u]$, we have $\frac{\mathfrak{M}\otimes_{\mathfrak{S}}\mathfrak{M}'}{u\text{-tor}}[1/u] \simeq M \otimes_{\mathcal{O}} M'$. By Proposition 2.4, we obtain $T_{\mathfrak{S}}(\mathfrak{M} \otimes \mathfrak{M}') \simeq T(M \otimes_{\mathcal{O}} M') \simeq T(M) \otimes_{\mathbb{Z}_p} T(M') \simeq T_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M}')$. The last assertion follows from Lemma 2.16. □

Proposition 2.18 (Scheme-theoretic closure, [Li1, Lemma 2.3.6])**.** *Let $f\colon \mathfrak{M} \to L$ be a morphism of φ-modules over $\mathfrak{S}$. Suppose that $\mathfrak{M}$ and L are p'-torsion free and $\mathfrak{M}$ is an object of $'\mathrm{Mod}^r_{/\mathfrak{S}}$. Then $\ker(f)$ and $\mathrm{im}(f)$ are étale and belong to $'\mathrm{Mod}^r_{/\mathfrak{S}}$. In particular, if $\mathfrak{M}$ is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, then $\ker(f)$ and $\mathrm{im}(f)$ are also objects of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$.*

There exists the $(\varphi, \hat{G})$-analogue of the above proposition.

Corollary 2.19. *Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}'$ be objects of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$). Let $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ be a morphism of $(\varphi, \hat{G})$-modules. Then, $\ker(f)$ and $\mathrm{im}(f)$ as φ-modules are objects of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Furthermore, the $\hat{G}$-action on $\hat{\mathfrak{M}}$ gives $\ker(f)$ a structure of a weak $(\varphi, \hat{G})$-module (resp. a $(\varphi, \hat{G})$-module) and the $\hat{G}$-action on $\hat{\mathfrak{M}}'$ gives $\mathrm{im}(f)$ a structure of a weak $(\varphi, \hat{G})$-module (resp. a $(\varphi, \hat{G})$-module).*

Proof. It is enough to prove only the case where $\hat{\mathfrak{M}}, \hat{\mathfrak{M}}'$ are objects of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. By Proposition 2.18, $\ker(f)$ and $\mathrm{im}(f)$ as φ-modules are in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Consider the image of f. Let $\hat{f}\colon \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ be the morphism induced from f. Since $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f) = \hat{f}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ (by Corollary 2.13) and $\hat{f}$ is compatible with $\hat{G}$-actions, we can define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f)$ such that the map $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f)$ induced from f is $\hat{G}$-equivariant. Since $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f)/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f))$ is surjective, it is a routine work to check that $\widehat{\mathrm{im}(f)} = (\mathrm{im}(f), \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module. The assertion for the kernel of f follows from the fact that, two exact sequences $0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \ker(f) \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \xrightarrow{\hat{f}} \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f) \to 0$ and $0 \to (\widehat{\mathcal{R}}/I_+) \otimes_{\varphi,\mathfrak{S}} \ker(f) \to (\widehat{\mathcal{R}}/I_+) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \xrightarrow{\hat{f}} (\widehat{\mathcal{R}}/I_+) \otimes_{\varphi,\mathfrak{S}} \mathrm{im}(f) \to 0$ arising from the exact sequence $0 \to \ker(f) \to \mathfrak{M} \to \mathrm{im}(f) \to 0$ are exact by Corollary 2.11 (here, we remark that $\widehat{\mathcal{R}}/I_+ \simeq W(k)$ is p-torsion free). □

Corollary 2.20. *Let $0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$ be an exact sequence in ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. If $\hat{\mathfrak{M}}$ is an object of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, then $\hat{\mathfrak{M}}'$ and $\hat{\mathfrak{M}}''$ are also objects of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.*

Proof. This immediately follows from Corollary 2.11. □

3 Cartier duality for $(\varphi, \hat{G})$-modules

In this subsection, we give the Cartier duality for $(\varphi, \hat{G})$-modules. Throughout this section, we fix an integer $r < \infty$.

3.1 Cartier duality for Kisin modules

In this subsection, we recall Liu's results on duality theorems for Kisin modules (cf. Section 3 of [Li1].

Example 3.1. Let $\mathfrak{S}^\vee = \mathfrak{S} \cdot \mathfrak{f}^r$ be the rank-1 free $\mathfrak{S}$-module with $\varphi(\mathfrak{f}^r) = c_0^{-r} E(u)^r \cdot \mathfrak{f}^r$ where pc_0 is the constant coefficient of $E(u)$. We denote by $\varphi^\vee$ this Frobenius φ. Then $(\mathfrak{S}^\vee, \varphi^\vee)$ is a free Kisin module of height r and there exists an isomorphism $T_{\mathfrak{S}}(\mathfrak{S}^\vee) \simeq \mathbb{Z}_p(r)$ as $\mathbb{Z}_p[G_\infty]$-modules (see [Li1, Example 2.3.5]). Put $\mathfrak{S}^\vee_\infty = \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^\vee = \mathfrak{S}_\infty \cdot \mathfrak{f}^r$ (resp. $\mathfrak{S}^\vee_n = \mathbb{Z}_p/p^n\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^\vee = \mathfrak{S}_n \cdot \mathfrak{f}^r$ for any integer $n \geq 0$). The Frobenius φ on $\mathfrak{S}^\vee$ induces Frobenii $\varphi^\vee$ on $\mathfrak{S}^\vee_\infty$ and $\mathfrak{S}^\vee_n$.

Put $\mathcal{E}^\vee = \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{S}^\vee = \mathcal{E} \cdot \mathfrak{f}^r$ and equip $\mathcal{E}^\vee$ with a Frobenius $\varphi^\vee$ arising from those of $\mathcal{E}$ and $\mathfrak{S}^\vee$. Similarly, we put $\mathcal{O}^\vee = \mathcal{O} \cdot \mathfrak{f}^r, \mathcal{O}^\vee_\infty = \mathcal{O}_\infty \cdot \mathfrak{f}^r, \mathcal{O}^\vee_n = \mathcal{O}_n \cdot \mathfrak{f}^r$ and equip them with Frobenii $\varphi^\vee$ which arise from that of $\mathcal{E}^\vee$. We define $\mathcal{O}^{\mathrm{ur},\vee}$, $\mathcal{O}^{\mathrm{ur},\vee}_\infty$ $\mathcal{O}^{\mathrm{ur},\vee}_n$, and Frobenii $\varphi^\vee$ on them by the analogous way.

Let $\mathfrak{M}$ be a Kisin module of height r and denote by $M = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ the corresponding étale φ-module. Put

$$\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_\infty),\ M^\vee = \mathrm{Hom}_{\mathcal{O},\varphi}(M, \mathcal{O}_\infty) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}),\ M^\vee = \mathrm{Hom}_{\mathcal{O},\varphi}(M, \mathcal{O}) \quad \text{if } \mathfrak{M} \text{ is free.}$$

We then have natural pairings

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S}^\vee_\infty,\ \langle \cdot, \cdot \rangle \colon M \times M^\vee \to \mathcal{O}^\vee_\infty \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p$$

and

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S}^\vee,\ \langle \cdot, \cdot \rangle \colon M \times M^\vee \to \mathcal{O}^\vee \quad \text{if } \mathfrak{M} \text{ is free.}$$

The Frobenius $\varphi^\vee_{\mathfrak{M}}$ on $\mathfrak{M}^\vee$ (resp. $\varphi^\vee_M$ on $M^\vee$) is defined to be

$$\langle \varphi_{\mathfrak{M}}(x), \varphi^\vee_{\mathfrak{M}}(y) \rangle = \varphi^\vee(\langle x, y \rangle) \quad \text{for } x \in \mathfrak{M}, y \in \mathfrak{M}^\vee.$$

$$(\text{resp. } \langle \varphi_M(x), \varphi^\vee_M(y) \rangle = \varphi^\vee(\langle x, y \rangle) \quad \text{for } x \in M, y \in M^\vee.)$$

Theorem 3.2 ([Li1])**.** *Let $\mathfrak{M}$ be a Kisin module of height r, $M = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ the corresponding étale φ-module and $\langle \cdot, \cdot \rangle$ the pairing as above.*
(1) *$(\mathfrak{M}^\vee, \varphi^\vee_{\mathfrak{M}})$ is a Kisin module of height r. Similarly, $M^\vee$ is an étale φ-module.*
(2) *A natural map $\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}^\vee \to M^\vee$ is an isomorphism and $\varphi^\vee_M = \varphi_{\mathcal{O}} \otimes \varphi^\vee_{\mathfrak{M}}$.*
(3) *The assignment $\mathfrak{M} \mapsto \mathfrak{M}^\vee$ is an anti-self-equivalence on the category of torsion Kisin-modules (resp. free Kisin-modules) of height r, and the natural map $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ is an isomorphism.*
(4) *All pairings $\langle \cdot, \cdot \rangle$ appeared above are perfect.*
(5) *The dual preserves short exact sequences of torsion Kisin modules (resp. free Kisin modules, resp. torsion étale φ-modules, resp. free étale φ-modules).*

Remark 3.3. The assertion (2) of the above theorem says that we have a natural isomorphism $\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}^\vee \simeq (\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M})^\vee = M^\vee$ which is compatible with φ-structures. In fact, the pairing $\langle \cdot, \cdot \rangle$ for M is equal to the pairing which is obtained by tensoring $\mathcal{O}$ to the pairing $\langle \cdot, \cdot \rangle$ for $\mathfrak{M}$.

3.2 Construction of dual objects

Put

$$\widehat{\mathfrak{S}}^\vee=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{S}^\vee=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}(\mathfrak{S}\cdot\mathfrak{f}^r)=\widehat{\mathcal{R}}\cdot\mathfrak{f}^r,$$

$$\widehat{\mathfrak{S}}^\vee_n=\mathbb{Z}_p/p^n\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\widehat{\mathfrak{S}}^\vee=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{S}^\vee_n=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}(\mathfrak{S}_n\cdot\mathfrak{f}^r)=\widehat{\mathcal{R}}_n\cdot\mathfrak{f}^r\quad\text{for any integer } n\geq 0$$

and

$$\widehat{\mathfrak{S}}^\vee_\infty=\mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\widehat{\mathfrak{S}}^\vee=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{S}^\vee_\infty=\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}(\mathfrak{S}_\infty\cdot\mathfrak{f}^r)=\widehat{\mathcal{R}}_\infty\cdot\mathfrak{f}^r,$$

and we equip them with natural Frobenii arising from those of $\widehat{\mathcal{R}}$ and $\mathfrak{S}^\vee$. By Theorem 2.7, we can define a unique $\hat{G}$-action on $\widehat{\mathfrak{S}}^\vee$ such that $\widehat{\mathfrak{S}}^\vee$ has a structure of a $(\varphi,\hat{G})$-module of height r and there exists an isomorphism

$$\hat{T}(\widehat{\mathfrak{S}}^\vee)\simeq\mathbb{Z}_p(r)\tag{3.2.1}$$

as $\mathbb{Z}_p[G]$-modules. This $\hat{G}$-action on $\widehat{\mathfrak{S}}^\vee$ induces $\hat{G}$-actions on $\widehat{\mathfrak{S}}^\vee_n$ and $\widehat{\mathfrak{S}}^\vee_\infty$. Then it is not difficult to see that $\widehat{\mathfrak{S}}^\vee_n$ has a structure of a torsion $(\varphi,\hat{G})$-module of height r and there exists an isomorphism

$$\hat{T}(\widehat{\mathfrak{S}}^\vee_n)\simeq\mathbb{Z}_p/p^n\mathbb{Z}_p(r)\tag{3.2.2}$$

as $\mathbb{Z}_p[G]$-modules. We may say that $\widehat{\mathfrak{S}}^\vee$ (resp. $\widehat{\mathfrak{S}}^\vee_n$) is the dual $(\varphi,\hat{G})$-module of $\widehat{\mathfrak{S}}$ (resp. $\widehat{\mathfrak{S}}_n$) since (3.2.1) and (3.2.2) hold.

Remark 3.4. If $K_{p^\infty}\cap K_\infty=K$ (which automatically holds in the case $p>2$), then the $\hat{G}$-actions on $\widehat{\mathfrak{S}}^\vee,\widehat{\mathfrak{S}}^\vee_n$ and $\widehat{\mathfrak{S}}^\vee_\infty$ can be written explicitly as follows (see Example 3.2.3 of [Li3]): If $K_{p^\infty}\cap K_\infty=K$, we have $\hat{G}=G_{p^\infty}\rtimes H_K$ (see Lemma 5.1.2 in [Li2]). Fixing a topological generator $\tau\in G_{p^\infty}$, we define $\hat{G}$-actions on the above three modules by the relation $\tau(\mathfrak{f}^r)=\hat{c}^r\cdot\mathfrak{f}^r$. Here $\hat{c}=\frac{c}{\tau(c)}=\prod_{n=1}^\infty\varphi^n(\frac{E(u)}{\tau(E(u))})$, $c=\prod_{n=0}^\infty\varphi^n(\frac{\varphi(c_0^{-1}E(u))}{p})$. Example 3.2.3 of [Li3] says that $c\in A^\times_{\mathrm{cris}}$ and $\hat{c}\in\widehat{\mathcal{R}}^\times$. It follows from straightforward calculations that $\widehat{\mathfrak{S}}^\vee$ and $\widehat{\mathfrak{S}}^\vee_n$ are $(\varphi,\hat{G})$-modules of height r.

Recall that, for any $\mathbb{Z}$-module M, we put $M_\infty=M\otimes_{\mathbb{Z}_p}\mathbb{Q}_p/\mathbb{Z}_p$.

Lemma 3.5. *Let A be an $\mathfrak{S}$-algebra with characteristic coprime to p. Let $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (resp. $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$). Then there exists a natural isomorphism:*

$$A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}^\vee\xrightarrow{\sim}\mathrm{Hom}_A(A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M},A_\infty)\quad\text{if } \mathfrak{M} \textit{ is killed by some power of } p,$$

$$(\textit{resp.}\quad A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}^\vee\xrightarrow{\sim}\mathrm{Hom}_A(A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M},A)\quad \textit{if } \mathfrak{M} \textit{ is free}).$$

Proof. If $\mathfrak{M}$ is free, the statement is clear. If $p\mathfrak{M}=0$, then we may regard $\mathfrak{M}$ as a finite free $\mathfrak{S}_1$-module and thus the statement is clear. Suppose that $\mathfrak{M}$ is a (general) torsion Kisin module of height r. By Proposition 2.9 of [Li1], there exists a sequence of extensions of φ-modules

$$0=\mathfrak{M}_0\subset\mathfrak{M}_1\subset\cdots\subset\mathfrak{M}_n=\mathfrak{M}$$

such that, for all $1\leq i\leq n$, $\mathfrak{M}_i/\mathfrak{M}_{i-1}\in\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ is a finite free $\mathfrak{S}/p\mathfrak{S}=k[\![u]\!]$-module. Furthermore, we have $\mathfrak{M}_i\in\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ by Lemma 2.3.1 in [Li1]. We show that the natural map

$$A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}_i^\vee\longrightarrow\mathrm{Hom}_A(A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}_i,A_\infty),\quad a\otimes f\mapsto(a\otimes x\mapsto af(x))$$

where $a\in A$, $f\in\mathfrak{M}_i^\vee$ and $x\in\mathfrak{M}_i$, is an isomorphism by induction on i. For $i=0$, it is obvious. Suppose that the above map is an isomorphism for $i-1$. We consider the exact sequence of $\mathfrak{S}$-modules

$$0\to\mathfrak{M}_{i-1}\to\mathfrak{M}_i\to\mathfrak{M}_i/\mathfrak{M}_{i-1}\to 0.\tag{3.2.3}$$

By Corollary 3.1.5 of [Li1], we know that the sequence

$$0 \to (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee \to \mathfrak{M}_i^\vee \to \mathfrak{M}_{i-1}^\vee \to 0$$

is also exact as $\mathfrak{S}$-modules. Therefore, we have the following exact sequence of A-modules:

$$A \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee \to A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i^\vee \to A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{i-1}^\vee \to 0. \tag{3.2.4}$$

On the other hand, the exact sequence (3.2.3) induces an exact sequence of A-modules

$$0 \to \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i/\mathfrak{M}_{i-1}, A_\infty) \to \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i, A_\infty) \to \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{i-1}, A_\infty). \tag{3.2.5}$$

Combining the sequences (3.2.4) and (3.2.5), we obtain the following commutative diagram of A-modules:

$$\begin{array}{ccccccc} & & A \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}_i/\mathfrak{M}_{i-1})^\vee & \longrightarrow & A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i^\vee & \longrightarrow & A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{i-1}^\vee \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i/\mathfrak{M}_{i-1}, A_\infty) & \to & \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_i, A_\infty) & \to & \mathrm{Hom}_A(A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{i-1}, A_\infty) \end{array}$$

where the two rows are exact. Furthermore, the first and the third columns are isomorphisms by the induction hypothesis. By the snake lemma, we obtain that the second column is an isomorphism, too. $\square$

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a torsion (resp. free) weak $(\varphi, \hat{G})$-module of height r and $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}}^\vee)$ the dual Kisin module of $(\mathfrak{M}, \varphi_{\mathfrak{M}})$. By Lemma 3.5, we have isomorphisms

$$\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee) \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p, \tag{3.2.6}$$

$$\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}^\vee) \quad \text{if } \mathfrak{M} \text{ is free.} \tag{3.2.7}$$

We define a $\hat{G}$-action on $\mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$ (resp. $\mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}^\vee)$) by

$$(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$$

for $\sigma \in \hat{G}, x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $f \in \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$ (resp. $f \in \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}^\vee)$) and equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$ with a $\hat{G}$-action via the isomorphism (3.2.6) (resp. (3.2.7)).

Theorem 3.6. *Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be a torsion (resp. free) weak $(\varphi, \hat{G})$-module of height r and equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$ with a $\hat{G}$-action as above. Then the triple $\hat{\mathfrak{M}}^\vee = (\mathfrak{M}^\vee, \varphi_{\mathfrak{M}}^\vee, \hat{G})$ is a torsion (resp. free) weak $(\varphi, \hat{G})$-module of height r. If $\hat{\mathfrak{M}}$ is a $(\varphi, \hat{G})$-module of height r, then so is $\hat{\mathfrak{M}}^\vee$.*

Definition 3.7. Let $\hat{\mathfrak{M}}$ be a weak $(\varphi, \hat{G})$-module (resp. a $(\varphi, \hat{G})$-module). We call $\hat{\mathfrak{M}}^\vee$ in Theorem 3.6 the *Cartier dual of* $\hat{\mathfrak{M}}$.

To prove Theorem 3.6, we need the following easy property for $\widehat{\mathcal{R}}_\infty = \widehat{\mathcal{R}}[1/p]/\widehat{\mathcal{R}}$.

Lemma 3.8. (1) *For any integer n, we have*

$$\widehat{\mathcal{R}}[1/p] \cap p^n W(\mathrm{Fr}R) = \widehat{\mathcal{R}} \cap p^n W(R) = p^n \widehat{\mathcal{R}}.$$

(2) *The following properties for $a \in \widehat{\mathcal{R}}[1/p]$ are equivalent:*
- (i) *If $x \in \widehat{\mathcal{R}}[1/p]$ satisfies that $ax = 0$ in $\widehat{\mathcal{R}}_\infty$, then $x = 0$ in $\widehat{\mathcal{R}}_\infty$.*
- (ii) *$a \notin p\widehat{\mathcal{R}}$.*
- (iii) *$a \notin pW(R)$.*
- (iv) *$a \notin pW(\mathrm{Fr}R)$.*

Proof. (1) The result follows from relations

$$\widehat{\mathcal{R}}[1/p] \cap p^n W(\mathrm{Fr}R) = \widehat{\mathcal{R}}[1/p] \cap (W(R)[1/p] \cap p^n W(\mathrm{Fr}R)) = \widehat{\mathcal{R}}[1/p] \cap p^n W(R)$$

and

$$p^n \widehat{\mathcal{R}} \subset \widehat{\mathcal{R}}[1/p] \cap p^n W(R) \subset \widehat{\mathcal{R}}_{K_0} \cap p^n W(R) = p^n (\widehat{\mathcal{R}}_{K_0} \cap W(R)) = p^n \widehat{\mathcal{R}}.$$

(2) The equivalence of (ii), (iii) and (iv) follows from the assertion (1). Suppose the condition (iv) holds. Take any $x \in \widehat{\mathcal{R}}[1/p]$ such that $ax \in \widehat{\mathcal{R}}$. Then we have

$$\frac{1}{a}\widehat{\mathcal{R}} \cap \widehat{\mathcal{R}}[1/p] \subset \frac{1}{a} W(\mathrm{Fr}R) \cap W(\mathrm{Fr}R)[1/p] \subset W(\mathrm{Fr}R)$$

since $a \notin pW(\mathrm{Fr}R)$. Thus we obtain

$$x \in \frac{1}{a}\widehat{\mathcal{R}} \cap \widehat{\mathcal{R}}[1/p] = \frac{1}{a}\widehat{\mathcal{R}} \cap \widehat{\mathcal{R}}[1/p] \cap W(\mathrm{Fr}R) \subset \widehat{\mathcal{R}}[1/p] \cap W(\mathrm{Fr}R) = \widehat{\mathcal{R}},$$

which implies the assertion (i) (the last equality follows from (1)). Suppose the condition (ii) does not hold, that is, $a \in p\widehat{\mathcal{R}}$. Then $\widehat{\mathcal{R}}[1/p] \cap \frac{1}{a}\widehat{\mathcal{R}} \supset \frac{1}{p}\widehat{\mathcal{R}} \supsetneq \widehat{\mathcal{R}}$ and this implies that (i) does not hold. $\square$

Proof of Theorem 3.6. We only prove the case where $\hat{\mathfrak{M}}$ is a torsion $(\varphi, \hat{G})$-module (the free case can be checked by almost the same method).

We check the properties (1) to (5) of Definition 2.6 for $\hat{\mathfrak{M}}^\vee$. It is clear that (1) and (2) hold for $\hat{\mathfrak{M}}^\vee$. Take any $f \in \mathfrak{M}^\vee$. Regard $\mathfrak{M}^\vee$ as a submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$. Then, in $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$, we see that f is equal to the map

$$\hat{f} \colon \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \widehat{\mathcal{R}} \cdot \mathfrak{t}^r, \quad a \otimes x \mapsto a\varphi(f(x)) \cdot \mathfrak{t}^r$$

for $a \in \widehat{\mathcal{R}}$ and $x \in \mathfrak{M}$. Since $\mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^{H_K}$, we have

$$\begin{aligned}(\sigma.\hat{f})(a \otimes x) &= \sigma(\hat{f}(\sigma^{-1}(a \otimes x))) = \sigma(\hat{f}(\sigma^{-1}(a)(1 \otimes x))) = \sigma((\sigma^{-1}(a)\hat{f}(1 \otimes x))) \\ &= a\sigma(\hat{f}(1 \otimes x)) = a\sigma(\varphi(f(x)) \cdot \mathfrak{t}^r) = a\varphi(f(x)) \cdot \mathfrak{t}^r = \hat{f}(a \otimes x).\end{aligned}$$

for any $a \in \widehat{\mathcal{R}}, x \in \mathfrak{M}$ and $\sigma \in H_K$. This implies $\mathfrak{M}^\vee \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee)^{H_K}$ and hence (4) holds for $\hat{\mathfrak{M}}^\vee$. Check the property (5), that is, the condition that $\hat{G}$ acts trivially on $\hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}}$. By Lemma 3.5, we know that there exists the following natural isomorphism:

$$\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee / I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}), \widehat{\mathfrak{S}}_\infty^\vee / I_+\widehat{\mathfrak{S}}_\infty^\vee),$$

which is in fact $\hat{G}$-equivariant by the definition of the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee$. Since $\hat{G}$ acts on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ and $\widehat{\mathfrak{S}}_\infty^\vee / I_+\widehat{\mathfrak{S}}_\infty^\vee$ trivially, we obtain the desired result.

Finally we prove the property (3) for $\hat{\mathfrak{M}}^\vee$. First we note that, if we take any $f \in \mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_\infty)$ and regard f as a map with values in $\mathfrak{S}_\infty^\vee$, then we have

$$\varphi^\vee(f) \circ \varphi_{\mathfrak{M}} = \varphi^\vee \circ f \colon \mathfrak{M} \to \mathfrak{S}_\infty^\vee. \tag{3.2.8}$$

Recall that there exists a natural isomorphism

$$\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee \simeq \mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$$

by Lemma 3.5. We equip $\mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$ with a φ-structure $\varphi^\vee$ via this isomorphism. Then it is enough to show that $\sigma\varphi^\vee = \varphi^\vee\sigma$ on $\mathrm{Hom}_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$ for any $\sigma \in \hat{G}$. Take any $\hat{f} \in \mathrm{Hom}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathfrak{S}}_\infty^\vee)$ and consider the following diagram:

$$\begin{tikzcd}
\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, "\varphi_{\hat{\mathfrak{M}}}"] \arrow[d, "\hat{f}"'] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[d, "\varphi^\vee(\hat{f})"] \\
\widehat{\mathfrak{S}}_\infty^\vee \arrow[r, "\varphi^\vee"'] & \widehat{\mathfrak{S}}_\infty^\vee.
\end{tikzcd} \tag{3.2.9}$$

By (3.2.8), we obtain that the diagram (3.2.9) is also commutative. To check the relation $\sigma(\varphi^\vee(\hat{f})) = \varphi^\vee(\sigma(\hat{f}))$, it suffices to show that $\sigma(\varphi^\vee(\hat{f}))(\varphi_{\hat{\mathfrak{M}}}(x)) = \varphi^\vee(\sigma(\hat{f}))(\varphi_{\hat{\mathfrak{M}}}(x))$ for any $x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ since $\mathfrak{M}$ is of finite $E(u)$-height and, for any $a \in \widehat{\mathcal{R}}_\infty$, $\varphi(E(u))a = 0$ if and only if $a = 0$ by Lemma 3.8. By (3.2.9), we have

$$\sigma(\varphi^\vee(\hat{f}))(\varphi_{\hat{\mathfrak{M}}}(x)) = \sigma(\varphi^\vee(\hat{f})(\sigma^{-1}(\varphi_{\hat{\mathfrak{M}}}(x)))) = \sigma(\varphi^\vee(\hat{f})(\varphi_{\hat{\mathfrak{M}}}(\sigma^{-1}(x)))) = \sigma(\varphi^\vee(\hat{f}(\sigma^{-1}(x)))).$$

By replacing $\hat{f}$ with $\sigma(\hat{f})$ in the diagram (3.2.9), we have

$$\varphi^\vee(\sigma(\hat{f}))(\varphi_{\hat{\mathfrak{M}}}(x)) = \varphi^\vee(\sigma(\hat{f}))(x) = \varphi^\vee(\sigma(\hat{f}(\sigma^{-1}(x)) = \sigma(\varphi^\vee(\hat{f}(\sigma^{-1}(x))))$$

and this finishes the proof. □

3.3 Cartier duality theorem

Let $\hat{\mathfrak{M}}$ be a weak $(\varphi, \hat{G})$-module of height r. We have natural pairings

$$\langle\cdot,\cdot\rangle\colon (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \times (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee) \to \widehat{\mathfrak{S}}^\vee_\infty \quad \text{if } \mathfrak{M} \text{ is killed by some power of } p \tag{3.3.1}$$

and

$$\langle\cdot,\cdot\rangle\colon (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \times (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee) \to \widehat{\mathfrak{S}}^\vee \quad \text{if } \mathfrak{M} \text{ is free.} \tag{3.3.2}$$

It is not difficult to see that these pairings commute with the Frobenii and the $\hat{G}$-actions.

Here we describe the Cartier duality theorem for $(\varphi, \hat{G})$-modules.

Theorem 3.9 (Cartier duality theorem)**.** *Let $\hat{\mathfrak{M}}$ be a weak $(\varphi, \hat{G})$-module (resp. a $(\varphi, \hat{G})$-module) of height r.*
(1) *The assignment $\hat{\mathfrak{M}} \mapsto \hat{\mathfrak{M}}^\vee$ is an anti-self-equivalence on the category of torsion weak $(\varphi, \hat{G})$-modules (resp. free weak $(\varphi, \hat{G})$-modules, resp. torsion $(\varphi, \hat{G})$-modules, resp. free weak $(\varphi, \hat{G})$-modules) of height r, and the natural map $\hat{\mathfrak{M}} \to (\hat{\mathfrak{M}}^\vee)^\vee$ is an isomorphism.*
(2) *Pairings (3.3.1) and (3.3.2) are perfect.*
(3) *The dual preserves short exact sequences of torsion weak $(\varphi, \hat{G})$-modules (resp. free weak $(\varphi, \hat{G})$-modules, resp. torsion $(\varphi, \hat{G})$-modules, resp. free weak $(\varphi, \hat{G})$-modules).*

Proof. By Theorem 3.2 (3), we have already known that the natural map $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ is an isomorphism as φ-modules. Furthermore, straightforward calculations show that the map $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ is compatible with the Galois actions after tensoring $\widehat{\mathcal{R}}$. Thus we obtain that $\hat{\mathfrak{M}} \to (\hat{\mathfrak{M}}^\vee)^\vee$ is an isomorphism, and the assertion (1) follows immediately. The assertion (3) follows from Theorem 3.2 (5). Consequently, we have to show the assertion (2). We leave a proof to the next section. □

3.4 Compatibility with Galois actions

The goal of this subsection is to prove the following which is equivalent to Theorem 3.9 (2):

Proposition 3.10. *Let $\hat{\mathfrak{M}}$ be a weak $(\varphi, \hat{G})$-module. Then we have*

$$\hat{T}(\hat{\mathfrak{M}}^\vee) \simeq \hat{T}^\vee(\hat{\mathfrak{M}})(r) \tag{3.4.1}$$

as $\mathbb{Z}_p[G]$-modules where $\hat{T}^\vee(\hat{\mathfrak{M}})$ is the dual representation of $\hat{T}(\hat{\mathfrak{M}})$ and the symbol "(r)" stands for the r-th Tate twist.

First we construct a covariant functor on the category of weak $(\varphi, \hat{G})$-modules. Recall that, if $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is a weak $(\varphi, \hat{G})$-module, we often abuse notations by writing $\hat{\mathfrak{M}}$ for the underlying module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Proposition 3.11. *Let $\hat{\mathfrak{M}}$ be a weak $(\varphi, \hat{G})$-module. Then the natural $W(\mathrm{Fr}R)$-linear map*

$$W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \to W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, \quad a \otimes x \mapsto ax, \tag{3.4.2}$$

for any $a \in W(\mathrm{Fr}R)$ and $x \in (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}$, is an isomorphism, which is compatible with the φ-structures and the G-actions.

Proof. A non-trivial assertion of this proposition is only the bijectivity of the map (3.4.2). First we note the following natural φ-equivariant isomorphisms:

$$\begin{aligned} W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} &\simeq W(\mathrm{Fr}R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \\ &\simeq W(\mathrm{Fr}R) \otimes_{\mathcal{O}} (\mathcal{O} \otimes_{\varphi, \mathfrak{S}} M) \\ &\xrightarrow{1 \otimes \varphi_M^*} W(\mathrm{Fr}R) \otimes_{\mathcal{O}} M \end{aligned}$$

where $M = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the étale φ-module corresponding to $\mathfrak{M}$. Here the bijectivity of $1 \otimes \varphi_M^*$, where φ_M^* is the $\mathcal{O}$-linearization of φ_M, follows from the étaleness of M. Combining the above isomorphisms with the relation (2.2.1), we obtain the following natural φ-equivariant bijective maps

$$W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \xrightarrow{\sim} W(\mathrm{Fr}R) \otimes_{\mathcal{O}} M \xleftarrow{\sim} W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}} M)^{\varphi=1} \tag{3.4.3}$$

and hence we obtain

$$(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \simeq (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}} M)^{\varphi=1}. \tag{3.4.4}$$

By (3.4.3) and (3.4.4), we obtain

$$W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1} \xrightarrow{\sim} W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$$

and the desired result follows from the fact that this isomorphism coincides with the natural map (3.4.2). □

For any weak $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$, we set

$$\hat{T}_*(\hat{\mathfrak{M}}) = (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}.$$

Since the Frobenius action on $W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$ commutes with the G-action, we see that $\hat{T}_*(\hat{\mathfrak{M}})$ is stable under the G-action. We have shown in the proof of Proposition 3.11 (see (3.4.4)) that

$$\hat{T}_*(\hat{\mathfrak{M}}) \simeq \mathcal{T}_*(M)$$

as $\mathbb{Z}_p[G_\infty]$-modules for $M = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ (the functor $\mathcal{T}_*$ is defined in Section 2.2). In particular, if $\hat{\mathfrak{M}}$ is free and $d = \mathrm{rank}_{\mathfrak{S}}(\mathfrak{M})$, $\hat{T}_*(\hat{\mathfrak{M}})$ is free of rank d as a $\mathbb{Z}_p$-module. The association $\hat{\mathfrak{M}} \mapsto \hat{T}_*(\hat{\mathfrak{M}})$ is a covariant functor from the category of $(\varphi, \hat{G})$-modules of height r to the category $\mathrm{Rep}_{\mathbb{Z}_p}(G)$ of finite $\mathbb{Z}_p[G]$-modules. By the exactness of the functor $\mathcal{T}_*$, the functor $\hat{T}_*$ is an exact functor.

Corollary 3.12. *The $\mathbb{Z}_p$-representation $\hat{T}_*(\hat{\mathfrak{M}})$ of G is the dual of $\hat{T}(\hat{\mathfrak{M}})$, that is,*

$$\hat{T}^\vee(\hat{\mathfrak{M}}) \simeq \hat{T}_*(\hat{\mathfrak{M}})$$

as $\mathbb{Z}_p[G]$-modules where $\hat{T}^\vee(\hat{\mathfrak{M}})$ is the dual representation of $\hat{T}(\hat{\mathfrak{M}})$.

Proof. Suppose $\hat{\mathfrak{M}}$ is killed by some power of p. By Proposition 3.11 and the relation $W(\mathrm{Fr}R)_\infty^{\varphi=1} = \mathbb{Q}_p/\mathbb{Z}_p$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}_*(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p) &\simeq \mathrm{Hom}_{W(\mathrm{Fr}R),\varphi}(W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}})^{\varphi=1}, W(\mathrm{Fr}R)_\infty) \\ &\simeq \mathrm{Hom}_{W(\mathrm{Fr}R),\varphi}(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty) \\ &\simeq \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty) = \hat{T}(\hat{\mathfrak{M}}). \end{aligned}$$

The last equality follows from the proof of [Li3, Lemma 3.1.1], but we include a proof here for the sake of completeness. Take any $h \in \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(\mathrm{Fr}R)_\infty)$. It is enough to prove that h has in fact its values in $W(R)_\infty$. Put $g = h|_{\mathfrak{M}}$. Since g is a $\varphi(\mathfrak{S})$-linear homomorphism from $\mathfrak{M}$ to $W(R)_\infty = \varphi(W(R)_\infty)$, there exists an $\mathfrak{S}$-linear homomorphism $\mathfrak{g}\colon \mathfrak{M} \to W(\mathrm{Fr}R)_\infty$ such that $\varphi(\mathfrak{g}) = g$. Furthermore, we see that $\mathfrak{g}$ is φ-equivariant. Note that $\mathfrak{g}(\mathfrak{M}) \subset W(\mathrm{Fr}R)_\infty$ is an $\mathfrak{S}$-finite type φ-stable submodule with $E(u)$-height r. By [Fo, Proposition B.1.8.3], we have $\mathfrak{g}(\mathfrak{M}) \subset \mathfrak{S}^{\mathrm{ur}}_\infty$. Since $h(a \otimes x) = a\varphi(\mathfrak{g}(x))$ for any $a \in \widehat{\mathcal{R}}$ and $x \in \mathfrak{M}$, we obtain that h has values in $W(R)_\infty$.

In the case $\hat{\mathfrak{M}}$ is free, we obtain the desired result by the same proof as above if we replace $W(\mathrm{Fr}R)_\infty$ (resp. $\mathbb{Q}_p/\mathbb{Z}_p$) with $W(\mathrm{Fr}R)$ (resp. $\mathbb{Z}_p$). $\square$

In the rest of this subsection, we prove Proposition 3.10. We only prove the case where $\mathfrak{M}$ is killed by p^n for some integer $n \geq 1$ (we can prove the free case by an analogous way and the free case is easier than the torsion case).

First we consider natural pairings

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^\vee \to \mathfrak{S}^\vee_n \tag{3.4.5}$$

and

$$\langle \cdot, \cdot \rangle \colon M \times M^\vee \to \mathcal{O}^\vee_n \tag{3.4.6}$$

which are perfect and compatible with φ-structures. Here $M = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ is the étale φ-module corresponding to $\mathfrak{M}$. We can extend the pairing (3.4.6) to a φ-equivariant perfect pairing

$$(\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M) \times (\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M^\vee) \to \mathcal{O}^{\mathrm{ur},\vee}_n.$$

Since the above pairing is φ-equivariant and $(\mathcal{O}^{\mathrm{ur},\vee}_n)^{\varphi=1} \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p(-r)$, we have a pairing

$$(\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M)^{\varphi=1} \times (\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M^\vee)^{\varphi=1} \to \mathbb{Z}_p/p^n\mathbb{Z}_p(-r) \tag{3.4.7}$$

compatible with the G_∞-actions. Liu showed in the proof of [Li1, Lemma 3.1.2] that this pairing is perfect. By a similar way, we obtain a pairing

$$(W(\mathrm{Fr}R) \otimes_{\mathcal{O}} M)^{\varphi=1} \times (W(\mathrm{Fr}R) \otimes_{\mathcal{O}} M^\vee)^{\varphi=1} \to \mathbb{Z}_p/p^n\mathbb{Z}_p(-r). \tag{3.4.8}$$

On the other hand, the pairing (3.4.5) induces a pairing

$$(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \times (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee) \to \widehat{\mathfrak{S}}^\vee_n. \tag{3.4.9}$$

We can extend the pairing (3.4.9) to a φ-equivariant perfect pairing

$$(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})) \times (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee)) \to W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{S}}^\vee_n.$$

Since the above pairing is φ-equivariant and $(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \widehat{\mathfrak{S}}^\vee_n)^{\varphi=1} \simeq \mathbb{Z}_p/p^n\mathbb{Z}_p(-r)$, we have a pairing

$$(W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}))^{\varphi=1} \times (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}^\vee))^{\varphi=1} \to \mathbb{Z}_p/p^n\mathbb{Z}_p(-r) \tag{3.4.10}$$

compatible with the G-actions. Since we have a natural isomorphism $\mathcal{O}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} (\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M)^{\varphi=1} \overset{\sim}{\longrightarrow} \mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M$, we obtain φ-equivariant isomorphisms

$$W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \overset{\sim}{\longrightarrow} W(\mathrm{Fr}R) \otimes_{\mathcal{O}} M \overset{\sim}{\longleftarrow} W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (\mathcal{O}^{\mathrm{ur}} \otimes_{\mathcal{O}} M)^{\varphi=1}. \tag{3.4.11}$$

Therefore, combining (3.4.7), (3.4.8), (3.4.10) and (3.4.11), we have the following diagram

$$
\begin{array}{ccccc}
(W(\mathrm{Fr}R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}})^{\varphi=1} & \times & (W(\mathrm{Fr}R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}}^{\vee})^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n\mathbb{Z}_p(-r) \\
\wr\downarrow & & \wr\downarrow & & \| \\
(W(\mathrm{Fr}R)\otimes_{\mathcal{O}} M)^{\varphi=1} & \times & (W(\mathrm{Fr}R)\otimes_{\mathcal{O}} M^{\vee})^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n\mathbb{Z}_p(-r) \\
\wr\uparrow & & \wr\uparrow & & \| \\
(\mathcal{O}^{\mathrm{ur}}\otimes_{\mathcal{O}} M)^{\varphi=1} & \times & (\mathcal{O}^{\mathrm{ur}}\otimes_{\mathcal{O}} M^{\vee})^{\varphi=1} & \longrightarrow & \mathbb{Z}_p/p^n\mathbb{Z}_p(-r)
\end{array}
$$

It is a straightforward calculation to check that the above diagram is commutative. Since the bottom pairing is perfect, we see that the top pairing is also perfect. This implies $\hat{T}_*(\hat{\mathfrak{M}}^{\vee})\simeq\hat{T}_*(\hat{\mathfrak{M}})(-r)$ and therefore, we have the desired result by Corollary 3.12.

4 Category of representations arising from torsion $(\varphi,\hat{G})$-modules

4.1 Relations between $(\varphi,\hat{G})$-modules and their representations

Choose an element $\mathfrak{t}\in\mathfrak{S}^{\mathrm{ur}}$ such that $\mathfrak{t}\notin p\mathfrak{S}^{\mathrm{ur}}$ and $\varphi(\mathfrak{t})=c_0^{-1}E(u)\mathfrak{t}$ where $pc_0=E(0)$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$, see Example 2.3.5 in [Li1] for details.

Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. We construct a map $\iota_{\mathfrak{S}}$ which connects $\mathfrak{M}$ to $T_{\mathfrak{S}}(\mathfrak{M})$ (cf. [Li1, Section 3.2]). First observe that there exists a natural isomorphism of $\mathbb{Z}_p[G_\infty]$-modules

$$T_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_\infty)\simeq\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_\infty)$$

where G_∞ acts on $\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_\infty)$ by $(\sigma.f)(x)=\sigma(f(\sigma^{-1}(x)))$ for $\sigma\in G_\infty, f\in\mathrm{Hom}_{\mathfrak{S}^{\mathrm{ur}},\varphi}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_\infty), x\in\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}$ and G_∞ acts on $\mathfrak{M}$ trivial. Thus we can define a morphism $\iota'_{\mathfrak{S}}\colon\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}\to\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}),\mathfrak{S}^{\mathrm{ur}}_\infty)$ by

$$x\mapsto(f\mapsto f(x)),\quad x\in\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}, f\in T_{\mathfrak{S}}(\mathfrak{M}).$$

Since $T_{\mathfrak{S}}(\mathfrak{M})\simeq\oplus_{i\in I}\mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$ as finite $\mathbb{Z}_p$-modules, we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}),\mathfrak{S}^{\mathrm{ur}}_\infty)\simeq\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M})$ where $T^{\vee}_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}),\mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of $T_{\mathfrak{S}}(\mathfrak{M})$. Composing this isomorphism with $\iota'_{\mathfrak{S}}$, we obtain the desired map

$$\iota_{\mathfrak{S}}\colon\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}\to\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M}).$$

For $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}}$, we also construct $\iota_{\mathfrak{S}}\colon\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}\to\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M})$ by the same way except only for replacing $\mathfrak{S}^{\mathrm{ur}}_\infty$ with $\mathfrak{S}^{\mathrm{ur}}$.

Lemma 4.1. *Let A be a ring with $\mathfrak{S}^{\mathrm{ur}}\subset A\subset W(\mathrm{Fr}R)$ which yields a ring extension $A_1\subset\mathrm{Fr}R$. Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ or $\mathrm{Mod}^r_{/\mathfrak{S}}$. Let $\iota_{\mathfrak{S}}$ be as above.*
(1) $\iota_{\mathfrak{S}}$ is G_∞-equivariant and φ-equivariant. Furthermore, $A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}$ is injective.
(2) If $r<\infty$, then $\mathfrak{t}^r(A\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M}))\subset(A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}})(A\otimes_{\mathfrak{S}}\mathfrak{M})$. If $r=\infty$, then $\mathfrak{t}^{r'}(A\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M}))\subset(A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}})(A\otimes_{\mathfrak{S}}\mathfrak{M})$ for $r'>0$ such that $\mathfrak{M}$ is of height r'.
(3) The map

$$W(\mathrm{Fr}R)\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}\colon W(\mathrm{Fr}R)\otimes_{\mathfrak{S}}\mathfrak{M}\to W(\mathrm{Fr}R)\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M})$$

is bijective.

Proof. We may suppose that $r<\infty$. The assertion that $\iota_{\mathfrak{S}}$ is G_∞-equivariant and φ-equivariant is a result of [Li1, Theorem 3.2.2]. Liu showed *loc. cit.*, that there exists a map $\iota^{\vee}_{\mathfrak{S}}\colon\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}T^{\vee}_{\mathfrak{S}}(\mathfrak{M})\to\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathfrak{S}}\mathfrak{M}$ such that $\iota^{\vee}_{\mathfrak{S}}\circ\iota_{\mathfrak{S}}=\mathfrak{t}^r$, in particular, $(A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota^{\vee}_{\mathfrak{S}})\circ(A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}})=\mathfrak{t}^r$. Moreover, in the proof *loc. cit*, Liu also showed that the composite $(\mathcal{O}^{\mathrm{ur}}\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}})\circ(\mathcal{O}^{\mathrm{ur}}\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota^{\vee}_{\mathfrak{S}})\colon\mathcal{O}^{\mathrm{ur}}\otimes_{\mathfrak{S}^{\mathrm{ur}}}(\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}$

$T^\vee_{\mathfrak{S}}(\mathfrak{M})) \to \mathcal{O}^{\mathrm{ur}} \otimes_{\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}))$ is equal to the map $\mathcal{O}^{\mathrm{ur}} \otimes_{\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{t}^r \otimes_{\mathbb{Z}_p} \mathrm{Id})$. Hence we obtain $\iota_{\mathfrak{S}} \circ \iota^\vee_{\mathfrak{S}} = \mathfrak{t}^r$ and then the assertion (2) follows.

We show the injectivity of $A \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$. Since $(A \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota^\vee_{\mathfrak{S}}) \circ (A \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}) = \mathfrak{t}^r$, if $\mathfrak{M}$ is free over $\mathfrak{S}$, we see that $A \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ is injective. Next we suppose that $\mathfrak{M}$ is killed by p. In this case, the proof is almost the same as the free case, except one needs to note that $\mathfrak{M}$ is free as a $k[\![u]\!]$-module, $\mathfrak{t} \neq 0$ in A_1 (since $\mathfrak{S}^{\mathrm{ur}}_1 \subset A_1$; see Remark 2.14) and A_1 is a domain (since $A_1 \subset \mathrm{Fr}R$). Suppose that $\mathfrak{M}$ is killed by some power of p. By Proposition 2.9 (4) and Remark 2.10, there exists a sequence of extensions

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_k = \mathfrak{M}$$

in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ such that $\mathfrak{M}_i,\ \mathfrak{M}_{i+1}/\mathfrak{M}_i \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $\mathfrak{M}_{i+1}/\mathfrak{M}_i$ is a finite free $k[\![u]\!]$-module. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes_{\mathfrak{S}} \mathfrak{M}_{i-1} \arrow[r] \arrow[d, "A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},i-1}"'] & A \otimes_{\mathfrak{S}} \mathfrak{M}_i \arrow[r] \arrow[d, "A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},i}"'] & A \otimes_{\mathfrak{S}} \mathfrak{M}_i/\mathfrak{M}_{i-1} \arrow[r] \arrow[d, "A\otimes_{\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S},i,i-1}"'] & 0 \\
0 \arrow[r] & A \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}_{i-1}) \arrow[r] & A \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}_i) \arrow[r] & A \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}_i/\mathfrak{M}_{i-1}) \arrow[r] & 0
\end{tikzcd}$$

where $\iota_{\mathfrak{S},i-1}, \iota_{\mathfrak{S},i}$ and $\iota_{\mathfrak{S},i,i-1}$ are the maps $\iota_{\mathfrak{S}}$ for $\mathfrak{M}_i, \mathfrak{M}_{i-1}$ and $\mathfrak{M}_i/\mathfrak{M}_{i-1}$, respectively. By Corollary 2.11 and the exactness of $T_{\mathfrak{S}}$, the two horizontal sequences are exact. By induction on i, we see that $A \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ (for $\mathfrak{M}$) is injective.

Finally, if we put $A = W(\mathrm{Fr}R)$, we see the bijectivity of $W(\mathrm{Fr}R) \otimes_{\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$ from (1), (2) and $\mathfrak{t} \in W(\mathrm{Fr}R)^\times$. $\square$

Let $\hat{\mathfrak{M}}$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. We construct a map $\hat{\iota}$ which connects $\hat{\mathfrak{M}}$ to $\hat{T}(\hat{\mathfrak{M}})$ (cf. [Li2, Section 3.1]). First, we recall that we abuse notations by writing $\hat{\mathfrak{M}}$ for the underlying module $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Observe that there exists a natural isomorphism of $\mathbb{Z}_p[G]$-modules

$$\hat{T}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\hat{\mathfrak{M}}, W(R)_\infty) \simeq \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R)_\infty)$$

where G acts on $\mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R)_\infty)$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G, f \in \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, W(R)_\infty), x \in W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$. Thus we can define a morphism $\hat{\iota}' \colon W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \to \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)_\infty)$ by

$$x \mapsto (f \mapsto f(x)), \quad x \in W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}, f \in \hat{T}(\hat{\mathfrak{M}}).$$

Since $\hat{T}(\hat{\mathfrak{M}}) \simeq \oplus_{i \in I} \mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$ as finite $\mathbb{Z}_p$-modules, we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)_\infty) \simeq W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}})$ where $\hat{T}^\vee(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of $\hat{T}(\hat{\mathfrak{M}})$. Composing this isomorphism with $\hat{\iota}'$, we obtain the desired map

$$\hat{\iota} \colon W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \to W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}}).$$

For $\hat{\mathfrak{M}} \in {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$, we also construct $\hat{\iota} \colon W(R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \to W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}})$ by the same way except only for replacing $W(R)_\infty$ with $W(R)$.

Lemma 4.2. *Let A be a ring with $\mathfrak{S}^{\mathrm{ur}} \subset A \subset W(\mathrm{Fr}R)$ which yields a ring extension $A_1 \subset \mathrm{Fr}R$. Suppose that A is $\varphi_{W(\mathrm{Fr}R)}$-stable. Let $\hat{\mathfrak{M}}$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ or ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$. Let $\hat{\iota}$ be as above.*
(1) $\hat{\iota} \simeq W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{S}}$, that is, the following diagram commutes:

$$\begin{tikzcd}
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r, "\hat{\iota}"] & W(R) \otimes_{\mathbb{Z}_p} \hat{T}^\vee(\hat{\mathfrak{M}}) \\
W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}) \arrow[r, "W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}"] \arrow[u, "\alpha\otimes\mathrm{Id}_{\mathfrak{M}}", "\wr"'] & W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}\mathfrak{M}). \arrow[u, "\alpha\otimes(\theta^\vee)^{-1}"', "\wr"]
\end{tikzcd}$$

Here, $\alpha\colon W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\mathfrak{S}^{\mathrm{ur}}\to W(R)$ *is the isomorphism given by* $\alpha(\sum_i a_i\otimes b_i)=\sum_i a_i\varphi(b_i)$ *with* $a_i\in W(R), b_i\in\mathfrak{S}^{\mathrm{ur}}$.
(2) $\hat{\iota}$ *is* G*-equivariant and* φ*-equivariant. Furthermore,* $A\otimes_{W(R)}\hat{\iota}$ *is injective.*
(3) *If* $r<\infty$*, then* $\varphi(\mathfrak{t})^r(A\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}))\subset(A\otimes_{W(R)}\hat{\iota})(A\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}})$*. If* $r=\infty$*, then* $\varphi(\mathfrak{t})^{r'}(A\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}))\subset(A\otimes_{W(R)}\hat{\iota})(A\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}})$ *for* $r'>0$ *such that* $\mathfrak{M}$ *is of* $E(u)$*-height* r'.
(4) *The map*

$$W(\mathrm{Fr}R)\otimes_{W(R)}\hat{\iota}\colon W(\mathrm{Fr}R)\otimes_{\widehat{\mathcal{R}}}\hat{\mathfrak{M}}\to W(\mathrm{Fr}R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}})$$

is bijective.

Proof. The statement (1) follows from the same proof as that of Proposition 3.1.3 (2) of [Li2]. To see that $A\otimes_{W(R)}\hat{\iota}$ is injective, by (1), it is enough to check that $A\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}}\iota_{\mathfrak{S}}\colon A\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to A\otimes_{\mathbb{Z}_p}T^\vee_{\mathfrak{S}}(\mathfrak{M})$ is injective. This can be checked by almost the same method as the proof of Lemma 4.1 (1). The rest statements follow from (1) and Lemma 4.1. $\square$

Let $\hat{\mathfrak{M}}$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ or ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$. Then $T_{\mathfrak{S}}(\hat{\mathfrak{M}})$ has a natural G-action via $\theta\colon T_{\mathfrak{S}}(\mathfrak{M})\overset{\sim}{\to}\hat{T}(\hat{\mathfrak{M}})$ (see Theorem 2.7).

Corollary 4.3. *Let* $\hat{\mathfrak{M}}$ *and* $\hat{\mathfrak{M}}'$ *be objects of* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *(resp.* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$*). Let* $f\colon\mathfrak{M}'\to\mathfrak{M}$ *be a morphism in* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$*. If* $T_{\mathfrak{S}}(f)$ *is* G*-equivariant, then* f *is in fact a morphism in* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *(resp.* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$*).*

Proof. Consider a commutative diagram

$$\begin{tikzcd}
W(R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}') \arrow[r] & W(R)\otimes_{\mathbb{Z}_p}\hat{T}^\vee(\hat{\mathfrak{M}}) \\
W(R)\otimes_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}') \arrow[r] \arrow[u, hook, "\hat{\iota}", "\wr"'] & W(R)\otimes_{\widehat{\mathcal{R}}}(\widehat{\mathcal{R}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}) \arrow[u, hook, "\hat{\iota}", "\wr"']
\end{tikzcd}$$

where the top and bottom arrows are morphisms induced from f. By our assumption on f and the result that $\hat{\iota}$ is injective, we see that the bottom arrow commutes with the G-actions and then we have done. $\square$

4.2 Proof of Theorem 1.1

Lemma 4.4. *Let* $0\to T'\to T\to T''\to 0$ *be an exact sequence in* $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$*. Suppose that there exist* $\mathfrak{M}\in\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *and an isomorphism* $\psi\colon T_{\mathfrak{S}}(\mathfrak{M})\xrightarrow{\sim}T$ *of* $\mathbb{Z}_p$*-representations of* G_∞*. Then there exists an exact sequence* $0\to\mathfrak{M}''\to\mathfrak{M}\to\mathfrak{M}'\to 0$ *in* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *which makes the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r] & T \arrow[r] & T'' \arrow[r] & 0 \\
0 \arrow[r] & T_{\mathfrak{S}}(\mathfrak{M}') \arrow[r] \arrow[u, "\wr"'] & T_{\mathfrak{S}}(\mathfrak{M}) \arrow[r] \arrow[u, "\psi", "\wr"'] & T_{\mathfrak{S}}(\mathfrak{M}'') \arrow[r] \arrow[u, "\wr"'] & 0.
\end{tikzcd}$$

Proof. Put $M=\mathcal{O}\otimes_{\mathfrak{S}}\mathfrak{M}$ and let Ψ be the composite $\mathcal{T}(M)\simeq T_{\mathfrak{S}}(\mathfrak{M})\overset{\psi}{\to}T$. By Proposition 2.1, there exists an exact sequence $0\to M''\to M\overset{g}{\to}M'\to 0$ in $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$ which makes the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r] & T \arrow[r] & T'' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{T}(M') \arrow[r] \arrow[u, "\wr"'] & \mathcal{T}(M) \arrow[r] \arrow[u, "\Psi", "\wr"'] & \mathcal{T}(M'') \arrow[r] \arrow[u, "\wr"'] & 0.
\end{tikzcd}$$

By abuse of notation we denote by g the composite $\mathfrak{M} \hookrightarrow M \xrightarrow{g} M'$. Put $\mathfrak{M}'' = \mathfrak{M} \cap M''$ and $\mathfrak{M}' = g(\mathfrak{M})$. Since $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and M' is p'-torsion free, it follows from Proposition 2.18 that $\mathfrak{M}'$ and $\mathfrak{M}''$ are in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. The inclusion map $\mathfrak{M} \hookrightarrow M$ induces an injection $\mathfrak{M}'' \hookrightarrow M''$ and thus we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M'' \arrow[r] & M \arrow[r] & M' \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}'' \arrow[r] \arrow[u] & \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[u, equal] & \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where the two horizontal sequences of étale φ-modules are exact. By a diagram chasing, we see that the map $\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}' \to M'$ is surjective. Since $\mathfrak{M}' \subset M'$ is φ-stable and finite as an $\mathfrak{S}$-module, we know that the map $\mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}' \to M'$ is injective (cf. [Fo, B. 1.4.2]) and thus, it is bijective. By the snake lemma, we know that the left vertical arrow of the above diagram is also bijective. Applying the functor T to the above diagram, we obtain the desired result. $\square$

Theorem 4.5. *Let $0 \to T' \to T \to T'' \to 0$ be an exact sequence in $\mathrm{Rep}_{\mathrm{tor}}(G)$. Suppose that there exist $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ and an isomorphism $\psi\colon \hat{T}(\hat{\mathfrak{M}}) \xrightarrow{\sim} T$ of $\mathbb{Z}_p$-representations of G. Then there exists an exact sequence $0 \to \hat{\mathfrak{M}}'' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}' \to 0$ in $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ which makes the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r] & T \arrow[r] & T'' \arrow[r] & 0 \\
0 \arrow[r] & \hat{T}(\hat{\mathfrak{M}}') \arrow[r] \arrow[u, "\wr"'] & \hat{T}(\hat{\mathfrak{M}}) \arrow[r] \arrow[u, "\psi", "\wr"'] & \hat{T}(\hat{\mathfrak{M}}'') \arrow[r] \arrow[u, "\wr"'] & 0.
\end{tikzcd}$$

Proof. A short argument shows that we may suppose $T = \hat{T}(\hat{\mathfrak{M}})$ and ψ is the identity map on T. Let

$$\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$$

be as in Section 2.4, which is a G_∞-equivariant isomorphism. By Lemma 4.4, we have an exact sequence $0 \to \mathfrak{M}'' \to \mathfrak{M} \to \mathfrak{M}' \to 0$ in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ which makes the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T' \arrow[r] & T \arrow[r] & T'' \arrow[r] & 0 \\
0 \arrow[r] & T_{\mathfrak{S}}(\mathfrak{M}') \arrow[r] \arrow[u, "\wr"'] & T_{\mathfrak{S}}(\mathfrak{M}) \arrow[r] \arrow[u, "\theta", "\wr"'] & T_{\mathfrak{S}}(\mathfrak{M}'') \arrow[r] \arrow[u, "\wr"'] & 0.
\end{tikzcd}$$

We want to equip $\mathfrak{M}'$ and $\mathfrak{M}''$ with structures of $(\varphi, \hat{G})$-modules. Combining the above diagram with Lemma 4.2, we obtain the following diagram all of whose squares commute:

$$\begin{tikzcd}
 & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[rr, hook, "\hat{\iota}"] & & W(R) \otimes_{\mathbb{Z}_p} T^\vee \arrow[dd, "\alpha^{-1}\otimes\theta^\vee", "\wr"'] \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \arrow[ur, hook] & & W(R) \otimes_{\mathbb{Z}_p} (T'')^\vee \arrow[ur, hook] \arrow[dd, "\wr"'] & \\
 & W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}) \arrow[uu, "\alpha\otimes\mathrm{Id}_{\mathfrak{M}}", "\wr"'] \arrow[rr, hook, "W(R)\otimes\iota_{\mathfrak{S}}"] & & W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M})) \\
W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}'') \arrow[uu, "\alpha\otimes\mathrm{Id}_{\mathfrak{M}''}", "\wr"'] \arrow[ur, hook] \arrow[rr, hook, "W(R)\otimes\iota_{\mathfrak{S}}"] & & W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} (\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T^\vee_{\mathfrak{S}}(\mathfrak{M}'')). \arrow[ur, hook] &
\end{tikzcd}$$

Here, $\alpha\colon W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \mathfrak{S}^{\mathrm{ur}} \to W(R)$ is the isomorphism given by $\alpha(\sum_i a_i \otimes b_i) = \sum_i a_i\varphi(b_i)$ with $a_i \in W(R), b_i \in \mathfrak{S}^{\mathrm{ur}}$. Define a map $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \to W(R) \otimes_{\mathbb{Z}_p} (T'')^\vee$ such that all squares in the above diagram commute. Tensoring $W(\mathrm{Fr}R)$ to the ceiling, we obtain the following diagram (note that all maps in the diagram are injective (cf. Corollary 2.11 and 2.12)):

$$
\begin{tikzcd}
\widehat{\mathfrak{M}} \arrow[r, hook] & W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}} \arrow[r, hook, "\iota"] & W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} \hat{T}^{\vee}(\hat{\mathfrak{M}}) \\
\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \arrow[r, hook] \arrow[u, hook] & W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'') \arrow[r, hook, "\iota''"] \arrow[u, hook] & W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (T'')^{\vee} \arrow[u, hook]
\end{tikzcd}
$$

Moreover, the map $\iota = W(\mathrm{Fr}R) \otimes_{W(R)} \hat{\iota}$ is bijective by Lemma 4.2 (4), and the map ι'' is also bijective by Lemma 4.1 (3). Define a G-action on $W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'')$ via ι''. Then the injection $W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'') \hookrightarrow W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$ is automatically G-equivariant. On the other hand, see the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \arrow[r] \arrow[d] & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[d] & \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \arrow[r] \arrow[d] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r] \arrow[d] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \arrow[r] & W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \arrow[r] & W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \arrow[r] & 0.
\end{tikzcd}
$$

By Corollary 2.11 and 2.12, we see that all the horizontal sequences are exact and all the vertical arrows are injective. Hence we may regard $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$ and $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$ as submodules of $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$. In particular, we have

$$
\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' = (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \cap (W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''). \tag{4.2.1}
$$

Since the G-actions on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$ are restrictions of the G-action on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}$, the equation (4.2.1) gives an well-defined G-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$. Since the G-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ factors through $\hat{G}$, the G-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$ also factors through $\hat{G}$. We also define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ via the natural isomorphism $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' \simeq (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'')/(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'')$. It is not difficult to check that triples $\hat{\mathfrak{M}}' = (\mathfrak{M}', \varphi, \hat{G})$ and $\hat{\mathfrak{M}}'' = (\mathfrak{M}'', \varphi, \hat{G})$ are weak $(\varphi, \hat{G})$-modules. Obviously, we have an exact sequences

$$
0 \to \hat{\mathfrak{M}}'' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}' \to 0 \tag{4.2.2}
$$

of weak $(\varphi, \hat{G})$-modules. By Corollary 2.20, we know that $\hat{\mathfrak{M}}'$ and $\hat{\mathfrak{M}}''$ are in fact $(\varphi, \hat{G})$-modules. Now we check that the exact sequence (4.2.2) satisfies the desired property. Projections $\mathfrak{M} \to \mathfrak{M}'$ and $\hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ induce injections $T_{\mathfrak{S}}(\mathfrak{M}') \hookrightarrow T_{\mathfrak{S}}(\mathfrak{M})$ of $\mathbb{Z}_p[G_\infty]$-modules and $\hat{T}(\mathfrak{M}') \hookrightarrow \hat{T}(\mathfrak{M})$ of $\mathbb{Z}_p[G]$-modules. Furthermore, the diagram below is commutative:

$$
\begin{tikzcd}
\hat{T}(\hat{\mathfrak{M}}') \arrow[d, hook] & T_{\mathfrak{S}}(\mathfrak{M}') \arrow[l, "\sim"', "\theta"] \arrow[r, "\sim"] \arrow[d, hook] & T' \arrow[d, hook] \\
\hat{T}(\hat{\mathfrak{M}}) & T_{\mathfrak{S}}(\mathfrak{M}) \arrow[l, "\sim"', "\theta"] \arrow[r, "\sim", "\theta"'] & T.
\end{tikzcd}
$$

This induces the commutative diagram

$$
\begin{tikzcd}
T' \arrow[r, hook] & T \\
\hat{T}(\hat{\mathfrak{M}}') \arrow[r, hook] \arrow[u, "\wr"] & \hat{T}(\hat{\mathfrak{M}}) = T \arrow[u, equal]
\end{tikzcd}
$$

and thus we see that the left vertical arrow in just the above square is G-equivariant. The desired result follows from this. $\square$

Remark 4.6. By using the theory of étale $(\varphi, \hat{G})$-modules, we will give a more natural interpretation of the sequence $0 \to \hat{\mathfrak{M}}'' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}' \to 0$ appeared in Theorem 4.5, see Remark 5.11.

By Theorem 4.5, the essential image of the functor $\hat{T} \colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is stable under subquotients. In particular, we see that the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is also stable under subquotients. Clearly, the category $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is also stable under direct sums. We show that $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is stable under the dual and tensor products.

Lemma 4.7. *The full subcategory* $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ *of* $\mathrm{Rep}_{\mathrm{tor}}(G)$ *is stable under the dual.*

Proof. Let $T \in \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ and take some $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (for some $r < \infty$) such that $T = \hat{T}(\hat{\mathfrak{M}})$. Take an integer $n \geq 0$ such that $\mathfrak{M}$ is killed by p^n. For any integer $k \geq 0$, denote by $\hat{\mathfrak{S}}_n(k)$ the Cartier dual of the trivial $(\varphi, \hat{G})$-module $\hat{\mathfrak{S}}_n$ in $\mathrm{Mod}^{k,\hat{G}}_{/\mathfrak{S}_\infty}$ and by $\mathfrak{S}_n(k)$ its underlying φ-module. Then it can be seen immediately that $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}_n(k)$ has a structure of a $(\varphi, \hat{G})$-module of height $r + k$, and if we denote it by $\hat{\mathfrak{M}}(k)$, then $\hat{T}(\hat{\mathfrak{M}}(k)) = \hat{T}(\hat{\mathfrak{M}})(k)$. Take an integer $m > r$ which is divisible by $p^{n-1}(p-1)$. Then

$$\begin{aligned} T^\vee &= \hat{T}(\hat{\mathfrak{M}}^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-r) = \hat{T}(\hat{\mathfrak{M}}^\vee) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(m-r) \\ &= \hat{T}(\hat{\mathfrak{M}}^\vee) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{S}}_n(m-r)) = \hat{T}(\hat{\mathfrak{M}}^\vee(m-r)) \end{aligned}$$

and we have done. □

Finally we consider the assertion related with a tensor product of Theorem 1.1. It is enough to prove the following lemma.

Lemma 4.8. *Let* $\hat{\mathfrak{M}} \in {}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *(resp.* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*) and* $\hat{\mathfrak{M}}' \in {}_{\mathrm{w}}\mathrm{Mod}^{r',\hat{G}}_{/\mathfrak{S}_\infty}$ *(resp.* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r',\hat{G}}_{/\mathfrak{S}_\infty}$*) for some* $r, r' \in \{0, 1, \ldots, \infty\}$*. Then* $\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}$ *is an object of* ${}_{\mathrm{w}}\mathrm{Mod}^{r+r'}_{/\mathfrak{S}_\infty}$ *(resp.* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r+r',\hat{G}}_{/\mathfrak{S}_\infty}$*) and has a structure of a weak* $(\varphi, \hat{G})$*-module (resp. a* $(\varphi, \hat{G})$*-module). If we put* $\hat{\mathfrak{M}} \otimes \hat{\mathfrak{M}}' = \widehat{\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}}$*, then there exists a canonical isomorphism* $\hat{T}(\hat{\mathfrak{M}} \otimes \hat{\mathfrak{M}}') \simeq \hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}}')$ *of* $\mathbb{Z}_p[G]$*-modules.*

Proof. Since $\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}$ is u-torsion free, we see $\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}} \in \mathrm{Mod}^{r+r'}_{/\mathfrak{S}_\infty}$ by Proposition 2.9. We equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}')$ (resp. $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}')$) with a $\hat{G}$-action (resp. a G-action) via the canonical isomorphism $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}') \simeq (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) \otimes_{\mathfrak{S}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}'))$. (resp. $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}') \simeq (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})) \otimes_{\mathfrak{S}} (W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'))$). If we denote by ($u$-tor) by the u-torsion part of $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'$, then we obtain the exact sequence

$$\widehat{\mathcal{R}} \otimes (u\text{-tor}) \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}') \xrightarrow{\eta} \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}) \to 0$$

as $\widehat{\mathcal{R}}$-modules. Note that u is a unit of $W(\mathrm{Fr}R)$. Since the natural map $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}} \to W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}} = W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}')$ is injective (cf. Corollary 2.12), we see that the equality $\ker(\eta) = \ker(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}') \to W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'))$ holds and thus $\ker(\eta)$ is stable under the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}')$. Therefore, we can define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}$ via the canonical isomorphism $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}} \simeq (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'))/\ker(\eta)$. Then it is not difficult to see that $\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}}$ has a structure of a $(\varphi, \hat{G})$-module. Finally we prove $\hat{T}(\hat{\mathfrak{M}} \otimes \hat{\mathfrak{M}}') \simeq \hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}}')$. By Proposition 3.11, we obtain φ-equivariant and G-compatible isomorphisms

$$\begin{aligned} W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} (\hat{T}_*(\hat{\mathfrak{M}}) \otimes \hat{T}_*(\hat{\mathfrak{M}}')) &\simeq W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\hat{\mathfrak{M}} \otimes_{\widehat{\mathcal{R}}} \hat{\mathfrak{M}}') \\ &\simeq W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}')) \\ &\simeq W(\mathrm{Fr}R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\frac{\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'}{u\text{-tor}})). \end{aligned}$$

Seeing "$\varphi = 1$"-part of the above modules, we have that $\hat{T}_*(\hat{\mathfrak{M}}) \otimes \hat{T}_*(\hat{\mathfrak{M}}') \simeq \hat{T}_*(\hat{\mathfrak{M}} \otimes \hat{\mathfrak{M}})$. Taking the dual of both sides, we obtain the desired result. $\square$

5 Maximal objects and minimal objects

Caruso and Liu defined maximal objects for Kisin modules and Breuil modules in [CL1] and they proved that the category of maximal objects can be regarded as a full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$. In this section, we discuss maximal objects for $(\varphi, \hat{G})$-modules and prove that the category of them can be regarded as a full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G)$.

5.1 Maximal objects and minimal objects for Kisin modules

In this subsection, we recall the theory of maximal (minimal) objects given in [CL1]. For $M \in \mathbf{\Phi M}_{/\mathcal{O}_\infty}$, we denote by $F^r_{\mathfrak{S}}(M)$ the (partially) ordered set (by inclusion) of $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ contained in M such that $\mathfrak{M}[1/u] = M$. Then $F^r_{\mathfrak{S}}(M)$ has a greatest element and a smallest element (cf. [CL1, Corollary 3.2.6]).

Definition 5.1. Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. We denote by $\mathrm{Max}^r(\mathfrak{M})$ the greatest element of $F^r_{\mathfrak{S}}(\mathfrak{M}[1/u])$. It is endowed with a morphism $\iota^{\mathfrak{M}}_{\max} \colon \mathfrak{M} \hookrightarrow \mathrm{Max}^r(\mathfrak{M})$ of Kisin modules. We say that $\mathfrak{M}$ is *maximal* if $\iota^{\mathfrak{M}}_{\max}$ is an isomorphism.

Maximal objects are characterized by the following universality ([CL1, Proposition 3.3.5]): Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. The couple $(\mathrm{Max}^r(\mathfrak{M}), \iota^{\mathfrak{M}}_{\max})$ is characterized by the following universal property:

- The morphism $T_{\mathfrak{S}}(\iota^{\mathfrak{M}}_{\max})$ is an isomorphism.
- For each couple $(\mathfrak{M}', f)$ where $\mathfrak{M}' \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $f \colon \mathfrak{M} \to \mathfrak{M}'$ becomes an isomorphism under $T_{\mathfrak{S}}$, there exists a unique map $g \colon \mathfrak{M}' \to \mathrm{Max}^r(\mathfrak{M})$ such that $g \circ f = \iota^{\mathfrak{M}}_{\max}$.

This property gives rise to a functor $\mathrm{Max}^r \colon \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. If we denote by $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ its essential image, Caruso and Liu proved

Theorem 5.2 ([CL1, Theorem 3.3.8])**.** *The category $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ is abelian and, if $r < \infty$, it is Artinian. Moreover, kernels, cokernels, images and coimages in the abelian category $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ have explicit descriptions.*

The restriction $T_{\mathfrak{S}}$ on $\mathrm{Max}^r_{/\mathfrak{S}_\infty}$ is exact and fully faithful (cf. [CL1, Corollary 3.3.10]):

$$\begin{tikzcd} \mathrm{Mod}^r_{/\mathfrak{S}_\infty} \arrow[rr, "T_{\mathfrak{S}}"] \arrow[dr, "\mathrm{Max}^r"'] & & \mathrm{Rep}_{\mathrm{tor}}(G_\infty) \\ & \mathrm{Max}^r_{/\mathfrak{S}_\infty}. \arrow[ur, hook, "T_{\mathfrak{S}}"'] & \end{tikzcd}$$

In the case $r < \infty$, we obtain the theory for minimal objects if we apply the "dual" to the above theory. By Proposition 5.6 of [CL2], if $r = \infty$, the functor $T_{\mathfrak{S}}$ gives an anti-equivalence of abelian categories:

$$T_{\mathfrak{S}} \colon \mathrm{Max}^\infty_{/\mathfrak{S}_\infty} \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{tor}}(G_\infty).$$

For more precise properties, see Section 3 of [CL1].

5.2 Étale $(\varphi, \hat{G})$-modules

In this subsection, we introduce a notion of étale $(\varphi, \hat{G})$-modules. The idea in this subsection follows from the (φ, τ)-theory given in [Ca4]. As one of main theorems in [Ca4], we prove that the category of various étale $(\varphi, \hat{G})$-modules are equivalent to the category of various $\mathbb{Z}_p$-representations of G, including the case where $p = 2$.

Here[2], we put $\mathcal{O}_{\hat{G}} = W(\mathrm{Fr}R)^{H_\infty}$, which is absolutely unramified and a complete discrete valuation ring with perfect residue field $\mathrm{Fr}R^{H_\infty}$. Furthermore $\mathcal{O}_{\hat{G}}$ is a closed subring of $W(\mathrm{Fr}R)$ for the weak topology. Put $\mathcal{E}_{\hat{G}} = \mathrm{Fr}\mathcal{O}_{\hat{G}} = \mathcal{O}_{\hat{G}}[1/p]$. By definition, $\varphi_{W(\mathrm{Fr}R)[1/p]}$ is stable on $\mathcal{O}_{\hat{G}}$ and $\mathcal{E}_{\hat{G}}$ which is bijective on themselves. Furthermore, $\hat{G}$ acts on $\mathcal{O}_{\hat{G}}$ and $\mathcal{E}_{\hat{G}}$ continuously. Since the inclusion $\mathcal{O} \to \mathcal{O}_{\hat{G}}$ (resp. $\mathcal{E} \to \mathcal{E}_{\hat{G}}$) is faithfully flat, for any étale φ-module M over $\mathcal{O}$ (resp. over $\mathcal{E}$), the natural map $M \to \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ (resp. $M \to \mathcal{E}_{\hat{G}} \otimes_{\mathcal{E}} M$) is an injection. By this embedding, we regard M as a sub $\mathcal{O}$-module of $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ (resp. a sub $\mathcal{E}$-module of $\mathcal{E}_{\hat{G}} \otimes_{\mathcal{E}} M$). Similarly, the natural map $M \to \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ (resp. $M \to \mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} M$) is an injection and by this embedding we regard M as a sub $\varphi(\mathcal{O}_{\hat{G}})$-module of $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ (resp. a sub $\varphi(\mathcal{E}_{\hat{G}})$-module of $\mathcal{E}_{\hat{G}} \otimes_{\varphi,\mathcal{E}} M$).

Definition 5.3. An *étale* $(\varphi, \hat{G})'$*-module over* $\mathcal{O}$ (resp. an *étale* $(\varphi, \hat{G})$*-module over* $\mathcal{O}$) is a triple $'\hat{M} = (M, \varphi_M, \hat{G})$ (resp. $\hat{M} = (M, \varphi_M, \hat{G})$) where

(1) (M, φ_M) is an étale φ-module over $\mathcal{O}$,

(2) $\hat{G}$ is a continuous $\mathcal{O}_{\hat{G}}$-semi-linear $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ (resp. $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$) for the weak topology,

(3) the $\hat{G}$-action commutes with $\varphi_{\mathcal{O}_{\hat{G}}} \otimes \varphi_M$,

(4) $M \subset (\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M)^{H_K}$ (resp. $M \subset (\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M)^{H_K}$).

If M is killed by some power of p, then $'\hat{M}$ (resp. $\hat{M}$) is called a *torsion étale* $(\varphi, \hat{G})'$*-module* (resp. a *torsion étale* $(\varphi, \hat{G})$*-module*). If M is a free $\mathcal{O}$-module, then $'\hat{M}$ (resp. $\hat{M}$) is called a *free étale* $(\varphi, \hat{G})'$*-module* (resp. a *free étale* $(\varphi, \hat{G})$*-module*).

By replacing $\mathcal{E}$ and $\mathcal{E}_{\hat{G}}$ with $\mathcal{O}$ and $\mathcal{O}_{\hat{G}}$, respectively, we define an *étale* $(\varphi, \hat{G})'$*-module over* $\mathcal{E}$ (resp. an *étale* $(\varphi, \hat{G})$*-module over* $\mathcal{E}$).

Denote by $'\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ (resp. $'\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$, resp. $'\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$) the category of torsion étale $(\varphi, \hat{G})'$-modules over $\mathcal{O}$ (resp. the category of free étale $(\varphi, \hat{G})'$-modules over $\mathcal{O}$, resp. the category of étale $(\varphi, \hat{G})'$-modules over $\mathcal{E}$). Similarly, we denote by $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ (resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$, resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$) the category of torsion étale $(\varphi, \hat{G})$-modules over $\mathcal{O}$ (resp. the category of free étale $(\varphi, \hat{G})$-modules over $\mathcal{O}$, resp. the category of étale $(\varphi, \hat{G})$-modules over $\mathcal{E}$).

If $'\hat{M}$ is an étale $(\varphi, \hat{G})'$-module over $\mathcal{O}$, then $\hat{G}$ acts on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}_{\hat{G}}} (\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M)$ by a natural way. Hence we obtain a $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ via the isomorphism

$$\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}_{\hat{G}}} (\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M) \simeq \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M, \quad a \otimes (b \otimes x) \mapsto a\varphi(b) \otimes x$$

where $a, b \in \mathcal{O}_{\hat{G}},\ x \in M$. This $\hat{G}$-action equips M with a structure of an étale $(\varphi, \hat{G})$-module over $\mathcal{O}$. Conversely, if $\hat{M}$ is an étale $(\varphi, \hat{G})$-module over $\mathcal{O}$, we obtain a $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ via the isomorphism

$$\mathcal{O}_{\hat{G}} \otimes_{\varphi^{-1},\mathcal{O}_{\hat{G}}} (\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M) \simeq \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, \quad a \otimes (b \otimes x) \mapsto a\varphi^{-1}(b) \otimes x$$

where $a, b \in \mathcal{O}_{\hat{G}},\ x \in M$. This $\hat{G}$-action equips M with a structure of an étale $(\varphi, \hat{G})'$-module over $\mathcal{O}$. Consequently, we have canonical equivalences of categories

$$'\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty} \simeq \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}, \quad '\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}} \simeq \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}. \tag{5.2.1}$$

By the same way, we obtain

$$'\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}} \simeq \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}. \tag{5.2.2}$$

In the following proposition, $\mathcal{M}$ and $\mathcal{T}$ are functors defined in Section 2.2.

[2] In [Ca4], rings $\mathcal{O}_{\hat{G}}$ and $\mathcal{E}_{\hat{G}}$ are denoted by $\mathcal{E}_\tau^{\mathrm{int}}$ and $\mathcal{E}_\tau$, respectively.

Lemma 5.4. (1) *For any finite torsion $\mathbb{Z}_p$-representation T of G_∞ (resp. finite free $\mathbb{Z}_p$-representation T of G_∞, resp. finite $\mathbb{Q}_p$-representation T of G_∞), the natural map*

$$\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R)_\infty)$$
$$(\text{resp. } \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R)),$$
$$\text{resp. } \mathcal{E}_{\hat{G}} \otimes_{\mathcal{E}} \mathcal{M}(T) \to \mathrm{Hom}_{\mathbb{Q}_p[H_\infty]}(T, W(\mathrm{Fr}R)[1/p]))$$

is an isomorphism.
(2) *For any torsion étale φ-module M over $\mathcal{O}$ (resp. free étale φ-module M over $\mathcal{O}$, resp. étale φ-module M over $\mathcal{E}$), the natural map*

$$\mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty)$$
$$(\text{resp. } \mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)),$$
$$\text{resp. } \mathcal{T}(M) \to \mathrm{Hom}_{\mathcal{E}_{\hat{G}},\varphi}(\mathcal{E}_{\hat{G}} \otimes_{\mathcal{E}} M, W(\mathrm{Fr}R)[1/p]))$$

is an isomorphism.

Proof. We only prove the torsion case. The rest cases can be checked by a similar manner. First we consider (1). Applying the tensor product $W(\mathrm{Fr}R)$ over $\widehat{\mathcal{O}^{\mathrm{ur}}}$ to (2.2.3) and picking up H_∞-fixed parts, we obtain a bijection

$$\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} (\mathcal{O}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} T)^{G_\infty} \to (W(\mathrm{Fr}R) \otimes_{\mathbb{Z}_p} T)^{H_\infty}. \tag{5.2.3}$$

If we replace T in (5.2.3) with its dual representation, we obtain the desired result. Using (2.2.1), we can check (2) by a similar way.

□

We define a contravariant functor $'\hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathrm{tor}}(G) \to {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}_\infty}$ as below: for any $T \in \mathrm{Rep}_{\mathrm{tor}}(G)$, define

$$'\hat{\mathcal{M}}(T) = \mathcal{M}(T) = \mathrm{Hom}_{G_\infty}(T, \mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}})$$

as a φ-module over $\mathcal{O}$, and we equip $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(T)$ with a $\hat{G}$-action via the isomorphism $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(T) \simeq \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R)_\infty)$ (cf. Lemma 5.4 (1)). Here $\hat{G}$ acts on the right hand side by the formula $(\sigma.f)(x) = \hat{\sigma}(f(\hat{\sigma}^{-1}(x)))$ for $\sigma \in \hat{G}$ and $\hat{\sigma} \in G$ any lift of σ, $f \in \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(T, W(\mathrm{Fr}R)_\infty), x \in T$.

On the other hand, we define a contravariant functor $'\hat{\mathcal{T}} \colon {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ as below: for any $'\hat{M} \in {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}_\infty}$, define

$$'\hat{\mathcal{T}}('\hat{M}) = \mathcal{T}(M) = \mathrm{Hom}_{\mathcal{O},\varphi}(M, \mathcal{E}^{\mathrm{ur}}/\mathcal{O}^{\mathrm{ur}})$$

as a $\mathbb{Z}_p$-module, and we equip $'\hat{\mathcal{T}}('\hat{M})$ with a G-action via the isomorphism $\mathcal{T}(M) \simeq \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty)$ (cf. Lemma 5.4 (2)). Here G acts on the right hand side by the formula $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G, f \in \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty), x \in \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$.

We also define a contravariant functor $'\hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathrm{fr}}(G) \to {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}}$ (resp. $'\hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G) \to {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{E}}$) and $'\hat{\mathcal{T}} \colon {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}_\infty} \to \mathrm{Rep}_{\mathrm{fr}}(G)$ (resp. $'\hat{\mathcal{T}} \colon {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{E}} \to \mathrm{Rep}_{\mathbb{Q}_p}(G)$) by a similar manner.

Combining $'\hat{\mathcal{T}}, '\hat{\mathcal{M}}$ with (5.2.1) or (5.2.2), we obtain contravariant functors

$$\hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathrm{tor}}(G) \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty},\ \hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathrm{fr}}(G) \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}},\ \hat{\mathcal{M}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G) \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$$

and

$$\hat{\mathcal{T}} \colon \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G),\ \hat{\mathcal{T}} \colon \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}} \to \mathrm{Rep}_{\mathrm{fr}}(G),\ \hat{\mathcal{T}} \colon \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}} \to \mathrm{Rep}_{\mathbb{Q}_p}(G).$$

Proposition 5.5. *The contravariant functor $\hat{\mathcal{T}}$ is an anti-equivalence of categories between $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ (resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}}$, resp. $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{E}}$) and $\mathrm{Rep}_{\mathrm{tor}}(G)$ (resp. $\mathrm{Rep}_{\mathrm{fr}}(G)$, resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G)$)). Furthermore, $\hat{\mathcal{M}}$ is a quasi-inverse of $\hat{\mathcal{T}}$.*

Proof. By Proposition 2.1, we have already known that, for an étale $(\varphi,\hat{G})$-module $\hat{M}$ and a representation T of G, natural morphisms $M \to \mathcal{M}(\mathcal{T}(M))$ and $T \to \mathcal{T}(\mathcal{M}(T))$ are isomorphisms as étale φ-modules and G_∞-representations, respectively. It is enough to prove that the former is compatible with $\hat{G}$-action and the latter is G-equivariant. In the following, we only prove the torsion case; the same proofs proceed for rest cases. It is enough to prove that functors $'\hat{\mathcal{T}}$ and $'\hat{\mathcal{M}}$ are inverses of each other. Take any $'\hat{M} \in {'\mathbf{\Phi M}}^{\hat{G}}_{/\mathcal{O}_\infty}$. We show the canonical isomorphism

$$\eta\colon \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M \to \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(\mathcal{T}(M))$$

is $\hat{G}$-equivariant. By definitions of functors $'\hat{\mathcal{T}}$ and $'\hat{\mathcal{M}}$, the following composition map

$$\begin{aligned}\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} \mathcal{M}(\mathcal{T}(M)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(\mathcal{T}(M), W(\mathrm{Fr}R)_\infty)\\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(\mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty), W(\mathrm{Fr}R)_\infty)\end{aligned}$$

is $\hat{G}$-equivariant. By composing this map with η, we obtain a bijection

$$\tilde{\eta}\colon \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p[H_\infty]}(\mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty), W(\mathrm{Fr}R)_\infty)$$

which is given by $x \mapsto (f \mapsto f(x))$ for $x \in \mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M,\ f \in \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M, W(\mathrm{Fr}R)_\infty)$. It is a straightforward calculation to check that $\tilde{\eta}$ is compatible with $\hat{G}$-actions, and thus so is η. Consequently, we obtain $'\hat{\mathcal{M}} \circ {'\hat{\mathcal{T}}} \simeq \mathrm{Id}$. By a similar argument we can check $'\hat{\mathcal{T}} \circ {'\hat{\mathcal{M}}} \simeq \mathrm{Id}$ and this finishes a proof.

□

Remark 5.6. By definitions of $\hat{\mathcal{T}}$ and $\hat{\mathcal{M}}$ and the theory of Fontaine's étale φ-modules, we see that these functors preserves various structures of categories. For example, these functors are exact and commute with formations of tensor products and the dual. Here the notion of the tensor product of étale $(\varphi,\hat{G})$-modules and that of dual étale $(\varphi,\hat{G})$-modules are defined by natural manners.

5.3 Link between Liu's $(\varphi,\hat{G})$-modules and étale $(\varphi,\hat{G})$-modules

In this subsection, we connect the theory of Liu's $(\varphi,\hat{G})$-modules and the theory of our étale $(\varphi,\hat{G})$-modules.

Let $\hat{\mathfrak{M}} = (\mathfrak{M},\varphi,\hat{G})$ be a $(\varphi,\hat{G})$-module, or a weak $(\varphi,\hat{G})$-module, in the sense of Definition 2.6. Extending the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ to $\mathcal{O}_{\hat{G}} \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ by a natural way, we see that $\mathfrak{M}[1/u] = \mathcal{O} \otimes_{\mathfrak{S}} \mathfrak{M}$ has a structure of an étale $(\varphi,\hat{G})$-module over $\mathcal{O}$ (recall that G acts on $W(\mathrm{Fr}R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ continuously for the weak topology by Definition 2.6). This is the reason why a $\hat{G}$-action in the definition of an étale $(\varphi,\hat{G})$-module is defined not on $\mathcal{O}_{\hat{G}} \otimes_{\mathcal{O}} M$ but on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$. In the below, we denote by $\widehat{\mathfrak{M}[1/u]}$ the étale $(\varphi,\hat{G})$-module over $\mathcal{O}$ obtained as above. Note that there exists a natural isomorphism of $\mathbb{Z}_p$-representations of G:

$$\hat{T}(\hat{\mathfrak{M}}) \simeq \hat{\mathcal{T}}(\widehat{\mathfrak{M}[1/u]}).$$

In fact, we have isomorphisms

$$\begin{aligned}\hat{\mathcal{T}}(\widehat{\mathfrak{M}[1/u]}) &\simeq \mathrm{Hom}_{\mathcal{O}_{\hat{G}},\varphi}(\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} (\mathfrak{M}[1/u]), W(\mathrm{Fr}R)_\infty)\\ &\simeq \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(\mathrm{Fr}R)_\infty)\\ &\simeq \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)_\infty) = \hat{T}(\hat{\mathfrak{M}})\end{aligned}$$

by Lemma 5.4 (1) and [Fo, Proposition B. 1.8.3] (see also the proof of Corollary 3.12).

In the below, we want to use various morphisms between Liu's $(\varphi,\hat{G})$-modules and étale $(\varphi,\hat{G})$-modules. To do this, we need to define some notions. Let $\mathrm{Mod}(\varphi,\hat{G})$ be the category whose objects are φ-modules $\mathfrak{M}$ over $\mathfrak{S}$ killed by a power of p equipped with an $\mathcal{O}_{\hat{G}}$-semilinear $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Morphisms in $\mathrm{Mod}(\varphi,\hat{G})$ are defined by a natural manner. Then categories ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ and $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ can be regarded as full subcategories of $\mathrm{Mod}(\varphi,\hat{G})$. We call a morphism $f\colon \mathfrak{M} \to M$ in the category $\mathrm{Mod}(\varphi,\hat{G})$ a *morphism of* $(\varphi,\hat{G})$*-modules*, and we often denote f by $f\colon \hat{\mathfrak{M}} \to \hat{M}$.

Definition 5.7. Let $\hat{\mathfrak{M}}$ be an object of ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ or $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, and $\hat{M}$ an object of $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ equipped with a morphism $f\colon \hat{\mathfrak{M}} \to \hat{M}$ of $(\varphi,\hat{G})$-modules. If f is an injection as a morphism of $\mathfrak{S}$-modules, then $\hat{\mathfrak{M}}$ can be regarded as a subobject of $\hat{M}$ in the category $\mathrm{Mod}(\varphi,\hat{G})$. In this case, (the image of) $\hat{\mathfrak{M}}$ is said to be a sub $(\varphi,\hat{G})$-module of $\hat{M}$.

Proposition 5.8 (Analogue of scheme theoretic closure)**.** *Let* $\hat{\mathfrak{M}}$ *be in* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$) *and* $\hat{M}$ *an object of* $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$. *Let* $f\colon \hat{\mathfrak{M}} \to \hat{M}$ *be a morphism of* $(\varphi,\hat{G})$*-modules. Then,* $\ker(f)$ *and* $\mathrm{im}(f)$ *as* φ*-modules are contained in* $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$. *Furthermore, the* $\hat{G}$*-action on* $\hat{\mathfrak{M}}$ *gives* $\ker(f)$ *a structure of a weak* $(\varphi,\hat{G})$*-module* (*resp. a* $(\varphi,\hat{G})$*-module*) *and the* $\hat{G}$*-action on* $\hat{M}$ *gives* $\mathrm{im}(f)$ *a structure of a weak* $(\varphi,\hat{G})$*-module* (*resp. a* $(\varphi,\hat{G})$*-module*)*.*

In this paper, we often denote $\widehat{\mathrm{im}(f)}$ *by* $f(\hat{\mathfrak{M}})$ *or* $\widehat{f(\mathfrak{M})}$*.*

Proof. The same proof as that of Corollary 2.19 proceeds. □

The above proposition gives us a result on a successive extension for $(\varphi,\hat{G})$-modules, which is an analogue of Proposition 2.9 (4).

Corollary 5.9. *Let* $\hat{\mathfrak{M}}$ *be an object of* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$)*. Then there exists a sequence of extensions*

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_k = \mathfrak{M}$$

in $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ *which satisfies the following; for any* i,
(i) $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *is a finite free* $k[\![u]\!]$*-module,*
(ii) $\mathfrak{M}_i$ *and* $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *have structures of weak* $(\varphi,\hat{G})$*-modules of height* r (*resp.* $(\varphi,\hat{G})$*-modules of height* r) *which make an exact sequence*

$$0 \to \hat{\mathfrak{M}}_{i-1} \to \hat{\mathfrak{M}}_i \to \widehat{\mathfrak{M}_i/\mathfrak{M}_{i-1}} \to 0$$

in ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$)*.*

Proof. Putting $M = \mathfrak{M}[1/u]$, we have seen that $\hat{M} = \widehat{\mathfrak{M}[1/u]}$ is an étale $(\varphi,\hat{G})$-module. We see that pM and M/pM have structures of étale $(\varphi,\hat{G})$-modules, and we have an exact sequence $0 \to \widehat{pM} \to \hat{M} \overset{\mathrm{pr}}{\to} \widehat{M/pM} \to 0$ of étale $(\varphi,\hat{G})$-modules. We also denote by pr a composition $\hat{\mathfrak{M}} \to \hat{M} \overset{\mathrm{pr}}{\to} \widehat{M/pM}$ which is a morphism of $(\varphi,\hat{G})$-modules. By Proposition 5.8, we know that $\mathfrak{M}' = \ker(\mathrm{pr}|_{\mathfrak{M}})$ and $\mathfrak{M}'' = \mathrm{pr}(\mathfrak{M})$ have structures of weak $(\varphi,\hat{G})$-modules of height r (resp. $(\varphi,\hat{G})$-modules of height r). Furthermore, we have an exact sequence $0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$ in ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (resp. $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$). Since $p^{n-1}\mathfrak{M}' = 0$ and $p\mathfrak{M}'' = 0$, we can obtain the desired sequence of extensions inductively. □

Before starting the maximal (minimal) theory, we give one result on the "cokernel" of a morphism of $(\varphi,\hat{G})$-modules, which will be used in the proof of Theorem 1.2.

Proposition 5.10. *Let* $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{N}}$ *be a morphism in* ${}_{\mathrm{w}}\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$)*. Denote by* $\mathrm{coker}(f)$ *the cokernel of* f *as a morphism of* φ*-modules. Then* $\frac{\mathrm{coker}(f)}{u\text{-tor}}$ *is an object of* $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$*. Furthermore,* $\frac{\mathrm{coker}(f)}{u\text{-tor}}$ *has a structure of a weak* $(\varphi,\hat{G})$*-module* (*resp. a* $(\varphi,\hat{G})$*-module*) *induced from* $\hat{\mathfrak{N}}$*.*

Proof. It is enough to check the case where f is a morphism in $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. Put $C = \mathrm{coker}(f)$ and denote by $C_{u\text{-tor}}$ the u-torsion part of C. By Proposition 2.9, we see that $\frac{C}{C_{u\text{-tor}}}$ is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Since C is finitely generated as an $\mathfrak{S}$-module, there exists an integer $n > 0$ such that $u^n C_{u\text{-tor}} = 0$. Then $C' = u^n C$ is u-torsion free and thus C' is a torsion Kisin module of finite height. By Corollary 2.13, we see that the natural map $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$ is injective. Since the composition map $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C \overset{1\otimes u^n}{\to} \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \hookrightarrow \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$ is the multiplication-by-u^{np} map, if we regard $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C'$ as a submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$, we obtain $u^{np}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C) \subset \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C'$. Since $C' \in \mathrm{Mod}^{\infty}_{/\mathfrak{S}_\infty}$, we know that $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \subset \mathcal{O} \otimes_{\varphi,\mathfrak{S}} C'$ and thus $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C'$ is u-torsion free. Therefore, denoting by $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C)_{u\text{-tor}}$ the u-torsion part of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$, we obtain

$$u^{np}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C)_{u\text{-tor}} = 0. \tag{5.3.1}$$

The exact sequence $0 \to C_{u\text{-tor}} \to C \overset{u^n}{\to} C' \to 0$ of $\mathfrak{S}$-modules induces an exact sequence

$$0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C_{u\text{-tor}} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C \overset{u^{np}}{\to} \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \to 0 \tag{5.3.2}$$

since $\mathrm{Tor}_1^{\mathfrak{S}}(C', \widehat{\mathcal{R}}) = 0$ (see Corollary 2.11). By (5.3.1) and (5.3.2), we obtain the equality $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C_{u\text{-tor}} = (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C)_{u\text{-tor}}$ in $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$. On the other hand, we remark that the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ induces that on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C$. Since this $\hat{G}$-action preserves $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C)_{u\text{-tor}}$, we can equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \frac{C}{C_{u\text{-tor}}}$ with a $\hat{G}$-action by using the exact sequence $0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C_{u\text{-tor}} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \frac{C}{C_{u\text{-tor}}} \to 0$. Then it is not difficult to check that $\frac{\mathrm{coker}(f)}{u\text{-tor}} = \frac{C}{C_{u\text{-tor}}}$ is a $(\varphi, \hat{G})$-module. $\square$

Remark 5.11. Let $0 \to T' \to T \to T'' \to 0$ and $\hat{\mathfrak{M}}$ be as in Theorem 4.5. Admitting notions of étale $(\varphi, \hat{G})$-modules, the proof of Theorem 4.5 implies that the sequence $(*) : 0 \to \hat{\mathfrak{M}}'' \to \hat{\mathfrak{M}} \overset{g}{\to} \hat{\mathfrak{M}}' \to 0$ appeared in the theorem is obtained by the following mannar: let $0 \to \hat{M}'' \to \hat{M} \to \hat{M}' \to 0$ be a sequence of étale $(\varphi, \hat{G})$-modules corresponding to $(*)$. Then $\hat{\mathfrak{M}}$ is a sub $(\varphi, \hat{G})$-module of $\hat{M}$, and $\mathfrak{M}' = g(\mathfrak{M})$ (resp. $\mathfrak{M}'' = \mathfrak{M} \cap M$) has a structure of a sub $(\varphi, \hat{G})$-module of $\hat{M}'$ (resp. $\hat{M}''$).

5.4 Definitions of maximality and minimality

In this subsection, we construct maximal objects (resp. minimal objects) for $(\varphi, \hat{G})$-modules by using the theory of étale $(\varphi, \hat{G})$-modules given in the previous section. Let $\hat{M} = (M, \varphi, \hat{G}) \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ be a torsion étale $(\varphi, \hat{G})$-module over $\mathcal{O}$. We denote by $F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$ the (partially) ordered set (by inclusion) of $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ which is a sub $(\varphi, \hat{G})$-modules of an étale $(\varphi, \hat{G})$-module M such that $\mathfrak{M}[1/u] = M$. Note that $\hat{\mathfrak{M}}$ is a sub $(\varphi, \hat{G})$-modules of M if and only if the natural inclusion[3] $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \hookrightarrow \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ is $\hat{G}$-equivariant.

Lemma 5.12. *Let $\hat{M}$ be a torsion étale $(\varphi, \hat{G})$-module. Let $\hat{\mathfrak{M}}_1$ and $\hat{\mathfrak{M}}_2$ be objects of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ endowed with injections $\hat{\mathfrak{M}}_1 \to \hat{M}$ and $\hat{\mathfrak{M}}_2 \to \hat{M}$ of $(\varphi, \hat{G})$-modules. Then $\mathfrak{M}_{12} = \mathfrak{M}_1 + \mathfrak{M}_2$ (resp. $\mathfrak{M}'_{12} = \mathfrak{M}_1 \cap \mathfrak{M}_2$) in M has a structure of a $(\varphi, \hat{G})$-module of height r. In particular, the ordered set $F^{r,\hat{G}}_{\mathfrak{S}}(M)$ has finite supremum and finite infimum.*

Proof. First we note that $\mathfrak{M}_{12}$ (resp. $\mathfrak{M}'_{12}$) is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and we have $\mathfrak{M}_{12}[1/u] = M$ (resp. $\mathfrak{M}'_{12}[1/u] = M$) (see the proof of [CL1, Proposition 3.2.3]). Furthermore, $\mathfrak{M}'_{12}$ is canonically isomorphic to the underlying Kisin module of the kernel of the morphism of $(\varphi, \hat{G})$-modules

$$\hat{\mathfrak{M}}_1 \oplus \hat{\mathfrak{M}}_2 \to \hat{\mathfrak{M}}_1 + \hat{\mathfrak{M}}_2 \subset \hat{M}, \quad (x, y) \mapsto x - y.$$

[3] The natural map $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ is injective by Corollary 2.12.

Thus we obtain the desired result for $\mathfrak{M}'_{12}$ by Proposition 5.8. Hence it is enough to prove the result for $\mathfrak{M}_{12}$. Since the $\hat{G}$-actions on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1$ and $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2$ are restrictions of the $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$, we see that the $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathcal{O}} M$ preserves $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{12} = \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1 + \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2$. For any $\sigma \in \hat{G}$ and $x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{12}$, take $x_1 \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1$ and $x_2 \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2$ such that $x = x_1 + x_2$. Then we have $\sigma(x) - x = (\sigma(x_1) - x_1) + (\sigma(x_2) - x_2) \in I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1) + I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2) = I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{12})$ and thus $\hat{G}$ acts on $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{12})/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_{12})$ trivial. Hence $\hat{\mathfrak{M}}_{12} = (\mathfrak{M}_{12}, \varphi, \hat{G})$ is a $(\varphi, \hat{G})$-module and we obtain the desired result. $\square$

Proposition 5.13. *$F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$ has a maximum element. If $r < \infty$, then it also has a minimum element.*

Proof. Assume that $F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$ does not have a maximum element. Take any $\hat{\mathfrak{M}} = \hat{\mathfrak{M}}_0 \in F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$. Since $\hat{\mathfrak{M}}_0$ is not maximum, there exists an element $\hat{\mathfrak{M}}'_1 \in F^{r,\hat{G}}_{\mathfrak{S}}(M)$ such that $\mathfrak{M}_0 \not\subset \mathfrak{M}'_1$. Put $\mathfrak{M}_1 = \mathfrak{M}_0 + \mathfrak{M}'_1$ (the sum is taken in M). By Lemma 5.12, $\mathfrak{M}_1$ has a structure of $(\varphi, \hat{G})$-module. We denote this $(\varphi, \hat{G})$-module by $\hat{\mathfrak{M}}_1$. We see that $\hat{\mathfrak{M}}_1 \in F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$ and $\mathfrak{M}_0 \subsetneq \mathfrak{M}_1$. Inductively, we find $\hat{\mathfrak{M}}_i \in F^{r,\hat{G}}_{\mathfrak{S}}(M)$ with an infinite length increasing sequence

$$\mathfrak{M}_0 \subsetneq \mathfrak{M}_1 \subsetneq \mathfrak{M}_2 \subsetneq \cdots$$

in $F^r_{\mathfrak{S}}(M)$. However, this is a contradiction by [CL1, Lemma 3.2.4]. The proof of the assertion for a minimum element is the same except only for that we use [CL1, Lemma 3.2.5]. $\square$

Remark 5.14. If $F^{\infty,\hat{G}}_{\mathfrak{S}}(\hat{M})$ is not empty, then $F^{\infty,\hat{G}}_{\mathfrak{S}}(\hat{M})$ does not have a minimum element. In fact, if $\hat{\mathfrak{M}}$ is an object of $F^{\infty,\hat{G}}_{\mathfrak{S}}(\hat{M})$, then we obtain an infinite length decreasing sequence

$$\hat{\mathfrak{M}} > \widehat{u\mathfrak{M}} > \widehat{u^2\mathfrak{M}} > \cdots$$

in $F^{\infty,\hat{G}}_{\mathfrak{S}}(\hat{M})$.

Definition 5.15. Let $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. We denote by $\mathrm{Max}^r(\hat{\mathfrak{M}})$ (resp. $\mathrm{Min}^r(\hat{\mathfrak{M}})$) the maximum element (resp. minimum element) of $F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$. It is endowed with a morphism of $(\varphi, \hat{G})$-modules $\iota^{\hat{\mathfrak{M}}}_{\max}\colon \hat{\mathfrak{M}} \to \mathrm{Max}^r(\hat{\mathfrak{M}})$ (resp. $\iota^{\hat{\mathfrak{M}}}_{\min}\colon \mathrm{Min}(\hat{\mathfrak{M}}) \to \hat{\mathfrak{M}}$). We often denote by $\max^r(\hat{\mathfrak{M}})$ (resp. $\min^r(\hat{\mathfrak{M}})$) the underlying Kisin module of $\mathrm{Max}^r(\hat{\mathfrak{M}})$ (resp. $\mathrm{Min}^r(\hat{\mathfrak{M}})$). We say that $\hat{\mathfrak{M}}$ is *maximal* (resp. *minimal*) if $\iota^{\hat{\mathfrak{M}}}_{\max}$ (resp. $\iota^{\hat{\mathfrak{M}}}_{\min}$) is an isomorphism.

5.5 Maximal objects for $(\varphi, \hat{G})$-modules

In this section, we prove various properties of maximal objects.

Proposition 5.16. *Definition 5.15 gives rise to a functor* $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Proof. We have to prove that any map $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ induces a map $\mathrm{Max}^r(\hat{\mathfrak{M}}) \to \mathrm{Max}^r(\hat{\mathfrak{M}}')$. The map $g = f[1/u]\colon \widehat{\mathfrak{M}[1/u]} \to \widehat{\mathfrak{M}'[1/u]}$ is a morphism in $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$. By Corollary 5.8, $g(\mathrm{Max}^r(\hat{\mathfrak{M}}))$ is a sub $(\varphi, \hat{G})$-module over $\mathfrak{S}$ of $\widehat{\mathfrak{M}'[1/u]}$. Since $\hat{\mathfrak{M}}'$ is maximal and $g(\mathrm{Max}^r(\hat{\mathfrak{M}})) + \hat{\mathfrak{M}}'$ is an object of $F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}'[1/u]})$, we see the underlying Kisin module of $g(\mathrm{Max}^r(\hat{\mathfrak{M}}))$ is contained in $\mathfrak{M}'$ and we have done. $\square$

Denote by $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ the essential image of the functor $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. It is a full subcategory of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. The following two propositions can be shown by essentially the same method of [CL1] (cf. Proposition 3.3.2, 3.3.3, 3.3.4 and 3.3.5) and we omit proofs.

Proposition 5.17. (1) *The functor* $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is a projection, that is,* $\mathrm{Max}^r \circ \mathrm{Max}^r = \mathrm{Max}^r$.
(2) *The functor* $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is left exact.*
(3) *The functor* $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is a left adjoint to the inclusion functor* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Proposition 5.18. *Let* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. *Then the couple* $(\mathrm{Max}^r(\hat{\mathfrak{M}}), \iota^{\hat{\mathfrak{M}}}_{\max})$ *is characterized by the following universal property:*

- *the morphism* $\hat{T}(\iota^{\hat{\mathfrak{M}}}_{\max})$ *is an isomorphism;*
- *for any* $\hat{\mathfrak{M}}' \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *endowed with a morphism* $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ *such that* $\hat{T}(f)$ *is an isomorphism, there exists a unique map* $g\colon \hat{\mathfrak{M}}' \to \mathrm{Max}^r(\hat{\mathfrak{M}})$ *such that* $g \circ f = \iota^{\hat{\mathfrak{M}}}_{\max}$.

Here we are ready to prove the essential part of Theorem 1.2.

Theorem 5.19. *The category* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is abelian. More precisely, if* $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ *is a morphism in* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, *then*

(1) *if we denote the kernel of* f *as a morphism of* φ*-modules by* $\ker(f)$, *then* $\ker(f)$ *is an object of* $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ *and has a structure of a* $(\varphi, \hat{G})$*-module of height* r. *If we denote it by* $\widehat{\ker(f)}$, *then it is maximal and is the kernel of* f *in the abelian category* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$;

(2) *if we denote the cokernel of* f *as a morphism of* φ*-modules by* $\mathrm{coker}(f)$, *then* $\frac{\mathrm{coker}(f)}{u\text{-tor}}$ *is an object of* $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ *and has a structure of a* $(\varphi, \hat{G})$*-module of height* r. *If we denote it by* $\widehat{\frac{\mathrm{coker}(f)}{u\text{-tor}}}$, *then* $\mathrm{Max}^r(\widehat{\frac{\mathrm{coker}(f)}{u\text{-tor}}})$ *is the cokernel of* f *in the abelian category* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$; *moreover, if* f *is injective as a morphism of* φ*-modules, then* $\mathrm{coker}(f)$ *has no* u*-torsion;*

(3) *if we denote the image (resp. coimage) of* f *as a morphism of* φ*-modules by* $\mathrm{im}(f)$ *(resp.* $\mathrm{coim}(f)$*), then* $\mathrm{im}(f)$ *(resp.* $\mathrm{coim}(f)$*) is an object of* $\mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ *and has a structure of a* $(\varphi, \hat{G})$*-module of height* r. *If we denote it by* $\widehat{\mathrm{im}(f)}$ *(resp.* $\widehat{\mathrm{coim}(f)}$*), then* $\mathrm{Max}^r(\widehat{\mathrm{im}(f)})$ *(resp.* $\mathrm{Max}^r(\widehat{\mathrm{coim}(f)})$*) is the image (resp. coimage) of* f *in the abelian category* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Proof. (1) By Corollary 2.19, we know that $\ker(f)$ has a structure of a $(\varphi, \hat{G})$-module of height r. We have to show that $\widehat{\ker(f)}$ is maximal. Consider the diagram below:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r] \arrow[d, hook] & \mathfrak{M} \arrow[r] \arrow[d, hook] & \mathfrak{M}' \arrow[d, hook] \\
0 \arrow[r] & \max^r(\widehat{\ker(f)}) \arrow[r] \arrow[d, hook] & \max^r(\hat{\mathfrak{M}}) \arrow[r] \arrow[d, hook] & \max^r(\hat{\mathfrak{M}}') \arrow[d, hook] \\
0 \arrow[r] & \ker(f)[1/u] \arrow[r] & \mathfrak{M}[1/u] \arrow[r] & \mathfrak{M}'[1/u].
\end{tikzcd}$$

The top and bottom horizontal sequences are exact as φ-modules over $\mathfrak{S}$. Put $\mathfrak{M}_{\max} = \max^r(\widehat{\ker(f)}) + \mathfrak{M}$ in $\mathfrak{M}[1/u]$ and observe that $\mathfrak{M}_{\max} \in \mathrm{Mod}^{r}_{/\mathfrak{S}_\infty}$ and $\mathfrak{M}_{\max}$ has a structure of a $(\varphi, \hat{G})$-module with injection of $\hat{G}$-modules $\widehat{\mathfrak{M}_{\max}} \hookrightarrow \widehat{\mathfrak{M}[1/u]}$. Since $\mathfrak{M} \subset \mathfrak{M}_{\max} \subset \mathfrak{M}[1/u]$, we have $\mathfrak{M}_{\max}[1/u] = \mathfrak{M}[1/u]$ and thus $\widehat{\mathfrak{M}_{\max}} \in F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$. Since $\hat{\mathfrak{M}}$ is maximal, we obtain $\mathfrak{M}_{\max} \subset \mathfrak{M}$. Therefore, we have $\mathfrak{M}_{\max} \subset \mathfrak{M} \cap \ker(f)[1/u] = \ker(f)$ (where the equality $\mathfrak{M} \cap \ker(f)[1/u] = \ker(f)$ follows from the above diagram) and hence $\widehat{\ker(f)}$ is maximal.

(2) By Proposition 5.10, we know that $\frac{\mathrm{coker}(f)}{u-\mathrm{tor}}$ has a structure of a $(\varphi,\hat{G})$-module of height r induced from that of $\hat{\mathfrak{M}}'$. By Proposition 5.17 (3), it is not difficult to check that $\mathrm{Max}^r(\widehat{\frac{\mathrm{coker}(f)}{u-\mathrm{tor}}})$ is the cokernel of f in the category $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Next we prove the latter assertion; suppose $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ is injective as a morphism of φ-modules. Put $C = \mathrm{coker}(f)$ (as an $\mathfrak{S}$-module). The following diagram of exact sequences of φ-modules are commutative;

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{M} \arrow[r] \arrow[d, hook] & \mathfrak{M}' \arrow[r] \arrow[d, hook] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathfrak{M}[1/u] \arrow[r] & \mathfrak{M}'[1/u] \arrow[r] & C[1/u] \arrow[r] & 0.
\end{tikzcd}$$

Put $\mathfrak{N} = \mathfrak{M}[1/u] \cap \mathfrak{M}'$. We claim that $\mathfrak{M} = \mathfrak{N}$. If we admit this claim, we see that g is injective and thus C is u-torsion free, which is the desired result. Hence it suffices to prove the claim. The inclusion $\mathfrak{M} \subset \mathfrak{N}$ is clear. To prove $\mathfrak{N} \subset \mathfrak{M}$, it is enough to prove that $\mathfrak{N}$ has a structure of a $(\varphi,\hat{G})$-module and $\hat{\mathfrak{N}} \in F^{r,\hat{G}}_{\mathfrak{S}}(\widehat{\mathfrak{M}[1/u]})$. By the proof of [CL1, Proposition 3.3.4], we know that $\mathfrak{N} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Furthermore, we see that $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$ since $\mathfrak{M} \subset \mathfrak{N} \subset \mathfrak{M}[1/u]$. If we denote by C' the cokernel of the inclusion map $\mathfrak{N} \hookrightarrow \mathfrak{M}'$, then we know that $C'[1/u] = C[1/u]$ and $\mathfrak{M}' \hookrightarrow \mathfrak{M}'[1/u]$ induces an injection $C' \hookrightarrow C'[1/u]$, in particular, C' is u-torsion free and $C' \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. By Corollary 2.12 and 2.11, two horizontal sequences of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} \arrow[r] \arrow[d] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \arrow[r] \arrow[d] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{N}[1/u]) \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}'[1/u]) \arrow[r] & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (C'[1/u]) \arrow[r] & 0
\end{tikzcd}$$

are exact as $\widehat{\mathcal{R}}$-modules and all vertical arrows are injective. Since $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$, we obtain

$$\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} = (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{N}[1/u])) \cap (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}')$$

in $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}'[1/u])$. It is not difficult to check that the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ extends to $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}[1/u])$, which coincides with the restriction of the $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}'[1/u])$. Hence the $\hat{G}$-action on $\mathcal{O}_{\hat{G}} \otimes_{\varphi,\mathfrak{S}} (\mathfrak{M}'[1/u])$ preserves $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ and $\mathfrak{N}$ has a structure of a weak sub $(\varphi,\hat{G})$-module of $\hat{\mathfrak{M}}'$. Since $C' \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, the exact sequence $0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} C' \to 0$ gives C' a structure of a weak $(\varphi,\hat{G})$-module. By Corollary 2.20, we know that $\hat{\mathfrak{N}}$ is in fact a $(\varphi,\hat{G})$-module. Therefore, maximality of $\hat{\mathfrak{M}}$ implies that $\mathfrak{N} \subset \mathfrak{M}$. This proves the claim and we finish a proof of the latter assertion of (2).

(3) Let $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ be a morphism in $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. Corollary 2.19 says that $\mathrm{im}(f)$ has a structure of a sub $(\varphi,\hat{G})$-module of $\hat{\mathfrak{M}}'$. The map f induces a map $g\colon \widehat{\mathrm{im}(f)} \to \hat{\mathfrak{M}}'$. It is clear that $\mathrm{coker}(f) = \mathrm{coker}(g)$ as $\mathfrak{S}$-modules. Consider the map $\mathrm{Max}^r(g)\colon \mathrm{Max}(\widehat{\mathrm{im}(f)}) \to \hat{\mathfrak{M}}'$. By (2) and Proposition 5.10, we see that $\mathrm{coker}(\mathrm{Max}^r(g))$ (as an $\mathfrak{S}$-module) is u-torsion free and it has a structure of a $(\varphi,\hat{G})$-module induced from that of $\hat{\mathfrak{M}}'$. Note that there exists an isomorphism $\mathrm{coker}\widehat{(\mathrm{Max}^r}(g)) \simeq \widehat{\frac{\mathrm{coker}(f)}{u-\mathrm{tor}}}$ of $(\varphi,\hat{G})$-modules. We have the exact sequence of $(\varphi,\hat{G})$-modules below:

$$0 \to \mathrm{Max}^r(\widehat{\mathrm{im}(f)}) \to \hat{\mathfrak{M}}' \to \mathrm{coker}\widehat{(\mathrm{Max}^r}(g)) \to 0.$$

Since the functor $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is left exact (cf. Proposition 5.17), we obtain an exact sequence of $(\varphi,\hat{G})$-modules

$$0 \to \mathrm{Max}^r(\widehat{\mathrm{im}(f)}) \to \hat{\mathfrak{M}}' \to \mathrm{Max}^r(\mathrm{coker}\widehat{(\mathrm{Max}^r}(g))).$$

Combining this with the description of kernels and cokernels in the category $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, we obtain that $\mathrm{Max}^r(\widehat{\mathrm{im}(f)})$ is the image of f in the category $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. The assertion for the coimage can be checked by a similar way. $\square$

Lemma 5.20. *If $\alpha\colon \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ and $\beta\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}''$ two morphisms in $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ such that $\beta\circ\alpha = 0$. Then the sequence $0 \to \hat{\mathfrak{M}}' \overset{\alpha}{\to} \hat{\mathfrak{M}} \overset{\beta}{\to} \hat{\mathfrak{M}}'' \to 0$ is exact in (the abelian category) $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ if and only if $0 \to \widehat{\mathfrak{M}'[1/u]} \overset{\alpha[1/u]}{\to} \widehat{\mathfrak{M}[1/u]} \overset{\beta[1/u]}{\to} \widehat{\mathfrak{M}''[1/u]} \to 0$ is exact in $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$. Furthermore, the functor*

$$\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}, \quad \hat{\mathfrak{M}} \mapsto \widehat{\mathfrak{M}[1/u]}$$

is fully faithful.

Proof. Since α and β is assumed to be $\hat{G}$-equivariant, $0 \to \widehat{\mathfrak{M}'[1/u]} \to \widehat{\mathfrak{M}[1/u]} \to \widehat{\mathfrak{M}''[1/u]} \to 0$ is exact in $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ if and only if $0 \to \mathfrak{M}'[1/u] \to \mathfrak{M}[1/u] \to \mathfrak{M}''[1/u] \to 0$ is exact in $\mathbf{\Phi M}_{/\mathcal{O}_\infty}$. Thus the same proof as that of [CL1, Lemma 3.3.9] proceeds. $\square$

Corollary 5.21. *The functor $\hat{T}$ defined on $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is exact and fully faithful, and its essential image is stable under subquotients.*

Proof. The former assertion follows from the commutative triangle below:

$$\begin{tikzcd}
\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \arrow[rr, "\hat{T}"] \arrow[dr] & & \mathrm{Rep}_{\mathrm{tor}}(G_\infty) \\
& \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty} \arrow[ur, "\simeq", "\hat{\mathcal{T}}"'] &
\end{tikzcd}$$

Here, $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$ is a functor defined by the assignment $\hat{\mathfrak{M}} \mapsto \widehat{\mathfrak{M}[1/u]}$, which is exact and fully faithful (by Lemma 5.20). The latter assertion follows from Theorem 4.5. $\square$

Corollary 5.22. *The functor $\mathrm{Max}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is exact.*

Proof. This follows from Lemma 5.20. $\square$

Proposition 5.23. *The category $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is stable under extensions in $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, that is, if*

$$0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$$

is an exact sequence in $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ with $\hat{\mathfrak{M}}', \hat{\mathfrak{M}}'' \in \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, then $\hat{\mathfrak{M}} \in \mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Proof. The proof is essentially the same as that of [CL1, Proposition 3.3.13]. $\square$

Proposition 5.24. *Let $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ and $\varphi^*\colon \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ the $\mathfrak{S}$-linearization of φ. If $\mathrm{coker}(\varphi^*)$ is killed by u^{p-2}, then $\hat{\mathfrak{M}}$ is maximal.*

Proof. By Corollary 5.9 and Proposition 5.23, we can reduce the proof to the case where $p\mathfrak{M} = 0$, and then the proof is essentially the same as that of [CL1, Lemma 3.3.14]. $\square$

Remark 5.25. All results in this subsection hold even if we replace "$(\varphi, \hat{G})$-modules" with "weak $(\varphi, \hat{G})$-modules" (e.g. the existence of maximal objects for weak $(\varphi, \hat{G})$-modules). Proofs are easier than that for "$(\varphi, \hat{G})$-modules" since we may omit "modulo I_+" arguments.

5.6 Minimal objects for $(\varphi, \hat{G})$-modules

Throughout this subsection, we always assume that $r < \infty$. Here we study minimal objects of $(\varphi, \hat{G})$-modules. Many arguments in this subsection are very similar to those of the maximal case and of [CL1].

Proposition 5.26. *Definition 5.15 gives rise to a functor* $\mathrm{Min}^r \colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Proof. We have to show that any morphism $f \colon \hat{\mathfrak{M}} \to \hat{\mathfrak{N}}$ in $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ embeds $\min^r(\hat{\mathfrak{M}})$ into $\min^r(\hat{\mathfrak{N}})$. Put $\hat{M} = \widehat{\mathfrak{M}[1/u]}$ and $\hat{N} = \widehat{\mathfrak{N}[1/u]}$. Denote by $g = f[1/u] \colon \hat{M} \to \hat{N}$ the morphism induced from f. Then g induces $\mathrm{Max}^r(f) \colon \mathrm{Max}^r(\mathfrak{M}) \to \mathrm{Max}^r(\mathfrak{N})$, we also denote it by g. We know that the kernel $\mathfrak{K}$ of the map

$$h \colon \mathrm{Max}^r(\hat{\mathfrak{M}}) \oplus \mathrm{Min}^r(\hat{\mathfrak{N}}) \to \mathrm{Max}^r(\hat{\mathfrak{N}}), \quad (x, y) \mapsto g(x) - y$$

has a structure of a $(\varphi, \hat{G})$-module $\hat{\mathfrak{K}}$ of height r. Note that the composition map $\hat{\mathfrak{K}} \to \mathrm{Max}^r(\hat{\mathfrak{M}}) \oplus \mathrm{Min}^r(\hat{\mathfrak{N}}) \to \mathrm{Max}^r(\hat{\mathfrak{M}})$ is an isomorphism, where the first arrow is the natural embedding and the second arrow is the first projection. In particular, we obtain an isomorphism $\eta \colon \widehat{\mathfrak{K}[1/u]} \xrightarrow{\sim} \hat{M}$. If we identify $\widehat{\mathfrak{K}[1/u]}$ and $\hat{M}$ via η, then $\hat{\mathfrak{K}}$ is contained in $F^{r,\hat{G}}_{\mathfrak{S}}(\hat{M})$ and thus $\min^r(\mathfrak{M}) \subset \mathfrak{K}$. Taking any element $x = (x, y)$ of $\min^r(\mathfrak{M}) \subset \mathfrak{K}$, we have $h(x, y) = 0$ and thus $g(x) = y \in \min^r(\hat{\mathfrak{N}})$. This finishes the proof. $\square$

Denote by $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ the essential image of the functor $\mathrm{Min}^r \colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. The following can be checked by the same way as that of [CL1, Proposition 3.4.6].

Proposition 5.27. *Let* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. *Then the couple* $(\mathrm{Min}^r(\hat{\mathfrak{M}}), \iota^{\hat{\mathfrak{M}}}_{\min})$ *is characterized by the following universal property:*

- *the morphism* $\hat{T}(\iota^{\hat{\mathfrak{M}}}_{\min})$ *is an isomorphism;*
- *for any* $\hat{\mathfrak{M}}' \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *endowed with a morphism* $f \colon \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ *such that* $\hat{T}(f)$ *is an isomorphism, there exists a unique map* $g \colon \mathrm{Min}^r(\hat{\mathfrak{M}}) \to \hat{\mathfrak{M}}'$ *such that* $f \circ g = \iota^{\hat{\mathfrak{M}}}_{\min}$.

Since the couple $(\mathrm{Max}^r(\hat{\mathfrak{M}}^\vee)^\vee, (\iota^{\hat{\mathfrak{M}}^\vee}_{\max})^\vee)$ satisfies the universality appeared in Proposition 5.27, we obtain

Corollary 5.28. *For* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$, *we have natural isomorphisms*

$$\mathrm{Min}^r(\hat{\mathfrak{M}}^\vee) \simeq \mathrm{Max}^r(\hat{\mathfrak{M}})^\vee \quad \text{and} \quad \mathrm{Max}^r(\hat{\mathfrak{M}}^\vee) \simeq \mathrm{Min}^r(\hat{\mathfrak{M}})^\vee.$$

In particular, the duality on $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *permutes subcategories* $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *and* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

The following proposition can be proved by essentially the same method of [CL1] (cf. Proposition 3.4.3, 3.4.8, Lemma 3.4.4 and Corollary 3.4.5) and we omit the proof.

Proposition 5.29. (1) *The functor* $\mathrm{Min}^r \colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is a projection, that is,* $\mathrm{Min}^r \circ \mathrm{Min}^r = \mathrm{Min}^r$.
(2) *Let* $f \colon \hat{\mathfrak{M}} \to \hat{\mathfrak{N}}$ *be a morphism in* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. *Then* $f(\mathrm{Min}^r(\hat{\mathfrak{M}})) = \mathrm{Min}^r(\widehat{f(\mathfrak{M})})$. (*For some notations, see Proposition 5.8.*)
(3) *Let* $f \colon \hat{\mathfrak{M}} \to \hat{\mathfrak{N}}$ *be a morphism in* $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. *If* f *is surjective* (*resp. injective*) *as an* $\mathfrak{S}$*-module morphism, then* $\mathrm{Min}^r(f)$ *is also.*
(4) *The functor* $\mathrm{Min}^r \colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is a right adjoint to the inclusion functor* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$.

Theorem 5.30. *The category* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is abelian. More precisely, if* $f\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ *is a morphism in* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*, then*

(1) *if we denote the kernel of f as a morphism of φ-modules by* $\ker(f)$*, then* $\ker(f)$ *is an object of* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *and has a structure of a $(\varphi, \hat{G})$-module of height r. If we denote it by* $\widehat{\ker(f)}$*, then* $\mathrm{Min}^r(\widehat{\ker(f)})$ *is the kernel of f in the abelian category* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*;*

(2) *if we denote the cokernel of f as a morphism of φ-modules by* $\mathrm{coker}(f)$*, then* $\frac{\mathrm{coker}(f)}{u\text{-tor}}$ *is an object of* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *and has a structure of a $(\varphi, \hat{G})$-module of height r. If we denote it by* $\widehat{\frac{\mathrm{coker}(f)}{u\text{-tor}}}$*, then it is minimal and is the cokernel of f in the abelian category* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*;*

(3) *if we denote the image (resp. coimage) of f as a morphism of φ-modules by* $\mathrm{im}(f)$ *(resp.* $\mathrm{coim}(f)$*), then* $\mathrm{im}(f)$ *(resp.* $\mathrm{coim}(f)$*) is an object of* $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *and has a structure of a $(\varphi, \hat{G})$-module of height r. If we denote it by* $\widehat{\mathrm{im}(f)}$ *(resp.* $\widehat{\mathrm{coim}(f)}$*), then it is minimal and is the image (resp. coimage) of f in the abelian category* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$*.*

Proof. (1) Since the functor Min^r is right adjoint (Proposition 5.29 (4)), we see the desired result.
(2) Put $C = \frac{\mathrm{coker}(f)}{u\text{-tor}}$. Recall that C is an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and has a structure of a $(\varphi, \hat{G})$-module of height r (Proposition 5.10). If we denote by g the projection $\hat{\mathfrak{M}}' \to \hat{C}$, by Proposition 5.17 (3), we have

$$\hat{C} = g(\hat{\mathfrak{M}}') = g(\mathrm{Min}^r(\hat{\mathfrak{M}}')) = \mathrm{Min}^r(g(\hat{\mathfrak{M}}')) = \mathrm{Min}^r(\hat{C})$$

and thus $\hat{C}$ is minimal.
(3) Let $g\colon \hat{C} \to \hat{\mathfrak{M}}'$ be as in the proof of (2). By (1) and (2), we see that the image of f in the category $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is $\mathrm{Min}^r(\widehat{\ker(g)})$. Let $\mathfrak{M}_g$ be the underlying Kisin module of $\mathrm{Min}^r(\widehat{\ker(g)})$. Then $\mathfrak{M}_g$ is the inverse image of the u-torsion part of $\mathrm{coler}(f)$ with respect to the projection $\mathfrak{M}' \to \mathrm{coler}(f)$. Since $\mathfrak{M}_g$ is finitely generated as an $\mathfrak{S}$-module, there exists a positive integer N such that $u^N\mathfrak{M}_g \subset \mathrm{im}(f)$. Hence we obtain $\mathfrak{M}_g[1/u] = \mathrm{im}(f)[1/u]$. Consequently, by Proposition 5.29 (3), we have

$$\mathrm{Min}^r(\widehat{\ker(g)}) = \mathrm{Min}^r(\hat{\mathfrak{M}}_g) = \mathrm{Min}^r(f(\hat{\mathfrak{M}})) = f(\mathrm{Min}^r(\hat{\mathfrak{M}})) = f(\hat{\mathfrak{M}}) = \widehat{\mathrm{im}(f)}$$

and thus $\widehat{\mathrm{im}(f)}$ is minimal. The proof for coimage is similar and hence we omit it. □

Proofs for the following three results are similar to those of the maximal case.

Lemma 5.31. *If* $\alpha\colon \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ *and* $\beta\colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}''$ *two morphisms in* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *such that* $\beta \circ \alpha = 0$*. Then the sequence* $0 \to \hat{\mathfrak{M}}' \overset{\alpha}{\to} \hat{\mathfrak{M}} \overset{\beta}{\to} \hat{\mathfrak{M}}'' \to 0$ *is exact in (the abelian category)* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *if and only if* $0 \to \widehat{\mathfrak{M}'[1/u]} \overset{\alpha[1/u]}{\to} \widehat{\mathfrak{M}[1/u]} \overset{\beta[1/u]}{\to} \widehat{\mathfrak{M}''[1/u]} \to 0$ *is exact in* $\mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$*. Furthermore, the functor*

$$\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}, \quad \hat{\mathfrak{M}} \mapsto \widehat{\mathfrak{M}[1/u]}$$

is fully faithful.

Corollary 5.32. *The functor* $\hat{T}$ *defined on* $\mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is exact and fully faithful, and its essential image is stable under subquotients.*

Corollary 5.33. *The functor* $\mathrm{Min}^r\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ *is exact.*

Put $e = [K : K_0]$, the absolute ramification index of K. If $er < p-1$, then $F^r_{\mathfrak{S}}(\hat{M})$ contains at most one element (cf. [CL1], Remark just after Corollary 3.2.6) and hence all torsion $(\varphi, \hat{G})$-modules of height r are automatically maximal and minimal. Therefore, we obtain

Corollary 5.34. *Suppose $er < p-1$. Then $\mathrm{Max}^{r,\hat{G}}_{/\mathfrak{S}_\infty} = \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} = \mathrm{Min}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$. In particular, the category $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ is abelian and the functor $\hat{T}\colon \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$ is exact and fully faithful, and its essential image is stable under subquotients.*

Remark 5.35. Similar to Remark 5.25, all results in this subsection hold even if we replace "$(\varphi,\hat{G})$-modules" with "weak $(\varphi,\hat{G})$-modules".

5.7 Some remarks

First the reader should be careful that *there are no new results in this subsection.*

5.7.1 Connection with a lifting problem

Let $r \in \{0,1,2,\dots,\infty\}$. Let $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{fr}}(G)$ be the category of lattices inside semi-stable p-adic representations with Hodge-Tate weights in $[0,r]$. Let $\mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}(G)$ be the category of torsion $\mathbb{Z}_p$-representations T such that there exists lattices $\Lambda_1,\Lambda_2 \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{fr}}(G)$ satisfying $\Lambda_1 \subset \Lambda_2$ and $T \simeq \Lambda_2/\Lambda_1$. The pair $\Lambda_1 \subset \Lambda_2$ is called a *lift of T*. We are interested in the following question:

Question 5.36. *For any $T \in \mathrm{Rep}_{\mathrm{tor}}(G)$, does there exists an integer $r \geq 0$ such that $T \in \mathrm{Rep}^{\mathrm{st},r}_{\mathrm{tor}}(G)$?*

If T is a tamely ramified $\mathbb{F}_p$-representation, then Caruso and Liu proved that the question has an affirmative answer (cf. [CL2, Theorem 5.7]). If we fix the choice of $r < \infty$, they also proved that Question 5.36 has a non-affirmative answer, which follows from a result on ramification bounds of torsion representations (cf. [CL2, Theorem 5.4]).

We connect Question 5.36 to our results in this paper. Recall that $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ is the essential image of $\hat{T}\colon \mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G)$, which is an abelian full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G)$. For simplicity, put $\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G) = \mathrm{Rep}^{\mathrm{st},\infty}_{\mathrm{tor}}(G)$. Then inclusions

$$\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) \subset \mathrm{Rep}_{\mathrm{tor}}(G)$$

are known (cf. [CL2, Theorem 3.1.3]). Thus Question 5.36 has an affirmative answer if and only if $\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G) = \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ and $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) = \mathrm{Rep}_{\mathrm{tor}}(G)$. On the other hand, we have seen the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mod}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \arrow[r, "\mathrm{Max}^\infty"] \arrow[d, "\text{forgetful}"'] & \mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \arrow[r, hook, "\hat{T}"'] & \mathrm{Rep}_{\mathrm{tor}}(G) \arrow[d, "\text{restriction}"] \\
\mathrm{Mod}^{\infty}_{/\mathfrak{S}_\infty} \arrow[r, "\mathrm{Max}^\infty"] & \mathrm{Max}^{\infty}_{/\mathfrak{S}_\infty} \arrow[r, "\sim", "T_{\mathfrak{S}}"'] & \mathrm{Rep}_{\mathrm{tor}}(G_\infty).
\end{tikzcd}$$

Here, the equivalence between categories $\mathrm{Max}^{\infty}_{/\mathfrak{S}_\infty}$ and $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ in the above diagram is proved in [CL2, Proposition 5.6]. Since the essential image of $\hat{T}\colon \mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \hookrightarrow \mathrm{Rep}_{\mathrm{tor}}(G)$ is $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$, it seems natural to suggest

Question 5.37. *Is the functor $\hat{T}\colon \mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \hookrightarrow \mathrm{Rep}_{\mathrm{tor}}(G)$ essentially surjective, that is, an equivalence of categories? This is equivalent to say that, for any $\hat{M} \in \mathbf{\Phi M}^{\hat{G}}_{/\mathcal{O}_\infty}$, does there exist a sub $(\varphi,\hat{G})$-module $\hat{\mathfrak{M}}$, of finite height, of $\hat{M}$ such that $\mathfrak{M}[1/u] = M$?*

If this has an affirmative answer, then we obtain $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G) = \mathrm{Rep}_{\mathrm{tor}}(G)$. In particular, we obtain an equivalence of abelian categories $\mathrm{Max}^{\infty,\hat{G}}_{/\mathfrak{S}_\infty} \simeq \mathrm{Rep}_{\mathrm{tor}}(G)$, which implies that maximal objects of torsion $(\varphi,\hat{G})$-modules completely classify torsion p-adic representations of G. On the other hand, we ask following questions:

Question 5.38. *Does any torsion $(\varphi, \hat{G})$-module have a resolution of free $(\varphi, \hat{G})$-modules?*

Question 5.39. *Is the category* $\mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$ *closed under extensions in* $\mathrm{Rep}_{\mathrm{tor}}(G)$*?*

Theorem 4.5 might be related with Question 5.39. If one of these questions has an affirmative answer, then we obtain $\mathrm{Rep}^{\mathrm{st}}_{\mathrm{tor}}(G) = \mathrm{Rep}^{\hat{G}}_{\mathrm{tor}}(G)$.

5.7.2 Connection with torsion Breuil modules

If we obtain an explicit relation between the categories of torsion Breuil modules and the category of torsion $(\varphi, \hat{G})$-modules, then our main result in this paper will give a partial answer of Question 2 in [CL1].

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