ON THE HIGHER FITTING IDEALS OF IWASAWA MODULES OF IDEAL CLASS GROUPS OVER REAL ABELIAN FIELDS

TATSUYA OHSHITA

Abstract. Kurihara established a refinement of the minus-part of the Iwasawa main conjecture for totally real number fields by using the higher Fitting ideals ([Ku]). In this paper, by using Kurihara's methods and Mazur–Rubin theory, we study the higher Fitting ideals of the plus-part of Iwasawa modules associated with the cyclotomic $\mathbb{Z}_p$-extension of abelian fields for an odd prime number p. We define *the higher cyclotomic ideals*, which are ideals of the Iwasawa algebra defined by the Kolyvagin derivative classes of circular units. Then, we prove that the higher cyclotomic ideals give upper and lower bounds of the higher Fitting ideals in some sense, and determine the pseudo-isomorphism classes of the plus-part of Iwasawa modules. Our results can be regarded as analogues of Kurihara's results and a refinement of the plus-part of the Iwasawa main conjecture for abelian fields.

1. Introduction

The Iwasawa main conjecture describes the characteristic ideals of certain Iwasawa modules. The characteristic ideals are important invariants on the structure of finitely generated torsion Iwasawa modules, but they are not enough to determine the pseudo-isomorphism classes of Iwasawa modules completely (cf. §2).

The higher Fitting ideals have more detailed information on Iwasawa modules. For instance, the higher Fitting ideals determine the pseudo-isomorphism class and the least cardinality of generators of finitely generated torsion Iwasawa modules. (See Remark 2.3 and Remark 2.4.) In [Ku], Kurihara proved that all the higher Fitting ideals of the minus-part of the Iwasawa modules associated with the cyclotomic $\mathbb{Z}_p$-extension of certain CM-fields coincide with the higher Stickelberger ideals, which are defined by analytic objects arising from p-adic L-functions (cf. [Ku] Theorem 1.1). His result is a refinement of the minus-part of the Iwasawa main conjecture for totally real number fields.

In this paper, we study the higher Fitting ideals of *the plus-part* of the Iwasawa modules by similar methods as in [Ku]. We construct a collection $\{\mathfrak{C}_{i,\chi}\}_{i\geq 0}$ of ideals of the Iwasawa algebra Λ_χ, which are analogues of Kurihara's higher Stickelberger ideals, and prove that the ideals $\mathfrak{C}_{i,\chi}$ give upper and lower bounds of the higher Fitting ideals of the plus-part in some sense. (In certain cases, the ideals $\mathfrak{C}_{i,\chi}$ determine the pseudo-isomorphism class of the plus-part.) The main tool in [Ku] is the Kolyvagin system of Gauss sums. Instead, in this paper, we use the Euler system of circular units, so

2010 *Mathematics Subject Classification.* Primary 11R23. Secondary 11R18.

we can only treat the Iwasawa modules associated with the cyclotomic $\mathbb{Z}_p$-extension of subfields of cyclotomic fields.

In order to state the main theorem of this paper, we introduce notations. We fix an odd prime number p. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and K a totally real subfield of $\overline{\mathbb{Q}}$ which is a finite abelian extension of $\mathbb{Q}$. We assume that the prime number p is unramified in $K/\mathbb{Q}$. Let μ_n be the group of all n-th roots of unity contained in $\overline{\mathbb{Q}}$. For an integer m with $m \geq 0$, let F_m be the maximal totally real subfield of $K(\mu_{p^{m+1}})$ and $F_\infty := \bigcup_{m \geq 0} F_m$. We put $\Gamma_{m,n} := \mathrm{Gal}(F_m/F_n)$ and $\Gamma_m := \mathrm{Gal}(F_\infty/F_m)$. Especially, we write $\Gamma := \Gamma_0$. We fix a topological generator $\gamma \in \Gamma_0$. Let $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/\mathbb{Q})]] = \varprojlim \mathbb{Z}_p[\mathrm{Gal}(F_m/\mathbb{Q})]$.

Put $\Delta := \mathrm{Gal}(F_0/\mathbb{Q}) = \Delta_0 \times \Delta_p$, where Δ_0 is the maximal subgroup of Δ whose order is prime to p, and Δ_p is the p-Sylow subgroup of Δ. We denote by D_p the decomposition subgroup of Δ at p. (Note that D_p is uniquely determined since Δ is abelian.) We put $\widehat{\Delta} := \mathrm{Hom}(\Delta, \overline{\mathbb{Q}}_p^\times)$. For any character $\chi \in \widehat{\Delta}$, we denote by $\mathcal{O}_\chi$ the $\mathbb{Z}_p[\Delta]$-algebra, which is isomorphic to $\mathbb{Z}_p[\mathrm{Im}\,\chi]$ as a $\mathbb{Z}_p$-algebra, and Δ acts on it via χ. We denote the Λ-algebra $\mathcal{O}_\chi[[\Gamma_0]]$ by Λ_χ, and we identify Λ_χ with $\mathcal{O}_\chi[[T]]$ by the isomorphism $\Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ of $\mathcal{O}_\chi$-algebras defined by $\gamma \mapsto 1 + T$.

For any Λ-module M, we put $M_\chi := M \otimes_\Lambda \Lambda_\chi$. We define a Λ-module $X := \varprojlim A_m$, where $A_m := A_{F_m}$ is the p-Sylow subgroup of the ideal class group Cl_{F_m} of F_m, and the projective limit is taken with respect to the norm maps. It is well-known that the Λ_χ-module X_χ is finitely generated and torsion. In this paper, we study the higher Fitting ideals $\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi)$ of X_χ for any non-trivial character $\chi \in \widehat{\Delta}$. Let $X_{\chi,\mathrm{fin}}$ be the largest pseudo-null Λ_χ-submodule of X_χ, and $X'_\chi := X_\chi/X_{\chi,\mathrm{fin}}$. We treat X'_χ instead of X_χ in order to apply Kurihara's Euler system arguments, which work for finitely generated torsion Λ-modules whose structures are given by square matrices (cf. Lemma 2.6 and §7).

Note that when we study the plus-part of ideal class groups, we do not have good analogues of Stickelberger elements in the group rings of Galois groups. So, comparing our cases treating the plus-part with the cases of the minus part, which Kurihara studied, a problem lies in how to define the ideals which are substitutes of Kurihara's higher Stickelberger ideals. A key idea of this paper lies in the definition of the ideal $\mathfrak{C}_{i,\chi}$ of Λ_χ, called *the higher cyclotomic ideal* for each $i \in \mathbb{Z}_{\geq 0}$ for which Kurihara's arguments can be applied. We shall define these ideals in §4, by using the Euler system of circular units (cf. Definition 4.15). Roughly speaking, first, we shall define the ideals $\mathfrak{C}_{i,m,N,\chi}$ of the group ring $R_{m,N,\chi} := \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]_\chi$ generated by images of certain Kolyvagin derivatives $\kappa_{m,N}(\xi)$ by *all $R_{m,N,\chi}$-homomorphisms*

$$\left(F_m^\times/(F_m^\times)^{p^N}\right)_\chi \longrightarrow R_{m,N,\chi},$$

then we shall define $\mathfrak{C}_{i,\chi}$ by taking the projective limit of them.

Let I and J be ideals of Λ_χ. Then, we write $I \prec J$ if there exists a height two ideal A of Λ_χ (called an "error factor") satisfying $AI \subseteq J$. Note that for two ideals I and J of Λ_χ, we have $I \prec J$ if and only if $I\Lambda_{\chi,\mathfrak{p}} \subseteq J\Lambda_{\chi,\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of height one, where we denote the localization of Λ_χ at $\mathfrak{p}$ by $\Lambda_{\chi,\mathfrak{p}}$. We write $I \sim J$ if $I \prec J$ and $J \prec I$. The relation $\sim$ is an equivalence relation on ideals of Λ_χ.

The following theorem is a rough form of our main theorem in this paper.

Theorem 1.1 (See Theorem 7.1 and Theorem 8.1)**.** *We assume that the extension degree of $K/\mathbb{Q}$ is prime to p. Let $\chi \in \widehat{\Delta}$ be a character satisfying $\chi(p) \neq 1$. Then, we have*

$$\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \sim \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$. Moreover, we have

$$\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi} \tag{1}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 1.2. Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. By a property of the principal (the 0-th) Fitting ideals, we have

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_{\chi,\mathrm{fin}}) \subseteq \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}}).$$

Note that we have

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_{\chi,\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi) = \mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi)$$

since X'_χ is a Λ_χ-module of projective dimension one (cf. Corollary 2.8). So, we have

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) \subseteq \mathfrak{C}_{0,\chi}.$$

Remark 1.3. In this paper, we study upper bounds of the higher Fitting ideals without any conditions on $[K:\mathbb{Q}]$ and $\chi(p)$. In the general cases, we only get the upper bounds of the Fitting ideals. For more precise results on the upper bounds of the Fitting ideals, see Theorem 7.1.

Theorem 1.1 implies that the higher cyclotomic ideals give "true" upper bounds in some special cases.

Corollary 1.4. *Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Assume that X_χ has no non-trivial pseudo-null Λ_χ-submodules. Then, we have*

$$\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

Remark 1.5. In all known examples, X_χ is pseudo-null (cf. Greenberg conjecture, for example, see [Gree] Conjecture 3.4). We have no non-trivial example for Corollary 1.4 at present.

When X_χ is a pseudo-null Λ_χ-module, the estimates (1) for $i = 0$ and Remark 1.2 imply the following corollary. (See Corollary 4.19. See also Example 4.20, which is an application of Corollary 4.19 for the study of the ideal class groups of F_m.)

Corollary 1.6. *Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Assume that X_χ is a pseudo-null Λ_χ-module. Then, we have*

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) = \mathrm{ann}_{\Lambda_\chi}(X_\chi).$$

We prove the inequality

$$\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

in §7 by an Euler system argument using analogues of Kurihara's elements. (See Theorem 7.1 and Corollary 7.2.) Then, in §8, we show

$$\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \succ \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$ by using the results of Mazur–Rubin on Kolyvagin systems. (See Theorem 8.1.) Note that the first assertion of Theorem 1.1 implies that the higher cyclotomic ideals determine the pseudo-isomorphism class of X_χ (cf. Remark 2.4). Related to our results, the following were previously known.

- By using usual Euler system argument and the Iwasawa main conjecture without Kurihara's elements, we can obtain estimates
$$\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \prec \mathfrak{C}_{i,\chi} \quad (i \in \mathbb{Z}_{\geq 0}),$$
(See Remark 8.5). But by such arguments without Kurihara's elements, we cannot obtain any explicit bounds of error factors, so these estimates are weaker than the inequalities (1) in Theorem 1.1.
- By Mazur–Rubin's theory of Kolyvagin systems in [MR] §5, it turns out that the pseudo-isomorphism class of X_χ is completely determined by the Λ-primitive Kolyvagin systems of $(\Lambda_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \mathcal{F}_\Lambda)$. (See Theorem 8.3, Corollary 8.11 and Corollary 8.14.) But the results in [MR] do not give explicit estimates of higher Fitting ideals of X_χ in terms of ideals of Λ_χ.

What is essentially new in this paper is the definition of the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ and to give stronger estimates (1) of higher Fitting ideals which contain more refined information (e.g. Corollary 1.6) than the pseudo-isomorphism class of X_χ. The key slogan is that the usual Euler system arguments work well only when the relation matrix is diagonal, but Kurihara's arguments work well under a milder assumption that the relation matrix is a square matrix.

We remark on the relation between higher cyclotomic ideals and the structure of $A_{0,\chi} := A_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_\chi$. By Mazur–Rubin theory in [MR], the isomorphism class of the $\mathcal{O}_\chi$-module $A_{0,\chi}$ is determined by the Kolyvagin systems of the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module $\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$. By comparing Mazur–Rubin theory and higher cyclotomic ideals, we obtain the following results. (See Theorem 8.4 and Corollary 8.6.)

Proposition 1.7. *Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Then, the following hold.*

(i) *The image of $\mathfrak{C}_{i,\chi}$ in the ring*
$$R_{0,\chi} := \mathbb{Z}_p[\mathrm{Gal}(F_0/\mathbb{Q})]_\chi = \Lambda_\chi/(\gamma - 1) = \varprojlim_N R_{0,N,\chi} \simeq \mathcal{O}_\chi$$
coincides with the ideal $\mathfrak{C}_{i,F_0,\chi} := \varprojlim_N \mathfrak{C}_{i,0,N,\chi}$ for any $i \in \mathbb{Z}_{\geq 0}$.

(ii) *We have $\mathrm{Fitt}_{\mathcal{O}_\chi,i}(A_{0,\chi}) = \mathfrak{C}_{i,F_0,\chi}$ for any $i \in \mathbb{Z}_{\geq 0}$.*

So, by Nakayama's lemma, we obtain the following corollary. (See Corollary 8.7.)

Corollary 1.8. *Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Let r be a non-negative integer. Then, the following are equivalent.*

(i) *The least cardinality of generators of the Λ_χ-module X_χ is r.*
(ii) $\mathfrak{C}_{r-1,\chi} \neq \Lambda_\chi$ *and* $\mathfrak{C}_{r,\chi} = \Lambda_\chi$.

In §2 we recall the definition and some basic properties of higher Fitting ideals. In §3, we recall some preliminary results on Iwasawa theory. In §4, we define the higher cyclotomic ideals, and prove our main theorem for $i = 0$ (Theorem 4.16). In §5, we recall some basic facts on the Kolyvagin derivatives of the Euler system of circular units, and introduce some elements $x_{\nu,q}$ of $(F_m^\times/p^N)_\chi$, which are analogues of Kurihara's elements in [Ku]. The elements $x_{\nu,q}(n)$ play an important role in Kurihara's arguments in the proof of Theorem 1.1. Especially, Proposition 5.6 is one of the keys of the arguments. In §6, we prove Proposition 6.1, which is a key proposition in the induction arguments. In §7, we prove the estimate

$$\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \geq 0$ (see Theorem 7.1). We also treat the case $\Delta_p \neq 0$ or $\chi(p) = 1$. In §8, we compare the higher Fitting ideals with Mazur–Rubin theory. We apply Mazur–Rubin theory, and prove Theorem 8.4, which is a result on the ground level, and complete the proof of the remaining part of Theorem 1.1.

Notation. In this paper, we use the following notation.

For a perfect field F, we fix an algebraic closure $\overline{F}$ of F. We denote the absolute Galois group of F by $G_F := \mathrm{Gal}(\overline{F}/F)$. For a topological abelian group T with continuous G_F-action, let $H^*(F,T) = H^*(G_F,T)$ be the continuous Galois cohomology group.

In this paper, we fix an odd prime p. An algebraic number field is a subfield F of a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ such that the extension degree of $F/\mathbb{Q}$ is finite. For a finite set Σ of places of $\mathbb{Q}$, we denote by $\mathbb{Q}_\Sigma$ the maximal extension of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$ unramified outside Σ. For any algebraic number field F, we denote the ring of integers of F by $\mathcal{O}_F$, and the p-Sylow subgroup of the ideal class group of F by A_F.

We define $\mathbb{Q}_\infty/\mathbb{Q}$ to be the cyclotomic $\mathbb{Z}_p$-extension inside $\overline{\mathbb{Q}}$. For any $m \in \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{Q}_m$ the unique subfield of $\mathbb{Q}_\infty$ whose extension degree over $\mathbb{Q}$ is p^m. Note that the field F_m introduced in the beginning of this section is the composite field of $\mathbb{Q}_m$ and F_0 for any $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We identify $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ as $\Gamma = \mathrm{Gal}(F_\infty/F_0)$ by the natural isomorphism $\Gamma \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$.

Let L/K be a finite Galois extension of algebraic number fields. Let λ be a prime ideal of $\mathcal{O}_K$, and λ' a prime ideal of $\mathcal{O}_L$ above λ. We denote the completion of K at λ by K_λ. If λ is unramified in L/K, the arithmetic Frobenius at λ' is denoted by $(\lambda', L/K) \in \mathrm{Gal}(L/K)$. We fix a family of embeddings $\{\ell_{\overline{\mathbb{Q}}}\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell:\mathrm{prime}}$ satisfying the condition (Chb) as follows:

(Chb) *For any subfield $F \subset \overline{\mathbb{Q}}$ which is a finite Galois extension of $\mathbb{Q}$ and any element $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, there exist infinitely many prime numbers ℓ such that ℓ is unramified in $F/\mathbb{Q}$ and $(\ell_F, F/\mathbb{Q}) = \sigma$, where ℓ_F is the finite place of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$.*

The existence of a family satisfying the condition (Chb) follows easily from the Chebotarev density theorem. We also fix an embedding $\infty_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

Let ℓ be a prime number. For an algebraic number field F, let ℓ_F be the finite place of F corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$. Then, if $L \supseteq F$ is an extension of algebraic number fields, we have $\ell_L | \ell_F$.

For an abelian group M and a positive integer n, we write M/n in place of M/nM for simplicity. In particular, for the multiplicative group $K^\times$ of a field K, we write $K^\times/p^N$ in place of $K^\times/(K^\times)^{p^N}$. We denote by M_{tor} the kernel of the natural homomorphism $M \longrightarrow M \otimes \mathbb{Q}$.

For an abelian group M with action of a group G, we denote the G-invariants (resp. G-coinvariants) of M by M^G (resp. M_G).

Acknowledgment

The author would like to thank Professors Kazuya Kato, Masato Kurihara, Masataka Chida and Tetsushi Ito for their helpful advices. The author also thanks Kenji Sakugawa for fruitful conversations and discussion with him. This work is supported by Grant-in-Aid for JSPS Fellows (22-2753) from Japan Society for the Promotion of Science.

2. Higher Fitting ideals

We use the same notation as in the previous section. In particular, we fix a finite abelian field K, and we define $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(F_\infty/\mathbb{Q})]]$, where F_∞ is the maximal totally real subfield of $K(\mu_{p^\infty})$.

Here, we briefly recall the definition and some basic properties of higher Fitting ideals.

Definition 2.1 (higher Fitting ideals, see [No] §3.1)**.** Let R be a commutative ring, and M a finitely presented R-module. Let

$$R^m \xrightarrow{f} R^n \longrightarrow M \longrightarrow 0$$

be an exact sequence of R-modules. For each $i \geq 0$, we define *the i-th Fitting ideal* $\mathrm{Fitt}_{R,i}(M)$ as follows.

- When $0 \leq i < n$ and $m \geq n - i$, we define $\mathrm{Fitt}_{R,i}(M)$ to be the ideal of R generated by all $(n-i) \times (n-i)$ minors of the matrix corresponding to f.
- When $0 \leq i < n$ and $m < n - i$, we define $\mathrm{Fitt}_{R,i}(M) := 0$.
- When $i \geq n$, we define $\mathrm{Fitt}_{R,i}(M) := R$.

When $i = 0$, the 0-th Fitting ideal is also called *the principal Fitting ideal.* The definition of these ideals depends only on M, and does not depend on the choice of the above exact sequence.

Remark 2.2. Let R be a commutative ring, S a commutative R-algebra, and M a finitely presented R-module, Then, by the definition of the higher Fitting ideals and the right exactness of tensor products, we have

$$\mathrm{Fitt}_{S,i}(M \otimes_R S) = \mathrm{Fitt}_{R,i}(M)S$$

for any $i \geq 0$.

Remark 2.3. Let R be a commutative ring, and M a finitely presented R-module. If we have $\mathrm{Fitt}_{R,i}(M) \neq R$, then the least cardinality of generators of M is greater than $i+1$. Note that when R is a local ring or a PID, the least cardinality of generators of M is $i+1$ if and only if $\mathrm{Fitt}_{R,i}(M) \neq R$ and $\mathrm{Fitt}_{R,i+1}(M) = R$.

Remark 2.4. Fix an arbitrary character $\chi \in \widehat{\Delta}$, and let M and N be Λ_χ-modules. We say that M is *pseudo-null* if the order of M is finite. We write $M \sim_{\text{p.i.}} N$ if there exists a homomorphism $M \longrightarrow N$ whose kernel and cokernel are both pseudo-null. We say that M is *pseudo-isomorphic* to N. Note the relation $\sim_{\text{p.i.}}$ is an equivalence relation on finitely generated torsion Λ_χ-modules. Assume

$$M \sim_{\text{p.i.}} \bigoplus_{i=1}^{n} \Lambda_\chi / f_i \Lambda_\chi$$

for non-zero elements $f_i \in \Lambda_\chi$, and f_i divides f_{i+1} for $1 \leq i \leq n-1$. Then, we have

$$\mathrm{Fitt}_{\Lambda_\chi,i}(M) \sim \begin{cases} (\prod_{k=1}^{n-i} f_k) & (\text{if } i < n) \\ \Lambda_\chi & (\text{if } i \geq n) \end{cases}$$

for any $i \geq 0$ (cf. [Ku] Lemma 8.2). In particular, the pseudo-isomorphism class of M is determined by the higher Fitting ideals $\{\mathrm{Fitt}_{\Lambda_\chi,i}(M)\}_{i\geq 0}$.

Remark 2.5. For a finitely generated torsion Λ_χ-module M, the characteristic ideal $\mathrm{char}_{\Lambda_\chi}(M)$ is the minimal principal ideal of Λ_χ containing $\mathrm{Fitt}_{\Lambda_\chi,0}(M)$.

The following lemma on principal Fitting ideals of Iwasawa modules plays an important role when we apply Kurihara's arguments in §7.

Lemma 2.6 (for example, see [Ku] Theorem 8.1)**.** *Let $R = \Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ and M a finitely generated torsion R-module. Suppose M has no non-trivial pseudo-null R-submodule. Then, there exists an exact sequence*

$$0 \longrightarrow R^n \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some integer $n > 0$, and we have

$$\mathrm{Fitt}_{R,0}(M) = \mathrm{char}_R(M).$$

Let us recall basic properties of higher Fitting ideals.

Lemma 2.7. *Let R be a commutative ring, and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of finitely generated torsion R-modules. We assume one of the following conditions:

- *R is a PID, or*
- *$R = \Lambda_\chi$, and N has no non-trivial pseudo-null R-submodule.*

Then, we have the following:

(i) $\mathrm{Fitt}_{R,i}(M) \subseteq \mathrm{Fitt}_{R,i}(L)$ *for any* $i \geq 0$.
(ii) $\mathrm{Fitt}_{R,i}(M) \subseteq \mathrm{Fitt}_{R,i}(N)$ *for any* $i \geq 0$.
(iii) $\mathrm{Fitt}_{R,i}(L)\,\mathrm{Fitt}_{R,0}(N) \subseteq \mathrm{Fitt}_{R,i}(M)$ *for any* $i \geq 0$.
(iv) $\mathrm{Fitt}_{R,0}(L)\,\mathrm{Fitt}_{R,i}(N) \subseteq \mathrm{Fitt}_{R,i}(M)$ *for any* $i \geq 0$.

Proof. Consider free resolutions

$$R^s \xrightarrow{f} R^r \longrightarrow L \longrightarrow 0,$$
$$R^{s'} \xrightarrow{g} R^{r'} \longrightarrow N \longrightarrow 0$$

of R-modules L and N. Note that by our assumption, we may assume $r' = s'$. Let $A \in M_{r,s}(R)$ (resp. $B \in M_{r',r'}(R)$) be the matrix representing the R-linear map f (resp. g) for standard basis. Then, we have an exact sequence

$$R^{s+s'} \xrightarrow{h} R^{r+r'} \longrightarrow M \longrightarrow 0$$

such that the $(r + r') \times (s + r')$ matrix C representing h is given by

$$C = \begin{pmatrix} A & * \\ 0 & B \end{pmatrix}.$$

Since B is a square matrix, all assertions of the lemma follow immediately from the computation of minors of the matrix C. $\square$

By Lemma 2.7, we obtain the following corollary.

Corollary 2.8. *Let $R = \Lambda_\chi \simeq \mathcal{O}_\chi[[T]]$ and M a finitely generated torsion R-module. We denote the maximal pseudo-null R-submodule of M by M_{fin}. Then, we have*

$$\mathrm{Fitt}_{R,0}(M) = \mathrm{Fitt}_{R,0}(M_{\mathrm{fin}})\,\mathrm{Fitt}_{R,0}(M/M_{\mathrm{fin}}).$$

Later, we also use the following fact on the principal Fitting ideals of pseudo-null Λ_χ-modules.

Lemma 2.9 (Proposition 3 in Appendix of [MW]). *Let M be a pseudo-null Λ_χ-module. Then, we have*

$$\mathrm{Fitt}_{\Lambda_\chi,0}(M) = \mathrm{Fitt}_{\Lambda_\chi,0}\left(\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)\right).$$

3. Preliminaries on Iwasawa modules

In this section, we recall some preliminary results on certain Iwasawa modules.

3.1. In this subsection, we give some remarks on "χ-quotients" of Λ-modules. Recall we denote the p-Sylow subgroup of $\Delta := \mathrm{Gal}(F_0/\mathbb{Q})$ by Δ_p, and the maximal subgroup of Δ of order prime to p by Δ_0. Note that $\Lambda_{\chi_0} := \mathcal{O}_{\chi_0}[[\Gamma]][\Delta_p]$ is flat over Λ for any $\chi_0 \in \widehat{\Delta}_0$. In particular, if the extension degree of $K/\mathbb{Q}$ is prime to p, then Λ_χ is flat over Λ for any $\chi \in \widehat{\Delta}$. When the degree of $K/\mathbb{Q}$ is divisible by p, we have to treat such Λ-algebras more carefully. (The reader may skip this subsection if he or she is interested in the case when $\Delta_p = 0$.)

Let $S_{\widehat{\Delta}}$ be a set of representatives of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-conjugacy classes of $\widehat{\Delta}$. We consider the natural homomorphism

$$\iota_{S_{\widehat{\Delta}}} : \Lambda \longrightarrow \prod_{\chi \in S_{\widehat{\Delta}}} \Lambda_\chi.$$

Note that the homomorphism $\iota_{S_{\widehat{\Delta}}}$ is injective, and $\mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}}$ is annihilated by $|\Delta_p|$. We use the following elementary lemma.

Lemma 3.1. *Let M be a Λ-module. The kernel and the cokernel of the natural homomorphism*

$$\iota_{M, S_{\widehat{\Delta}}} : M \longrightarrow \prod_{\chi \in S_{\widehat{\Delta}}} M_\chi.$$

are annihilated by $|\Delta_p|$.

***Proof*.** We consider the following exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\iota_{S_{\widehat{\Delta}}}} \prod_{\chi} \Lambda_\chi \longrightarrow \mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}} \longrightarrow 0.$$

Then, we obtain the exact sequence

$$\mathrm{Tor}_1^{\Lambda}(\mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}}, M) \longrightarrow M \longrightarrow \prod_{\chi} M_\chi \longrightarrow \mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}} \otimes_\Lambda M \longrightarrow 0.$$

Since the Λ-module $\mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}}$ is annihilated by $|\Delta_p|$, the Λ-modules $\mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}} \otimes_\Lambda M$ and $\mathrm{Tor}_1^{\Lambda}(\mathrm{Coker}\, \iota_{S_{\widehat{\Delta}}}, M)$ are annihilated by $|\Delta_p|$. ☐

Corollary 3.2. *Let M be a Λ-module with no non-zero $\mathbb{Z}_p$-torsion elements. Then, the Λ_χ-module $M_{\chi,\mathrm{tor}}$ consisting of all $\mathbb{Z}_p$-torsion elements is annihilated by $|\Delta_p|$.*

***Proof*.** We consider the commutative diagram of natural homomorphisms

$$\begin{tikzcd} M \arrow[r, hook, "f"] \arrow[d, "\iota_{M,S_{\widehat{\Delta}}}"'] & M \otimes \mathbb{Q} \arrow[d, "\iota_{M\otimes\mathbb{Q},S_{\widehat{\Delta}}}", "\simeq"'] \\ \prod_\chi M_\chi \arrow[r, "\prod_\chi f_\chi"] & \prod_\chi (M \otimes \mathbb{Q})_\chi. \end{tikzcd}$$

Then, the corollary follows from this commutative diagram and Lemma 3.1. ☐

Corollary 3.3. *Let M and N be Λ-modules, and $f\colon M \longrightarrow N$ a homomorphism of Λ-modules. We consider the commutative diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\iota_{M,S_{\widehat{\Delta}}}"'] & N \arrow[d, "\iota_{N,S_{\widehat{\Delta}}}"] \\ \prod_\chi M_\chi \arrow[r, "\prod_\chi f_\chi"] & \prod_\chi N_\chi \end{tikzcd} \tag{2}$$

induced by f.

- *We have*
$$\iota_{M,S_{\widehat{\Delta}}}(\operatorname{Ker} f) \supseteq |\Delta_p|^2 \cdot \operatorname{Ker}(\prod_\chi f_\chi).$$
In particular, for each character $\chi \in \widehat{\Delta}$, $|\Delta_p|^2 \operatorname{Ker} f_\chi$ is contained in the image of the kernel of f in M_χ.
- *The natural homomorphism $\operatorname{Coker}(f_\chi) \longrightarrow \big(\operatorname{Coker}(f)\big)_\chi$ is an isomorphism of Λ_χ-modules for any character $\chi \in \widehat{\Delta}$.*

Proof. The first assertion follows from the diagram (2) and Lemma 3.1. The second assertion is clear. □

3.2. In this and the next subsections, we recall some preliminary results on Iwasawa modules. Here, we recall some results on unit groups.

Let $m \in \mathbb{Z}_{\geq 0}$. We put $U_m := \big(\mathcal{O}_{F_m} \otimes \mathbb{Z}_p\big)^\times$ to be the group of semi-local units at p of F_m, and U_m^1 to be the maximal pro-p-subgroup of U_m. We denote the group of units of $\mathcal{O}_{F_m}$ by E_m, and the group of Sinnott's circular units in F_m by C_m (cf. [Si] §4). We define E_m^{cl} (resp. C_m^{cl}) to be the closure of E_m (resp. C_m) in U_m, and E_m^1 (resp. C_m^1) by $E_m^{\mathrm{cl}} \cap U_m^1$ (resp. $C_m^{\mathrm{cl}} \cap U_m^1$). We define $U_\infty := \varprojlim U_m^1$ and $E_\infty := \varprojlim E_m^1$, where the projective limits are taken with respect to the norm maps. Similarly, we define the limit $C_\infty := \varprojlim C_m^1$.

Remark 3.4. By Leopoldt's conjecture for abelian fields (cf. [Wa] Corollary 5.32), we have the natural isomorphisms $E_m \otimes \mathbb{Z}_p \xrightarrow{\simeq} E_m^1$ and $C_m \otimes \mathbb{Z}_p \xrightarrow{\simeq} C_m^1$.

The following proposition is well-known. (For example, see [Grei] p. 476 or [Wa] Lemma 15.41 for some special cases.)

Proposition 3.5. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. There exists a homomorphism $\varphi\colon E_{\infty,\chi} \longrightarrow \Lambda_\chi$ of Λ_χ-modules whose cokernel has finite order, and whose kernel is annihilated by a power of p. (Note that if the extension degree of $K/\mathbb{Q}$ is prime to p, then E_∞ has no non-trivial p-torsion element.)*

Let $\chi \in \widehat{\Delta}$. We denote the restriction of χ to Δ_0 by χ_0. We define an integer a_χ by
$$a_\chi := \begin{cases} 0 & \text{if } \chi_0(p) \neq 1; \\ 2 & \text{if } \chi_0(p) = 1. \end{cases}$$

For each $m \in \mathbb{Z}_{\geq 0}$, we consider the natural homomorphism $P_m^E\colon (E_\infty)_{\Gamma_m} \longrightarrow E_m^1$. We define the homomorphism $P_m^F\colon (E_\infty)_{\Gamma_m} \longrightarrow F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to be the composite of P_m^E and the inclusion map $E_m^1 \simeq \mathcal{O}_{F_m}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For each character $\chi \in \widehat{\Delta}$, the Λ_χ-homomorphisms P_m^E and P_m^F induce homomorphisms

$$\begin{aligned} P_{m,\chi}^E&\colon (E_{\infty,\chi})_{\Gamma_m} \longrightarrow E_m^1, \\ P_{m,\chi}^F&\colon (E_{\infty,\chi})_{\Gamma_m} \longrightarrow (F_m^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi. \end{aligned}$$

By Corollary 3.3, we have

$$|\Delta_p|^2 \operatorname{Ker} P_{m,\chi}^F \subseteq \operatorname{Ker} P_{m,\chi}^E \subseteq \operatorname{Ker} P_{m,\chi}^F.$$

Proposition 3.6. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Then, there exist ideals $I_{P_\chi^F}$ and $J_{P_\chi^E}$ of Λ_χ of finite indices such that*

$$\begin{aligned} (\gamma-1)^{a_\chi/2}\, |\Delta_p|^2\, I_{P_\chi^F} \operatorname{Ker} P_{m,\chi}^F &= \{0\}, \\ (\gamma-1)^{a_\chi/2} J_{P_\chi^E} \operatorname{Coker} P_{m,\chi}^E &= \{0\} \end{aligned}$$

for any $m \in \mathbb{Z}_{\geq 0}$.

Later, we introduce finer results than this proposition for the case when $\Delta_p = 0$ and $\chi_0(p) \neq 1$. (See Proposition 3.7.)

***Proof*.** For any finite cyclic group G and any G-module M, we denote the Tate cohomology groups by $\hat{H}^i(G, M)$. Fix a non-negative integer m. For $m' \geq m$, we have the exact sequence

$$0 \longrightarrow \varprojlim \hat{H}^{-1}(\Gamma_{m',m}, E_{m'}) \longrightarrow (E_\infty)_{\Gamma_m} \xrightarrow{P_m^E} E_m^1 \longrightarrow \varprojlim \hat{H}^0(\Gamma_{m',m}, E_{m'}) \longrightarrow 0$$

of Λ-modules. This exact sequence and Corollary 3.3 imply that

$$\begin{aligned} \operatorname{Coker} P_{m,\chi}^E &= \varprojlim \hat{H}^0(\Gamma_{m',m}, E_{m'})_\chi, \\ \operatorname{ann}_{\Lambda_\chi}(\operatorname{Ker} P_{m,\chi}^E) &\supseteq |\Delta_p|^2 \operatorname{ann}_{\Lambda_\chi}(\varprojlim \hat{H}^{-1}(\Gamma_{m',m}, E_{m'})_\chi). \end{aligned}$$

By Lemma 1.2 of [Ru1], there exists an integer k satisfying

$$\left|(\gamma-1)\hat{H}^i(\Gamma_{m',m}, E_{m'})\right| \leq p^k$$

for all $i \in \mathbb{Z}$ and $m', m \in \mathbb{Z}_{\geq 0}$ with $m' \geq m$. (Note that the setting of Lemma 1.2 of [Ru1] seems to be different from ours, but the argument in the proof of this lemma works in our case.) Therefore, the assertion for characters $\chi \in \widehat{\Delta}$ satisfying $\chi_0(p) = 1$ follows.

We assume that $\chi \in \widehat{\Delta}$ is a character satisfying $\chi_0(p) \neq 1$. By Corollary 3.3, it is sufficient to show that for any $i \in \mathbb{Z}$, the order of the Λ_{χ_0}-module

$$\hat{H}^i(\Gamma_{m',m}, E_{m'})_{\chi_0} = \hat{H}^i(\Gamma_{m',m}, (E_{m'} \otimes \mathbb{Z}_p)_{\chi_0})$$

is finite and bounded by a constant independent of m and m'.

Since the $\mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]$-module $E^1_{m'} \oplus \mathbb{Z}_p$ contains a submodule of finite index which is free of rank one, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})] \xrightarrow{f} E^1_{m'} \oplus \mathbb{Z}_p \longrightarrow N \longrightarrow 0,$$

where N is a $\mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]$-module of finite order. This exact sequence induces the exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0} \xrightarrow{f_{\chi_0}} E^1_{m',\chi_0} \longrightarrow N_{\chi_0} \longrightarrow 0.$$

Note that we have $(\mathbb{Z}_p)_{\chi_0} = 0$ since χ_0 is non-trivial. We consider the Herbrand quotients, and obtain

$$\begin{aligned}\frac{\#\hat{H}^0(\Gamma_{m',m}, E^1_{m',\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, E^1_{m',\chi_0})} &= \frac{\#\hat{H}^0(\Gamma_{m',m}, \mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, \mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]_{\chi_0})} \cdot \frac{\#\hat{H}^0(\Gamma_{m',m}, N_{\chi_0})}{\#\hat{H}^{-1}(\Gamma_{m',m}, N_{\chi_0})} \\ &= 1.\end{aligned}$$

Let $E^{(p)}_{m'}$ be the unit group of $\mathcal{O}_{F_{m'}}[1/p]^\times$. Then, we have an exact sequence

$$0 \longrightarrow E_{m'} \otimes \mathbb{Z}_p \xrightarrow{i} E^{(p)}_{m'} \otimes \mathbb{Z}_p \longrightarrow S_{m'} \longrightarrow 0, \tag{3}$$

where $S_{m'}$ is a $\mathbb{Z}_p[\mathrm{Gal}(F_{m'}/\mathbb{Q})]$-submodule of $\mathcal{I}^p_{F_{m'}} \otimes \mathbb{Z}_p$, which is a free $\mathbb{Z}_p$-module generated by all places of $F_{m'}$ above p. Note that the group D_p acts trivially on $S_{m'}$. So, the natural homomorphism

$$i_{\chi_0} \colon E^1_{m',\chi_0} \simeq (E_{m'} \otimes \mathbb{Z}_p)_{\chi_0} \longrightarrow (E^{(p)}_{m'} \otimes \mathbb{Z}_p)_{\chi_0}$$

is an isomorphism. By Corollary in §5.4 of [Iw], there exists an integer r such that

$$\left|\hat{H}^{-1}(\Gamma_{m',m}, E^{(p)}_{m'})\right| \leq p^r$$

for all $m', m \in \mathbb{Z}$ satisfying $m' \geq m \geq 0$. These results and the above arguments using Herbrand quotients imply that we have

$$\left|\hat{H}^0(\Gamma_{m',m}, E^1_{m',\chi_0})\right| = \left|\hat{H}^0(\Gamma_{m',m}, (E^{(p)}_{m'} \otimes \mathbb{Z}_p)_{\chi_0})\right| \leq p^r$$

for all $m', m \in \mathbb{Z}$ satisfying $m' \geq m \geq 0$. Therefore, the order of the Λ_{χ_0}-modules $\hat{H}^{-1}(\Gamma_{m',m}, E_{m'})_{\chi_0}$ and $\hat{H}^0(\Gamma_{m',m}, E_{m'})_{\chi_0}$ are bounded by a constant independent of m and m'. This completes the proof of Proposition 3.6. □

We can prove more refined results than Proposition 3.6 when $\Delta_p = 0$ and $\chi(p) \neq 1$ by a similar argument to the proof of [Ru3] Theorem 7.6. (We have to replace X_∞ in [Ru3] by $\mathrm{Gal}(M_\infty/F_\infty)$ and U_∞ in [Ru3] by our U_∞, where M_∞ is the maximal pro-p extension field of F_∞ unramified outside the places above p.)

Proposition 3.7. *Assume that the extension degree of $K/\mathbb{Q}$ is prime to p, and the character $\chi \in \widehat{\Delta}$ satisfies $\chi(p) \neq 1$. Then, we can take $I_{P^F_\chi} = \Lambda_\chi$ and $J_{P^E_\chi} = ann_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$ in Proposition 3.6.*

3.3. In this subsection, we recall some results on the ideal class groups and the statement of the plus part of the Iwasawa main conjecture.

Recall that we denote the p-Sylow subgroup of the ideal class group of F_m by A_{F_m}, and we define the Λ-module X by $X := \varprojlim A_{F_m}$, where the projective limits are taken with respect to the norm maps. Note that X is a finitely generated Λ-torsion module.

Recall that we assume p is unramified in $K/\mathbb{Q}$. For the Iwasawa module X, the following results are well-known.

Proposition 3.8 (cf. [Wa] Lemma 13.15)**.** (i) *For each $m \in \mathbb{Z}_{\geq 0}$, the natural homomorphism $X_{\Gamma_m} \longrightarrow A_{F_m}$ is surjective.*
(ii) *There exists a Λ-submodule Y of X such that $(\gamma - 1)X \subseteq Y \subseteq X$, and the kernel of the canonical homomorphism $X_{\Gamma_m,\chi} \longrightarrow A_{F_m,\chi}$ is annihilated by $I_A := ann_\Lambda(Y/(\gamma - 1)X)$ for any $m \in \mathbb{Z}_{\geq 0}$.*

When $\Delta_p = 0$ and $\chi(p) \neq 1$, we have more refined results than Proposition 3.8. Proposition 3.9 is proved by a similar method to [Ru3] Theorem 5.4 (i).

Proposition 3.9. *Assume that the extension degree of $K/\mathbb{Q}$ is prime to p, and the character $\chi \in \widehat{\Delta}$ satisfies $\chi(p) \neq 1$. Then, the natural homomorphism*

$$X_{\Gamma_m,\chi} \longrightarrow A_{F_m,\chi}$$

is an isomorphism for any $m \in \mathbb{Z}_{\geq 0}$.

Let us recall the statement of the plus-part of the Iwasawa main conjecture briefly:

> *Let the Λ-modules E_∞, C_∞ and X be as above. Let $\chi \in \widehat{\Delta}$ be an arbitrary character. Then, we have* $\mathrm{char}_{\Lambda_\chi}(X_\chi) = \mathrm{char}_{\Lambda_\chi}\left((E_\infty/C_\infty)_\chi\right)$.

(See [CS], [MW], [Ru2], [Grei] Theorem 3.1 and loc. cit. Remark c), et al.) We use the Iwasawa main conjecture in the proof of our main results.

4. Higher cyclotomic ideals

In this section, we define the ideal $\mathfrak{C}_{i,\chi}$ of Λ_χ for each $i \in \mathbb{Z}_{\geq 0}$ by using circular units, and prove Theorem 1.1 for $i = 0$. We call the ideal $\mathfrak{C}_{i,\chi}$ *the i-th cyclotomic ideal.*

4.1. Here, we define some special circular units in order to define the ideals $\mathfrak{C}_{i,\chi}$.

In §1, we have fixed an embedding $\infty_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We regard $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$. For each positive integer n, we define

$$\zeta_n := \exp(2\pi\sqrt{-1}/n) \in \overline{\mathbb{Q}} \subset \mathbb{C},$$

which is a primitive n-th root of unity. Note that we have $\zeta_{mn}^m = \zeta_n$ for any positive integers m and n.

For each integer $N > 0$, we define

$$\mathcal{S}_N := \{\ell \mid \ell \text{ is a prime number splitting completely in } K(\mu_{p^N})/\mathbb{Q}\},$$

$$\mathcal{N}_N := \left\{ \prod_{i=1}^{r} \ell_i \,\middle|\, r \in \mathbb{Z}_{>0},\ \ell_i \in \mathcal{S}_N\ (i = 1, \ldots, r),\ \text{and } \ell_i \neq \ell_j \text{ if } i \neq j \right\} \cup \{1\}.$$

In particular, if $\ell \in \mathcal{S}_N$, then we have $\ell \equiv 1 \mod p^N$.

Let m be a non-negative integer, and put $F := F_m$. We denote the conductor of $F/\mathbb{Q}$ by $\mathfrak{f}_F = \mathfrak{f}_{F/\mathbb{Q}}$. For a positive integer n prime to $\mathfrak{f}_F$, we define $H_{F,n} := \mathrm{Gal}\big(F(\mu_n)/F\big)$. For simplicity, we write $H_n := H_{\mathbb{Q},n}$. If n is decomposed as $n = \prod_{i=1}^{r} \ell_i^{e_i}$, where $\ell_1, \ldots, \ell_r$ are distinct prime numbers and $e_i > 0$ for each i, then we have natural isomorphisms

$$\mathrm{Gal}(F(\mu_n)/\mathbb{Q}) \simeq \mathrm{Gal}(F/\mathbb{Q}) \times H_{F,n},$$
$$H_{F,n} \simeq H_n \simeq H_{\ell_1^{e_1}} \times \cdots \times H_{\ell_r^{e_r}}.$$

We identify these groups by the canonical isomorphisms.

Definition 4.1. Let m be a non-negative integer, and n a positive integer prime to $p\mathfrak{f}_K$.

(i) For each $d \in \mathbb{Z}_{>1}$ dividing $\mathfrak{f}_K$, we define

$$\eta_m^d(n) := \mathrm{N}_{\mathbb{Q}(\mu_{p^{m+1}nd})/\mathbb{Q}(\mu_{p^{m+1}nd}) \cap F_m(\mu_n)}(1 - \zeta_d^{p^{-m}} \zeta_{np^{m+1}}) \in F_m(\mu_n)^\times.$$

(ii) For each $a \in \mathbb{Z}$ with $(a, p) = 1$, we define

$$\eta_m^{1,a}(n) := \mathrm{N}_{K(\mu_{p^{m+1}n})/F_m(\mu_n)}\left(\frac{1 - \zeta_n^{p^{-m}} \zeta_{p^{m+1}}^a}{1 - \zeta_n^{p^{-m}} \zeta_{p^{m+1}}}\right) \in F_m(\mu_n)^\times.$$

In this paper, we call the elements $\eta_m^d(n)$ and $\eta_m^{1,a}(n)$ *basic circular units* of $F_m(\mu_n)$.

The following lemma is well-known, and easily verified.

Lemma 4.2. *Let m be a non-negative integer, n a positive integer prime to $p\mathfrak{f}_K$, and ℓ a prime divisor of n. Let $\eta_m^\bullet(n)$ be a basic circular unit of $F_m(\mu_n)$. Namely, $\eta_m^\bullet(n)$ denotes $\eta_m^d(n)$ or $\eta_m^{1,a}(n)$. Then, the following hold.*

(i) *We have*

$$\mathrm{N}_{F_m(\mu_n)/F_m(\mu_{n/\ell})}\big(\eta_m^\bullet(n)\big) = \eta_m^\bullet(n/\ell)^{1-\mathrm{Fr}_\ell^{-1}},$$

where Fr_ℓ *is the arithmetic Frobenius element at* ℓ *in* $\mathrm{Gal}\big(F_m(\mu_{n/\ell})/\mathbb{Q}\big)$.

(ii) *We have*

$$\mathrm{N}_{F_{m+1}(\mu_n)/F_m(\mu_n)}\big(\eta_{m+1}^\bullet(n)\big) = \eta_m^\bullet(n).$$

Remark 4.3. Let $\mathcal{K}$ be the composite field of $\mathbb{Q}_\infty$ and $\mathbb{Q}(\mu_n)$ for all positive integers n satisfying $(n, p\mathfrak{f}_{K/\mathbb{Q}}) = 1$. We fix a circular unit

$$\eta := \prod_{d \mid \mathfrak{f}_K} \eta_0^d(1)^{u_d} \times \prod_{i=1}^{r} \eta_0^{1,a_i}(1)^{v_i} \in F_0^\times,$$

where $r \in \mathbb{Z}_{>0}$, $u_d, v_i \in \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$ for positive integers d and i satisfying $d \mid \mathfrak{f}_K$ and $1 \leq i \leq r$, and $a_1, \dots, a_r$ are integers prime to p. For any non-negative integer m and any positive integer n satisfying $(n, \mathfrak{f}_{K/\mathbb{Q}}) = 1$, we put

$$\eta_m(n) := \prod_{d \mid \mathfrak{f}_K} \eta_m^d(n)^{u_d} \times \prod_{i=1}^{r} \eta_m^{1,a_i}(n)^{v_i} \in F_m^\times.$$

We also denote by $\eta_m(n)_\chi$ the image of $\eta_m(n)_\chi$ in $H^1\big(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)\big)$ by the natural homomorphism

$$(F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi = H^1\big(\mathbb{Q}_m, \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)\big)_\chi \longrightarrow H^1\big(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)\big).$$

Then, the collection

$$\{\eta_m(n)_\chi \in H^1\big(\mathbb{Q}_m, \mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)\big)\}_{m,n}$$

of Galois cohomology classes defines an Euler system for $(\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \mathcal{K}/\mathbb{Q}, p\mathfrak{f}_{K/\mathbb{Q}})$ in the sense of [Ru4].

In particular, Lemma 4.2 (ii) implies that $(\eta_m^d(1))_{m\geq 0}$ is a norm compatible system, so it defines an element of C_∞. Later, we use the following result.

Proposition 4.4 (See [Grei] Lemma 2.3)**.** *The Λ-module C_∞ is generated by*

$$\left\{(\eta_m^d(1))_{m\geq 0} \;\middle|\; d \in \mathbb{Z}_{>1}, d \mid \mathfrak{f}_K\right\} \cup \left\{(\eta_m^{1,a}(1))_{m\geq 0} \;\middle|\; a \in \mathbb{Z}, (a,p) = 1\right\}.$$

Moreover, when $\Delta_p = 0$, the following result is known.

Proposition 4.5 ([Tsu] Lemma 6.2)**.** *Assume that the extension degree of $K/\mathbb{Q}$ is prime to p, and the character $\chi \in \widehat{\Delta}$ is non-trivial. Then, the Λ_χ-module $C_{\infty,\chi}$ is free of rank one.*

Remark 4.6. If the extension degree of $K/\mathbb{Q}$ is prime to p, and a character $\chi \in \widehat{\Delta}$ satisfies $\chi(p) \neq 1$, then we can easily show that any circular unit $\eta_\chi \in C^1_{0,\chi}$ extends to an element

$$\{\eta_{m,\chi}\}_m \in C_{\infty,\chi} = \varprojlim_m C^1_{m,\chi}$$

satisfying $\eta_{0,\chi} = \eta_\chi$. This fact implies that any circular unit $\eta \in C^1_{0,\chi}$ extends to an Euler system $\{\eta_m(n)_\chi\}_{m,n}$ for $(\mathcal{O}_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \Sigma)$ in the sense of Definition A.4 which consists of Λ_χ-linear combination of basic circular units, and satisfies $\eta_0(1)_\chi = \eta_\chi$. Here, Σ is a set of places of $\mathbb{Q}$ defined by

$$\Sigma := \{p, \infty\} \cup \{\ell \mid \ell \text{ ramifies in } K/\mathbb{Q}\}.$$

4.2. In this subsection, we define the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ by using Kolyvagin derivatives $\kappa^d_{m,N}(n)$ of Euler systems of circular units. First, let us recall the notion of Kolyvagin derivatives. Let ℓ be a prime number contained in $\mathcal{S}_N$. We shall take a generator σ_ℓ of the cyclic group $H_\ell = \mathrm{Gal}\big(\mathbb{Q}(\mu_\ell)/\mathbb{Q}\big)$ as follows. We put $N_{\{\ell\}} := \mathrm{ord}_p(\ell - 1)$, where ord_p is the additive valuation at p normalized by $\mathrm{ord}_p(p) = 1$. Then, we have $N_{\{\ell\}} \geq N \geq 1$. By the fixed embedding $\ell_{\overline{\mathbb{Q}}}\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we regard $\mu_{p^{N_{\{\ell\}}}}$ as a subset of $\mathbb{Q}_\ell$. We identify $\mathrm{Gal}\big(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell\big)$ with $H_\ell = \mathrm{Gal}\big(\mathbb{Q}(\mu_\ell)/\mathbb{Q}\big)$ by the isomorphism defined by $\ell_{\overline{\mathbb{Q}}}$. Let F be the maximal subextension of $\mathbb{Q}(\mu_\ell)/\mathbb{Q}$ such that $[F:\mathbb{Q}]$ is a power of p, and π a uniformizer of F_{ℓ_F}. We fix a generator σ_ℓ of H_ℓ such that

$$\pi^{\sigma_\ell - 1} \equiv \zeta_{p^{N\{\ell\}}} \pmod{\mathfrak{m}_\ell},$$

where $\mathfrak{m}_\ell$ is the maximal ideal of F_{ℓ_F}, and $\zeta_{p^{N\{\ell\}}}$ is a primitive $p^{N\{\ell\}}$-th root of unity defined as above. Note that the definition of σ_ℓ does not depend on the choice of π.

Definition 4.7. (i) For $\ell \in \mathcal{S}_N$, we define

$$D_\ell := \sum_{k=1}^{\ell-2} k\sigma_\ell^k \in \mathbb{Z}[H_\ell].$$

(ii) Let $n = \prod_{i=1}^r \ell_i \in \mathcal{N}_N$, where $\ell_i \in \mathcal{S}_N$ for each i. Then, we define

$$D_n := \prod_{i=1}^r D_{\ell_i} \in \mathbb{Z}[H_n].$$

In order to define the Kolyvagin derivatives of circular units, we use the following well-known lemma.

Lemma 4.8. *Let $n \in \mathcal{N}_N$. Then, for each $d \in \mathbb{Z}_{>1}$ dividing $\mathfrak{f}_K$ and for each $a \in \mathbb{Z}$ prime to p, the images of $\eta_m^d(n)^{D_n}$ and $\eta_m^{1,a}(n)^{D_n}$ in $F_m(\mu_n)^\times/p^N$ are fixed by H_n.*

Note that $H^0(F_m(\mu_n), \mu_{p^N}) = 0$ in our situation, so by Kummer theory and Hochschild-Serre spectral sequence, the natural homomorphism

$$F_m^\times/p^N \longrightarrow \left(F_m(\mu_n)^\times/p^N\right)^{H_n}$$

is an isomorphism. By Lemma 4.8, we define Kolyvagin derivatives $\kappa_{m,N}^d(n)$ of (basic) circular units as follows.

Definition 4.9. Let $n \in \mathcal{N}_N$. For each $d \in \mathbb{Z}_{>1}$ dividing $\mathfrak{f}_K$ (resp. $a \in \mathbb{Z}$ prime to p), we define

$$\kappa_{m,N}^d(n) \in F_m^\times/p^N \quad (\text{resp. } \kappa_{m,N}^{1,a}(n) \in F_m^\times/p^N)$$

to be the unique element whose image in $F_m(\mu_n)^\times/p^N$ is $\eta_m^d(n)^{D_n}$ (resp. $\eta_m^{1,a}(n)^{D_n}$).

Now, let us define the higher cyclotomic ideals $\{\mathfrak{C}_{i,\chi}\}_{i\geq 0}$. First, we fix integers m and N satisfying $N \geq m+1 > 0$. Let $\chi \in \widehat{\Delta}$, and put

$$R_{m,N} := \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})];$$
$$R_{m,N,\chi} := R_{m,N} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi \simeq \mathcal{O}_\chi/p^N[\Gamma_{m,0}].$$

Then, we have

$$R_{m,N} \simeq \Lambda_{\Gamma_m}/p^N, \quad R_{m,N,\chi} \simeq \Lambda_{\chi,\Gamma_m}/p^N,$$

where Λ_{Γ_m} (resp. Λ_{χ,Γ_m}) denotes the Γ_m-coinvariant of Λ (resp. Λ_χ). As in [Ku], we use the notion of *well-ordered* integers.

Definition 4.10. Let $n \in \mathcal{N}_N$. We call n *well-ordered* if n has a factorization $n = \prod_{i=1}^r \ell_i$ with $\ell_i \in \mathcal{S}_N$ for each i such that ℓ_{i+1} splits in $F_m\bigl(\mu_{\prod_{j=1}^i \ell_j}\bigr)/\mathbb{Q}$ for $i =$

$1, \ldots, r-1$. In other words, n is well-ordered if and only if n has a factorization $n = \prod_{i=1}^{r} \ell_i$ such that

$$\ell_{i+1} \equiv 1 \pmod{p^N \prod_{j=1}^{i} \ell_j}$$

for $i = 1, \ldots, r-1$. We denote by $\mathcal{N}_N^{\mathrm{w.o.}}$ the set of all well-ordered elements in $\mathcal{N}_N$.

Let $n \in \mathcal{N}_N^{\mathrm{w.o.}}$ with the decomposition $n = \prod_{i=1}^{r} \ell_i$, where $\ell_i \in \mathcal{S}_N$ for each i. We put $\epsilon(n) := r$. We define $\mathcal{W}_{m,N,\chi}(n)$ to be the $R_{m,N,\chi}$-submodule of $(F_m^{\times}/p^N)_\chi$ generated by the image of

$$\left\{\kappa_{m,N}^{d}(n) \;\middle|\; d \in \mathbb{Z}_{>0} \text{ dividing } \mathfrak{f}_K\right\} \cup \left\{\kappa_{m,N}^{1,a}(n) \;\middle|\; a \in \mathbb{Z} \text{ prime to } p\right\}.$$

We put $\mathcal{H}_{m,N,\chi} := \mathrm{Hom}_{R_{m,N,\chi}}\left((F_m^{\times}/p^N)_\chi, R_{m,N,\chi}\right)$.

Definition 4.11. We define $\mathfrak{C}_{i,m,N,\chi}$ to be the ideal of $R_{m,N,\chi}$ generated by

$$\bigcup_{f \in \mathcal{H}_{m,N,\chi}} \bigcup_{n} f\left(\mathcal{W}_{m,N,\chi}(n)\right),$$

where n runs through all elements of $\mathcal{N}_N^{\mathrm{w.o.}}$ satisfying $\epsilon(n) \leq i$.

Remark 4.12. Note that the $R_{m,N,\chi}$-module $\mathrm{Hom}(\mathcal{O}_\chi/p^N[\Gamma_{m,0}], \mathbb{Q}_p/\mathbb{Z}_p)$ is injective and free of rank one. So, $R_{m,N,\chi}$ is an injective $R_{m,N,\chi}$-module. In particular, the restriction map

$$\mathcal{H}_{m,N,\chi} \longrightarrow \mathcal{H}_{m,N,\chi}(n) := \mathrm{Hom}_{R_{m,N,\chi}}\left(\mathcal{W}_{m,N,\chi}(n), R_{m,N,\chi}\right)$$

is surjective. This implies that the ideal $\mathfrak{C}_{i,m,N,\chi}$ coincides with the ideal of $R_{m,N,\chi}$ generated by

$$\bigcup_{n} \bigcup_{f \in \mathcal{H}_{m,N,\chi}(n)} \mathrm{Im}(f),$$

where $\mathrm{Im}(f)$ is the image of f, and n runs through all elements of $\mathcal{N}_N^{\mathrm{w.o.}}$ satisfying $\epsilon(n) \leq i$.

In order to define the higher cyclotomic ideals, we need the following Lemma 4.13. (Note that in the construction of higher cyclotomic ideals, we only use the first assertion of Lemma 4.13 for $n = 1$. Other assertions of the lemma are used later, in §8.)

Lemma 4.13. *Let m_1, m_2, N_1 and N_2 be integers satisfying $m_2 \geq m_1$ and $N_2 \geq N_1$. Take a positive integer n prime to $p\mathfrak{f}_{K/\mathbb{Q}}$. Then, the following hold.*

(i) *For any $R_{m_2,N_2,\chi}[H_n]$-homomorphism*

$$f_2 \colon \left(F_{m_2}(\mu_n)^{\times}/p^{N_2}\right)_\chi \longrightarrow R_{m_2,N_2,\chi}[H_n],$$

there exists an $R_{m_2,N_2,\chi}[H_n]$-homomorphism

$$f_1 \colon \left(F_{m_1}(\mu_n)^{\times}/p^{N_1}\right)_\chi \longrightarrow R_{m_1,N_1,\chi}[H_n]$$

which makes the diagram

$$\begin{tikzcd}
\left(F_{m_2}(\mu_n)^\times/p^N\right)_\chi \arrow[r, "f_2"] \arrow[d, "N_{F_{m_2}/F_{m_1}}"'] & R_{m_2,N_2,\chi}[H_n] \arrow[d, "\mathrm{mod}\ (\gamma^{p^{m_1}}-1,p^{N_1})"] \\
\left(F_{m_1}(\mu_n)^\times/p^N\right)_\chi \arrow[r, dashed, "f_1"] & R_{m_1,N_1,\chi}[H_n]
\end{tikzcd}$$

commute.

(ii) *Assume* $\Delta_p = 0$ *and* $N_1 = N_2 =: N$. *Then, for any* $R_{m_1,N,\chi}[H_n]$*-homomorphism*

$$f_1\colon \left(F_{m_1}(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_{m_1,N,\chi}[H_n],$$

there exists an $R_{m_2,N,\chi}[H_n]$*-homomorphism*

$$f_2\colon \left(F_{m_2}(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_{m_2,N,\chi}[H_n]$$

which makes the diagram

$$\begin{tikzcd}
\left(F_{m_2}(\mu_n)^\times/p^N\right)_\chi \arrow[r, dashed, "f_2"] \arrow[d, "N_{F_{m_2}/F_{m_1}}"'] & R_{m_2,N,\chi}[H_n] \arrow[d, "\mathrm{mod}\ (\gamma^{p^{m_1}}-1)"] \\
\left(F_{m_1}(\mu_n)^\times/p^N\right)_\chi \arrow[r, "f_1"] & R_{m_1,N,\chi}[H_n]
\end{tikzcd}$$

commute.

***Proof*.** Let us prove the first assertion of the lemma. Note that we can easily reduce the proof of this claim to the following two cases:

(A) $(m_2, N_2) = (m_1, N_1 + 1)$;
(B) $(m_2, N_2) = (m_1 + 1, N_1)$.

In the case (A), our lemma is clear. We shall show the lemma in the case (B). We put $m = m_1$, $N = N_1 = N_2$, $R_1 = R_{m,N}[H_n]$, $R_2 = R_{m+1,N}[H_n]$, and the natural surjection $\mathrm{pr}\colon R_2 \longrightarrow R_1$. We denote by ι_χ the natural homomorphism

$$\iota\colon (F_m(\mu_n)^\times/p^N)_\chi \longrightarrow (F_{m+1}(\mu_n)^\times/p^N)_\chi.$$

We define an element

$$N_{m+1/m} := \sum_{\sigma\in \mathrm{Gal}(F_{m+1}/F_m)} \sigma \in R_2.$$

Then, there is a unique isomorphism

$$\nu_{m+1/m}\colon R_1 \xrightarrow{\simeq} N_{m+1/m}R_2 = (R_2)^{\Gamma_{m+1,m}}$$

of R_1-modules satisfying $1 \mapsto N_{m+1/m}$. Let $\mathcal{NF}$ be the image of $(F_{m+1}(\mu_n)^\times/p^N)_\chi$ in $(F_m(\mu_n)^\times/p^N)_\chi$ by the norm map. Note the composite map

$$\nu_{m+1/m}\circ \mathrm{pr}\colon R_2 \longrightarrow R_2$$

coincides with the scalar multiplication by $N_{m+1/m}$, so there exists a unique R_2-linear homomorphism

$$f_1^{(0)}\colon \iota_\chi(\mathcal{NF}) = N_{m+1/m}(F_{m+1}(\mu_n)^\times/p^N)_\chi \longrightarrow R_1$$

which makes the diagram

$$\begin{tikzcd}
 & (F_{m+1}(\mu_n)^\times/p^N)_\chi \arrow[r, "f_2"] \arrow[dl, "N_{F_m/F_{m+1}}"'] \arrow[d, "\times N_{m+1,m}"] & R_2 \arrow[r, "\times N_{m/m+1}"] \arrow[d, "\mathrm{pr}"] & R_2 \\
\mathcal{NF} \arrow[r, "\iota_\chi"] & \iota_\chi(\mathcal{NF}) \arrow[r, dashed, "f_1^{(0)}"] & R_1 \arrow[ur, hook, "\nu_{m+1/m}"'] &
\end{tikzcd}$$

commute. By the injectivity of R_1, we can extend $f_1^{(0)} \circ \iota_\chi$ to a homomorphism

$$f_1\colon \left(F_m(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_1$$

satisfying $f_1 |_{\mathcal{NF}}= f_1^{(0)} \circ \iota$. This completes the proof of the first assertion.

Next, let us prove the second assertion. It is sufficient to show it when $(m_2, N_2) = (m_1 + 1, N_1)$. Here, we use the same notation as in the proof of the first assertion. Let $f_1\colon \left(F_m(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_1$ be a given R_1-homomorphism. Note that we assume that p is odd, so we have $H^0(F_{m+1}(\mu_n), \mu_{p^N}) = 0$. This implies that the natural homomorphism

$$\iota\colon F_m(\mu_n)^\times/p^N \longrightarrow F_{m+1}(\mu_n)^\times/p^N$$

is injective. Since we assume $\Delta_p = 0$ here, we can regard $\left(F_m(\mu_n)^\times/p^N\right)_\chi$ as an R_2-submodule of $\left(F_{m+1}(\mu_n)^\times/p^N\right)_\chi$ by the natural injection ι_χ. Note that R_2 is an injective R_2-module, so we can extend the homomorphism

$$\nu_{m+1/m} \circ f_1\colon \left(F_m(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_2$$

to an R_2-homomorphism $f_2\colon \left(F_{m+1}(\mu_n)^\times/p^N\right)_\chi \longrightarrow R_2$. By the definition of f_2, we obtain the following commutative diagram of R_2-modules:

$$\begin{tikzcd}
\left(F_{m+1}(\mu_n)^\times/p^N\right)_\chi \arrow[r, dashed, "f_2"] \arrow[d, "\times N_{m+1/m}"'] & R_2 \arrow[r, "\times N_{m+1/m}"] \arrow[d, "\mathrm{pr}"'] & R_2 \\
\left(F_m(\mu_n)^\times/p^N\right)_\chi \arrow[r, "f_1"] & R_1 \arrow[ur, hook, "\nu_{m+1/m}"'] &
\end{tikzcd}$$

This completes the proof. □

Let m_1, m_2, N_1 and N_2 be integers satisfying $N_1 \geq m_1 + 1$, $N_2 \geq m_2 + 1$, $m_2 \geq m_1$ and $N_2 \geq N_1$. Let $n \in \mathcal{N}_{N_2}^{\mathrm{w.o.}}$ be an element satisfying $\epsilon(n) \leq i$. It follows from Lemma 4.2 that the image of $\mathcal{W}_{m_2,N_2,\chi}(n)$ by the norm map

$$N_{F_{m_2}/F_{m_1}} : (F_{m_2}^\times/p^{N_2})_\chi \longrightarrow (F_{m_1}^\times/p^{N_1})_\chi$$

is contained in $\mathcal{W}_{m_1,N_1,\chi}(n)$. Hence we obtain the following corollary of Lemma 4.13.

Corollary 4.14. *Let m_1, m_2, N_1 and N_2 be integers satisfying $N_1 \geq m_1 + 1$, $N_2 \geq m_2 + 1$, $m_2 \geq m_1$ and $N_2 \geq N_1$. Then, the image of $\mathfrak{C}_{i,m_2,N_2,\chi}$ by the projection $R_{m_2,N_2,\chi} \longrightarrow R_{m_1,N_1,\chi}$ is contained in $\mathfrak{C}_{i,m_1,N_1,\chi}$. Moreover, if we assume $\Delta_p = 0$ and $N_1 = N_2 =: N$, then the image of $\mathfrak{C}_{i,m_2,N,\chi}$ in $R_{m_1,N,\chi}$ coincides with $\mathfrak{C}_{i,m_1,N,\chi}$.*

Now, we can define the higher cyclotomic ideals.

Definition 4.15. We define *the i-th cyclotomic ideal* $\mathfrak{C}_{i,\chi}$ to be the ideal of Λ_χ by

$$\mathfrak{C}_{i,\chi} := \varprojlim \mathfrak{C}_{i,m,N,\chi},$$

where the projective limit is taken with respect to the system of the natural homomorphisms $\mathfrak{C}_{i,m_2,N_2,\chi} \longrightarrow \mathfrak{C}_{i,m_1,N_1,\chi}$ for integers m_1, m_2, N_1 and N_2 satisfying $N_1 \geq m_1+1$, $N_2 \geq m_2+1$, $m_2 \geq m_1$ and $N_2 \geq N_1$.

4.3. We take a generator $\theta \in \Lambda_\chi$ of $\mathrm{char}_{\Lambda_\chi}\left(E_{\infty,\chi}/C_{\infty,\chi}\right)$. Then, we denote the ideal of Λ_χ generated by

$$\bigcup_{\varphi \in \mathrm{Hom}_{\Lambda_\chi}(E_{\infty,\chi},\Lambda_\chi)} \theta^{-1}\varphi(C_{\infty,\chi})$$

by I_C. Note that I_C is an ideal of Λ_χ of finite index. Moreover, by Proposition 4.5, we have $I_C = \Lambda_\chi$ if the extension degree of $K/\mathbb{Q}$ is prime to p.

Here, we will prove the following theorem, which is a part of Theorem 1.1 for $i = 0$.

Theorem 4.16. *Let $\chi \in \widehat{\Delta}$ be any character. Then, we have*

(i) $\mathfrak{C}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
(ii) $(\gamma-1)^{a_\chi}\left|\Delta_p\right|^4 I_C I_{P^F_\chi} J_{P^E_\chi} \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi) \subseteq \mathfrak{C}_{0,\chi}$.

In order to prove Theorem 4.16, by Iwasawa main conjecture, it is enough to prove the following proposition.

Proposition 4.17. *Let χ be a non-trivial character in $\widehat{\Delta}$. Then,*

(i) $\mathfrak{C}_{0,\chi} \subseteq \mathrm{char}_{\Lambda_\chi}\left(E_{\infty,\chi}/C_{\infty,\chi}\right)$.
(ii) $(\gamma-1)^{a_\chi}\left|\Delta_p\right|^4 I_C I_{P^F_\chi} J_{P^E_\chi} \mathrm{char}_{\Lambda_\chi}\left(E_{\infty,\chi}/C_{\infty,\chi}\right) \subseteq \mathfrak{C}_{0,\chi}$.

***Proof*.** Let us prove Proposition 4.17. First, we prove the assertion (ii). By Proposition 3.5, we can take a Λ_χ-homomorphism $\varphi\colon E_{\infty,\chi} \longrightarrow \Lambda_\chi$ whose cokernel has finite order. This induces a homomorphism

$$\bar{\varphi}_{m,N,\chi}\colon (E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow R_{m,N,\chi}.$$

We take arbitrary elements $\delta_1 \in I_{P^F_\chi}$ and $\delta_2 \in J_{P^E_\chi}$. We need the following Lemma 4.18, which follows from Corollary 3.3 and Proposition 3.6 immediately.

Lemma 4.18. *Let $\mathcal{NO}_{m,N,\chi}$ be the image of the homomorphism*

$$(E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow \left(F_m^\times/p^N\right)_\chi$$

induced by the homomorphism $P^F_{m,\chi}$ defined in §3.2. Then, the kernel of the natural homomorphism is annihilated by $(\gamma-1)^{a_\chi}\left|\Delta_p\right|^4 I_{P^F_\chi} J_{P^E_\chi}$, and there exists a homomorphism $\psi\colon \mathcal{NO}_{m,N,\chi} \longrightarrow R_{m,N,\chi}$ which makes the diagram

$$\begin{tikzcd}
(C_{\infty,\chi})_{\Gamma_m}/p^N \arrow[r] \arrow[d] & (E_{\infty,\chi})_{\Gamma_m}/p^N \arrow[r, "(\gamma-1)^{a_\chi}|\Delta_p|^4\delta_1\delta_2\cdot\bar{\varphi}_{m,N,\chi}"] \arrow[d] & R_{m,N,\chi} \\
\mathcal{W}_{m,N,\chi}(1) \arrow[r, hook] & \mathcal{NO}_{m,N,\chi} \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

commute.

Lemma 4.18 implies the second assertion of Proposition 4.17. Indeed, since the image of $(C_{\infty,\chi})_{\Gamma_m}$ in $F_m^\times/p^N$ coincides with $\mathcal{W}_{m,N,\chi}(1)$ by Proposition 4.4, we have

$$(\gamma-1)^{a_\chi}\delta_1\delta_2\cdot\bar{\varphi}_{m,N,\chi}\bigl(\text{the image of }(C_{\infty,\chi})_{\Gamma_m}\bigr)\subseteq\psi(\mathcal{W}_{m,N,\chi}(1))\subseteq\mathfrak{C}_{0,m,N,\chi}.$$

Next, we prove the assertion (i). Let $\varphi\colon E_{\infty,\chi}\longrightarrow\Lambda_\chi$ be a Λ_χ-homomorphism whose cokernel has finite order, and $\bar{\varphi}_{m,N,\chi}\colon(E_{\infty,\chi})_{\Gamma_m}/p^N\longrightarrow R_{m,N,\chi}$ the homomorphism induced by φ. We take arbitrary elements $\delta_1'\in\mathrm{ann}_{\Lambda_\chi}(\mathrm{Ker}\,\varphi)$ and $\delta_2'\in\mathrm{ann}_{\Lambda_\chi}(\mathrm{Coker}\,\varphi)$. In particular, we may take some powers of p as δ_1' and δ_2'. We shall construct a homomorphism

$$\psi_{\delta_1',\delta_2'}\in\mathrm{Hom}_{R_{m,N,\chi}}(R_{m,N,\chi},(E_{\infty,\chi})_{\Gamma_m}/p^N),$$

which can be regarded as an "inverse" of $\bar{\varphi}_{m,N,\chi}$ in the following sense. For each $x\in R_{m,N,\chi}$, we take $y\in(E_{\infty,\chi})_{\Gamma_m}/p^N$ such that

$$\bar{\varphi}_{m,N,\chi}(y)=\delta_2'x.$$

Then, we define

$$\psi_{\delta_1',\delta_2'}(x):=\delta_1'y\in(E_{\infty,\chi})_{\Gamma_m}/p^N.$$

The definition of $\psi_{\delta_1',\delta_2'}(x)$ is independent of the choice of y, and $\psi_{\delta_1',\delta_2'}$ is contained in $\mathrm{Hom}_{R_{m,N,\chi}}(R_{m,N,\chi},(E_{\infty,\chi})_{\Gamma_m}/p^N)$.

Let $f\in\mathcal{H}_{m,N,\chi}(n)$ be an arbitrary homomorphism. Since $R_{m,N,\chi}$ is an injective $R_{m,N,\chi}$-module, there exists a homomorphism $\tilde{f}\colon\bigl(F_m^\times/p^N\bigr)_\chi\longrightarrow R_{m,N,\chi}$ whose restriction to $\mathcal{W}_{m,N,\chi}(1)$ coincides with f. Then, there exists a unique element $a\in R_{m,N,\chi}$ which makes the following diagram

$$\begin{tikzcd}
(E_{\infty,\chi})_{\Gamma_m}/p^N \arrow[ddd] & & & & R_{m,N,\chi} \arrow[llll, "\psi_{\delta_1',\delta_2'}"'] \arrow[ddd, dotted, "\times a"] \\
& & (C_{\infty,\chi})_{\Gamma_m}/p^N \arrow[ull, "\delta_1'\delta_2'\cdot i"'] \arrow[urr, "\bar{\varphi}_{m,N,\chi}\circ i"] \arrow[d, "\delta_1'\delta_2'\cdot j"] & & \\
& & \mathcal{W}_{m,N,\chi}(1) \arrow[dll, hook'] \arrow[drr, "f"] & & \\
\bigl(F_m^\times/p^N\bigr)_\chi \arrow[rrrr, dashed, "\tilde{f}"] & & & & R_{m,N,\chi}
\end{tikzcd}$$

commute, where the right vertical arrow $\times a$ is the scalar multiplication by a, and $i\colon(C_{\infty,\chi})_{\Gamma_m}/p^N\longrightarrow(E_{\infty,\chi})_{\Gamma_m}/p^N$ and $j\colon(C_{\infty,\chi})_{\Gamma_m}/p^N\longrightarrow\mathcal{W}_{m,N,\chi}(1)$ are the natural homomorphisms. From this diagram, we obtain

$$\delta_1'\delta_2'f(\mathcal{W}_{m,N,\chi}(1)_{\Gamma_m}/p^N)=a\bar{\varphi}_{m,N,\chi}\circ i((C_{\infty,\chi})_{\Gamma_m}/p^N).$$

Note that we may assume that δ_1' and δ_2' are powers of p. Since the characteristic ideal $\mathrm{char}_{\Lambda_\chi}\bigl(E_{\infty,\chi}/C_{\infty,\chi}\bigr)$ is a principal ideal of Λ_χ not containing p, the second assertion of Proposition 4.17 follows. $\square$

As mentioned in Introduction, we obtain the following corollary by Theorem 4.16 and Remark 1.2.

Corollary 4.19. *Assume $p \nmid [K:\mathbb{Q}]$ and $\chi(p) \neq 1$. Suppose that X_χ is a pseudo-null Λ_χ-module. Then, we have the equality*

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) = \mathrm{ann}_{\Lambda_\chi}(X_\chi).$$

***Proof*.** Here, we assume that X_χ is a pseudo-null Λ_χ-module, so we have $X_\chi = A_{m,\chi}$ and

$$E^1_{m,\chi}/C_{m,\chi} \xrightarrow{\simeq} E^1_{m+1,\chi}/C^1_{m+1,\chi}$$

for a sufficiently large integer m (cf. [KS] Proposition 2.2). Let $N > 0$ be an integer such that p^N annihilates $X_\chi = A_{m,\chi}$ and $E^1_{m,\chi}/C^1_{m,\chi}$. We put $R_{m,\chi} := \Lambda_{\chi,\Gamma_m} = \mathcal{O}_\chi[\Gamma_{m,0}]$. Since we assume $p \nmid [K:\mathbb{Q}]$ and $\chi(p) \neq 1$, Remark 4.6 implies that the image of any circular units of F_m in $(F_m^\times/p^N)_\chi$ is written as a Λ_χ-linear combination of basic circular units. In other words, in our case, the natural map

$$C_{\infty,\chi} \longrightarrow C^1_{m,\chi} = (C_m \otimes \mathbb{Z}_p)_\chi$$

is surjective (cf. Proposition 4.4). Then, it follows from Proposition 4.5 that $C^1_{m,\chi}$ is a free $R_{m,\chi}$-module of rank one. Note that arguments in the proof of [KS] Theorem 2.4 work if the $R_{m,\chi}$-module $C^1_{m,\chi}$ is free of rank one, so we obtain an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(E^1_{m,\chi}/C^1_{m,\chi}, \mathbb{Q}_p/\mathbb{Z}_p\right) \simeq R_{m,N,\chi}/\mathfrak{C}_{0,m,N,\chi}$$

as Λ_χ-module (for details, See loc. cit.). Therefore, we obtain

$$\mathrm{Fitt}_{R_{m,N,\chi},0}\left(\mathrm{Hom}_{\mathbb{Z}_p}(E^1_{m,\chi}/C^1_{m,\chi}, \mathbb{Q}_p/\mathbb{Z}_p)\right) = \mathfrak{C}_{0,m,N,\chi}. \tag{4}$$

for sufficiently large m and N.

Let n be an integer satisfying $n \geq m$. We denote the group of unit ideles of F_n by $U_{\mathbb{A}}(F_n)$. Note that the χ-quotients of the maximal pro-p quotients of the abelian groups $\mathbb{A}^\times_{F_n}/F_n^\times$ and $U_{\mathbb{A}}(F_n)$ are cohomologically trivial as $\Gamma_{n,m}$-modules since we assume $p \nmid [K:\mathbb{Q}]$ and $\chi(p) \neq 1$. The $\Gamma_{n,m}$-module $C^1_{n,\chi}$ is also cohomologically trivial since $C^1_{n,\chi}$ is a free $\mathcal{O}_\chi[\Gamma_{n,m}]$-module. So, by similar arguments to those in the proof of [KS] Proposition 2.6 and [CG] Proposition 11, we have

$$\hat{H}^q(\Gamma_{n,m}, E^1_{n,\chi}/C^1_{n,\chi}) \simeq \hat{H}^{q-2}(\Gamma_{n,m}, A_{n,\chi}) \simeq \hat{H}^q(\Gamma_{n,m}, A_{n,\chi})$$

for any $q \in \mathbb{Z}$. Since we assume X_χ is pseudo-null, we have

$$\hat{H}^0(\Gamma_{n,m}, E^1_{n,\chi}/C^1_{n,\chi}) = E^1_{n,\chi}/C^1_{n,\chi}$$
$$\hat{H}^0(\Gamma_{n,m}, A_{n,\chi}) = A_{n,\chi}$$

for sufficiently large m and $n \geq 2m$. This implies

$$E^1_{m,\chi}/C^1_{m,\chi} \simeq A_{m,\chi}$$

for sufficiently large m. Therefore, combining (4), we have

$$\mathrm{Fitt}_{R_{m,N,\chi},0}(X_\chi) = \mathrm{Fitt}_{R_{m,N,\chi},0}(A_{m,\chi}) = \mathfrak{C}_{0,m,N,\chi}. \tag{5}$$

Since the equality (5) holds for sufficiently large m and N, we obtain

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) = \mathrm{Fitt}_{\Lambda_\chi,0}(\mathrm{Hom}(X_\chi, \mathbb{Q}_p/\mathbb{Z}_p)) = \mathfrak{C}_{0,\chi}$$

by Lemma 2.9. Here, we assume $X_\chi = X_{\chi,\mathrm{fin}}$, so we have

$$\mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi) \subseteq \mathrm{ann}_{\Lambda_\chi}(X_\chi) \subseteq \mathfrak{C}_{0,\chi}.$$

by Theorem 4.16 and Remark 1.2. Hence we obtain the corollary. $\square$

In general, the computation of the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ is hard. In a certain very special case, we can determine the higher cyclotomic ideals explicitly and prove that they coincide with the higher Fitting ideals.

Example 4.20 (suggested by the Referee)**.** Let $r \in \mathbb{Z}_{\geq 2}$, and define a Λ_χ-module M by

$$M := (\Lambda_\chi/(p, \gamma - 1))^{\oplus r} = (\mathcal{O}_\chi/p\mathcal{O}_\chi)^{\oplus r}.$$

Then, Corollary 4.19 implies that such a Λ_χ-module M does not appear as an Iwasawa module X_χ since M is a pseudo-null Λ_χ-module satisfying $\mathrm{Fitt}_{\Lambda_\chi,0}(M) \neq \mathrm{ann}_{\Lambda_\chi}(M)$. For instance, we have the following observation. Assume $p = 3$ and $K := \mathbb{Q}(\sqrt{32009})$. Let $\chi \in \widehat{\Delta}$ be the even quadratic character corresponding to K. Then, we have

$$A_{0,\chi} = \mathrm{Cl}_K \otimes \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

But $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ cannot be the Iwasawa module X_χ. Therefore, we know that the Iwasawa module X_χ is bigger than $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

5. Kurihara's elements for circular units

In this section, we fix integers m and N satisfying $N \geq m + 1 > 0$. Here, we recall some basic facts on the Euler system of circular units, and define some elements $x_{\nu,q} \in (F_m^\times/p^N)_\chi$, which are analogues of Kurihara's elements in [Ku].

5.1. Let $\chi \in \widehat{\Delta}$ be a character. Recall we put $R_{m,N} := \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]$ and $R_{m,N,\chi} := R_{m,N} \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$. Here, for each $\ell \in \mathcal{S}_N$, we shall recall the definition of two homomorphisms

$$[\cdot]^\ell_{m,N,\chi} \colon (F_m^\times/p^N)_\chi \longrightarrow R_{m,N,\chi} \qquad \text{(cf. Definition 5.1)},$$
$$\bar{\phi}^\ell_{m,N,\chi} \colon (F_m^\times/p^N)_\chi \longrightarrow R_{m,N,\chi} \qquad \text{(cf. Definition 5.2)},$$

which are important in the induction part in the Euler system arguments. The homomorphism $[\cdot]^\ell_{m,N,\chi}$ is defined by the valuations at the places above ℓ, and $\bar{\phi}^\ell_{m,N,\chi}$ is defined by the local reciprocity maps.

First, we shall define $[\cdot]^\ell_{m,N,\chi}$. Let F be an algebraic number field. We define

$$\mathcal{I}_F := \mathrm{Div}\big(\mathrm{Spec}(\mathcal{O}_F)\big)$$

to be the divisor group, and we write its group law additively. We define the homomorphism $(\cdot)_F \colon F^\times \longrightarrow \mathcal{I}_F$ by

$$(x)_F = \sum_\lambda \mathrm{ord}_\lambda(x)\lambda,$$

where λ runs through all prime ideals of $\mathcal{O}_F$, and ord_λ is the normalized valuation at λ. For any prime number ℓ, we define $\mathcal{I}_F^\ell$ to be the subgroup of $\mathcal{I}_F$ generated by all prime ideals above ℓ. Then, we define $(\cdot)_F^\ell\colon F^\times \longrightarrow \mathcal{I}_F^\ell$ by

$$(x)_F^\ell = \sum_{\lambda|\ell} \mathrm{ord}_\lambda(x)\lambda.$$

Recall that we have fixed a family of embeddings $\{\ell_{\overline{\mathbb{Q}}}\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell\}_{\ell:\mathrm{prime}}$ (cf. §1 Notation). We denote the prime ideal of $\mathcal{O}_F$ corresponding to the embedding $\ell_{\overline{\mathbb{Q}}}|_F$ by ℓ_F for each prime number ℓ and algebraic number field F. Assume $F/\mathbb{Q}$ is a Galois extension, and ℓ splits completely in $F/\mathbb{Q}$. Then, $\mathcal{I}_F^\ell$ is a free $\mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})]$-module generated by ℓ_F, and we identify $\mathcal{I}_F^\ell$ with $\mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})]$ by the isomorphism $\iota\colon \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})] \xrightarrow{\simeq} \mathcal{I}_F^\ell$ defined by $x \longmapsto x \cdot \ell_F$ for $x \in \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})]$. We denote the composite map $F^\times \longrightarrow \mathcal{I}_F^\ell \xrightarrow{\iota^{-1}} \mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})]$ by $(\cdot)_F^\ell$.

Definition 5.1. We define the $R_{m,N,\chi}$-homomorphism

$$[\cdot]_{m,N,\chi}\colon (F_m^\times/p^N)_\chi \longrightarrow (\mathcal{I}_{F_m}/p^N)_\chi$$

to be the homomorphism induced by $(\cdot)_{F_m}\colon F_m^\times \longrightarrow \mathcal{I}_{F_m}$. Let $\ell \in \mathcal{S}_N$ be any element. We define the $R_{m,N,\chi}$-homomorphism

$$[\cdot]_{m,N,\chi}^\ell\colon (F_m^\times/p^N)_\chi \longrightarrow R_{m,N,\chi} = \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})] \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\chi$$

to be the homomorphism induced by $(\cdot)_{F_m}^\ell\colon F_m^\times \longrightarrow \mathcal{I}_{F_m}^\ell \longrightarrow \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$.

Next, we shall define $\bar{\phi}_{m,N,\chi}^\ell$. Let $\ell \in \mathcal{S}_N$ be any element. We have assumed $N \geq m+1$, so the prime number ℓ splits completely in $F_m/\mathbb{Q}$. We have $F_{m,\lambda} = \mathbb{Q}_\ell$ for any prime ideal λ of $\mathcal{O}_{F_m}$ above ℓ. There are natural isomorphisms of $\mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$-modules

$$\bigoplus_{\lambda|\ell} F_{m,\lambda}^\times \simeq \mathcal{I}_{F_m}^\ell \otimes_{\mathbb{Z}} \mathbb{Q}_\ell^\times,$$
$$\bigoplus_{\lambda|\ell} H_\ell \simeq \mathcal{I}_{F_m}^\ell \otimes_{\mathbb{Z}} H_\ell,$$

and we identify them. Here, we regard $\mathbb{Q}_\ell^\times$ as a $\mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$-module on which $\mathrm{Gal}(F_m/\mathbb{Q})$ acts trivially. We denote by

$$\phi_{\mathbb{Q}_\ell}\colon \mathbb{Q}_\ell^\times \longrightarrow \mathrm{Gal}\left(\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell\right) \simeq \mathrm{Gal}\left(\mathbb{Q}(\mu_\ell)/\mathbb{Q}\right) \simeq H_\ell$$

the reciprocity map of local class field theory. For any $x \in \mathbb{Z}_\ell^\times$, we have

$$\phi_{\mathbb{Q}_\ell}(x) = (\zeta_\ell \longmapsto \zeta_\ell^{-x}) \in H_\ell.$$

The homomorphism

$$\phi_m^\ell\colon F_m^\times \longrightarrow \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})] \otimes_{\mathbb{Z}} H_\ell$$

is defined to be the composite of the following three homomorphisms of $\mathbb{Z}[\mathrm{Gal}(F_m(\mu_n)/\mathbb{Q})]$-modules:

$$\mathrm{diag}\colon F_m^\times \longrightarrow \bigoplus_{\lambda|\ell} F_{m,\lambda}^\times,$$
$$\oplus\phi_{\mathbb{Q}_\ell}\colon \bigoplus_{\lambda|\ell} F_{m,\lambda}^\times \longrightarrow \bigoplus_{\lambda|\ell} H_\ell,$$
$$\iota_H^{-1}\colon \bigoplus_{\lambda|\ell} H_\ell \xrightarrow{\simeq} \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})] \otimes_{\mathbb{Z}} H_\ell,$$

which are defined as follows:

- the first homomorphism diag is the diagonal inclusion;
- the second homomorphism $\oplus\phi_{\mathbb{Q}_\ell}$ is the direct sum of the local reciprocity maps;
- the third isomorphism ι_H^{-1} is the inverse of the isomorphism
$$\iota_H\colon \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})] \otimes_{\mathbb{Z}} H_\ell \xrightarrow{\simeq} \bigoplus_{\lambda|\ell} H_\ell = \mathcal{I}_{F_m}^\ell \otimes_{\mathbb{Z}} H_\ell,$$
which is induced by $\iota\colon \mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})] \xrightarrow{\simeq} \mathcal{I}_{F_m}^\ell;\ x \longmapsto x \cdot \ell_{F_m}$.

Definition 5.2. Let $\ell \in \mathcal{S}_N$ be any element. We define
$$\phi_{m,N,\chi}^\ell\colon (F_m^\times/p^N)_\chi \longrightarrow \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]_\chi \otimes_{\mathbb{Z}} H_\ell$$
to be the homomorphism of $R_{m,N,\chi}$-modules induced by ϕ_m^ℓ. Since we have fixed a generator σ_ℓ of H_ℓ, we have an $R_{m,N,\chi}$-homomorphism
$$\bar{\phi}_{m,N,\chi}^\ell\colon (F_m^\times/p^N)_\chi \longrightarrow \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]_\chi = R_{m,N,\chi}.$$

The following formulas on Kolyvagin derivatives are well-known.

Proposition 5.3. *Let $n \in \mathcal{N}_N$ be an integer, $d \in \mathbb{Z}_{>1}$ an integer dividing $\mathfrak{f}_K$ and $a \in \mathbb{Z}$ an integer prime to p. For simplicity, we denote $\kappa_{m,N}^d(n)$ or $\kappa_{m,N}^{1,a}(n)$ by $\kappa^\bullet(n)$.*

(i) *If λ is a prime ideal of $\mathcal{O}_{F_m}$ not dividing n, the λ-component of $[\kappa^\bullet(n)_\chi]_{m,N,\chi}$ is 0. In particular, if $q \in \mathcal{S}_N$ is a prime number not dividing n, we have*
$$[\kappa^\bullet(n)_\chi]_{m,N,\chi}^q = 0.$$
(See [Grei] Lemma 3.6 and [Ru2] Proposition 2.4.)

(ii) *Let ℓ be a prime number dividing n. Then, we have*
$$[\kappa^\bullet(n)_\chi]_{m,N,\chi}^\ell = \bar{\phi}_{m,N,\chi}^\ell(\kappa^\bullet(n/\ell)_\chi).$$
(See [Grei] Lemma 3.6 and [Ru2] Proposition 2.4.)

(iii) *If n is well-ordered, then we have*
$$\bar{\phi}_{m,N,\chi}^\ell(\kappa^\bullet(n)_\chi) = 0$$
for each prime number ℓ dividing n. (See [Ku] Lemma 6.3.)

5.2. In this subsection, we will define Kurihara's elements $x_{\nu,q} \in (F_m^\times/p^N)_\chi$ which become a key of the proof of Theorem 7.1.

We fix circular units

$$\eta_m(n) := \prod_{d \mid \mathfrak{f}_K} \eta_m^d(n)^{u_d} \times \prod_{i=1}^{r} \eta_m^{1,a_i}(n)^{v_i} \in F_m(\mu_n)^\times$$

for any $m \in \mathbb{Z}_{\geq 0}$ and any $n \in \mathbb{Z}_{\geq 1}$ with $(n, p\mathfrak{f}_K) = 1$, where $r \in \mathbb{Z}_{>0}$, u_d and v_i are elements of $\mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$ for positive integers d and i with $d \mid \mathfrak{f}_K$ and $1 \leq i \leq r$, and $a_1, \dots, a_r$ are integers prime to p. Here, we assume that r, $a_1, \dots, a_r$, u_d's and v_i's are constant independent of m and n. Then, we put

$$\kappa_{m,N}(\eta; n) := \prod_{d \mid \mathfrak{f}_K} \kappa_{m,N}^d(n)^{u_d} \times \prod_{i=1}^{r} \kappa_{m,N}^{1,a_i}(n)^{v_i} \in F_m^\times/p^N.$$

Note that the χ-part of $\kappa_{m,N}(\eta; n)$ is the Kolyvagin derivative of the Euler system

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

of circular units. If no confusion arises, we write $\kappa(n) = \kappa_{m,N}(\eta; n)$ for simplicity.

Definition 5.4. Let $q\nu = q\prod_{i=1}^{r} \ell_i \in \mathcal{N}_N$, where $q, \ell_1, \dots, \ell_r$ are distinct prime numbers. For any $e \in \mathbb{Z}_{>0}$ dividing ν, we define $\tilde{\kappa}_{\{e,q\}} \in (F_m^\times/p^N)_\chi \otimes (\bigotimes_{\ell \mid e} H_\ell)$ by

$$\tilde{\kappa}_{\{e,q\}} := \kappa(qe)_\chi \otimes \Big(\bigotimes_{\ell \mid e} \sigma_\ell\Big).$$

Recall that we have fixed a generator σ_ℓ of H_ℓ in the beginning of §4.2.

Let $q\nu \in \mathcal{N}_N$ and assume $q\nu$ is *well-ordered*. Assume that for each prime number ℓ dividing ν, an element $w_\ell \in R_{m,N,\chi} \otimes H_\ell$ is given. Let $\bar{w}_\ell \in R_{m,N,\chi}$ be an element such that $w_\ell = \bar{w}_\ell \otimes \sigma_\ell$. Note that we will take $\{w_\ell\}_{\ell \mid \nu}$ explicitly later, but we take arbitrary elements here. For any $e \in \mathbb{Z}_{>0}$ dividing ν, we define

$$w_e := \bigotimes_{\ell \mid e} w_\ell \in R_{m,N,\chi} \otimes \Big(\bigotimes_{\ell \mid e} H_\ell\Big).$$

We also define the element $\bar{w}_e \in R_{m,N,\chi}$ by $w_e = \bar{w}_e \otimes \big(\bigotimes_{\ell \mid e} \sigma_\ell\big)$.

Note that we write the group law of $(F_m^\times/p^N)_\chi \otimes \big(\bigotimes_{\ell \mid e} H_\ell\big)$ multiplicatively.

Definition 5.5. We define the element $\tilde{x}_{\nu,q}$ by

$$\tilde{x}_{\nu,q} := \prod_{e \mid \nu} w_e \otimes \tilde{\kappa}_{\{\nu/e,q\}} \in (F_m^\times/p^N)_\chi \otimes \Big(\bigotimes_{\ell \mid \nu} H_\ell\Big).$$

The element $x_{\nu,q} \in (F_m^\times/p^N)_\chi$ is defined by $\tilde{x}_{\nu,q} = x_{\nu,q} \otimes \big(\bigotimes_{\ell \mid \nu} \sigma_\ell\big)$.

The following formulas follow from Proposition 5.3 immediately.

Proposition 5.6 (cf. [Ku] Proposition 6.1)**.** *Let $q\nu \in \mathcal{N}_N$ and we assume that $q\nu$ is well-ordered.*

(i) *If λ is a prime ideal of $\mathcal{O}_{F_m}$ not dividing n, the λ-component of $[x_{\nu,q}]_{m,N,\chi}$ is 0. In particular, if s is a prime number not dividing $q\nu$, we have*

$$[x_{\nu,q}]^{s}_{m,N,\chi} = 0.$$

(ii) *Let ℓ be a prime number dividing ν. Then, we have*

$$[x_{\nu,q}]^{\ell}_{m,N,\chi} = \bar{\phi}^{\ell}_{m,N,\chi}(x_{\nu/\ell,q}).$$

(iii) *Let ℓ be a prime number dividing ν. Then, we have*

$$\bar{\phi}^{\ell}_{m,N,\chi}(x_{\nu,q}) = \bar{w}_{\ell}\bar{\phi}^{\ell}_{m,N,\chi}(x_{\nu/\ell,q}).$$

6. Chebotarev Density Theorem

Recall that we have fixed a family of embeddings $\{\, \ell_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell \,\}_{\ell:\mathrm{prime}}$ satisfying the technical condition (Chb) for families of embeddings (see Introduction). We need the condition (Chb) in the proof of Proposition 6.1. As in the previous section, we fix integers m and N satisfying $N \geq m+1 > 0$.

Let $\chi \in \widehat{\Delta}$ be any element. Recall that we denote the restriction of χ to Δ_0 by χ_0, and we put

$$a_\chi := \begin{cases} 0 & \text{if } \chi_0(p) \neq 1; \\ 2 & \text{if } \chi_0(p) = 1. \end{cases}$$

Here, we shall prove Proposition 6.1, which plays a key role in the proof of Theorem 7.1. This proposition corresponds to Lemma 9.1 in [Ku].

Proposition 6.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Assume $qn = q\prod_{i=1}^{r} \ell_i \in \mathcal{N}_N$, where $q, \ell_1, \dots, \ell_r$ are prime numbers. Suppose the following are given:*

- *an $R_{m,N,\chi}$-submodule W of $(F_m^\times/p^N)_\chi$ of finite order;*
- *an $R_{m,N,\chi}$-homomorphism $\psi \colon W \longrightarrow R_{m,N,\chi}$.*

Then, there exist infinitely many $q' \in \mathcal{S}_N$ which split completely in $F_m(\mu_n)/\mathbb{Q}$, and satisfy all of the following properties.

(i) *the class of q'_{F_m} in $A_{m,\chi}$ coincides with the class of $|\Delta_p| \cdot q_{F_m}$. (Recall that we write the group law of $\mathcal{I}_F$ additively.)*

(ii) *there exists an element $z \in (F_m^\times \otimes \mathbb{Z}_p)_\chi$ such that*
   - *$(z)_{m,\chi} = (q'_{F_m} - |\Delta_p| \cdot q_{F_m})_\chi \in \left(\mathcal{I}_{F_m} \otimes \mathbb{Z}_p\right)_\chi$,*
   - *$\phi^{\ell_i}_{m,N,\chi}(z) = 0$ for each $i = 1, \dots, r$.*

(iii) *the group W is contained in the kernel of $[\cdot]^{q'}_{m,N,\chi}$, and*

$$\psi(x) = |\Delta_p|^2\, \bar{\phi}^{q'}_{m,N,\chi}(x)$$

for any $x \in W$.

Proof. The proof of this proposition is essentially the same as that of Lemma 9.1 in [Ku] though we need careful arguments when $|\Delta_p| \neq 1$. We shall prove this proposition in four steps.

The first step. Let v be a prime ideal of $\mathcal{O}_{F_m}$. We denote the ring of integers of the completion $F_{m,v}$ of F_m at v by $\mathcal{O}_{F_{m,v}}$, and define the subgroup $\mathcal{O}^1_{F_{m,v}}$ of $\mathcal{O}^\times_{F_{m,v}}$ by

$$\mathcal{O}^1_{F_{m,v}} := \{x | x \equiv 1 \pmod{\mathfrak{m}_v}\},$$

where $\mathfrak{m}_v$ is the maximal ideal of $\mathcal{O}_{F_{m,v}}$. We denote the residue field of F_m at v by $k(v)$. Let $F_m\{n\}$ be the maximal abelian p-extension of F_m unramified outside n. By global class field theory, we have the isomorphism

$$\frac{(\prod_{v|n} F^\times_{m,v}/\mathcal{O}^1_{F_{m,v}}) \times (\bigoplus_{u\nmid n} F^\times_{m,u}/\mathcal{O}^\times_{F_{m,u}})}{F^\times_m} \otimes \mathbb{Z}_p \xrightarrow{\simeq} \mathrm{Gal}\left(F_m\{n\}/F_m\right),$$

where u runs through all finite places not dividing n. Note that this isomorphism is compatible with the natural actions of $\mathrm{Gal}(F_m/\mathbb{Q})$. We denote by $F_m\{n\}_\chi$ the intermediate field of $F_m\{n\}/F_m$ with $\mathrm{Gal}\left(F_m\{n\}_\chi/F_m\right) = \mathrm{Gal}(F_m\{n\}/F_m)_\chi$, and $L := F_m\{n\}_\chi K(\mu_{np^N})$ the composite field. Clearly, the cokernel of the natural homomorphism

$$\mathrm{Gal}(L/F_m) \longrightarrow \mathrm{Gal}(L/F_m)_\chi \times \mathrm{Gal}(L/F_m)_1$$

is annihilated by $|\Delta_p|$ since $\chi \neq 1$. Note that the subgroup Δ of $\mathrm{Gal}(F_m/\mathbb{Q})$ acts on $\mathrm{Gal}\left(F_m\{n\}_\chi/F_m\right)$ via χ, so $\mathrm{Gal}\left(F_m\{n\}_\chi/F_m\right)$ is a quotient of $\mathrm{Gal}(L/F_m)_\chi$. On the other hand, since Δ acts on $\mathrm{Gal}\left(K(\mu_{np^N})/F_m\right)$ via the trivial character, $\mathrm{Gal}\left(K(\mu_{np^N})/F_m\right)$ is a quotient of $\mathrm{Gal}(L/F_m)_1$. Then, the cokernel of the natural homomorphism

$$\mathrm{Gal}(L/F_m) \longrightarrow \mathrm{Gal}\left(F_m\{n\}_\chi/F_m\right) \times \mathrm{Gal}\left(K(\mu_{np^N})/F_m\right)$$

is annihilated by $|\Delta_p|$. We take an element $\sigma \in \mathrm{Gal}\left(L/K(\mu_{np^N})\right)$ such that

$$\sigma|_{F_m\{n\}_\chi} = (q_{F_m\{n\}_\chi}, F_m\{n\}_\chi/F_m)^{|\Delta_p|}.$$

The second step. We fix a finite $R_{m,N}$-submodule $\mathcal{W}$ of $F^\times_m/p^N$ whose image in $(F^\times_m/p^N)_\chi$ is W. By Corollary 3.3, we take a homomorphism $\tilde{\psi} \in \mathrm{Hom}_{R_{m,N}}(\mathcal{W}, R_{m,N})$ which makes the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\tilde{\psi}} & R_{m,N} \\ \downarrow & & \downarrow \\ W & \xrightarrow{|\Delta_p|^2\psi} & R_{m,N,\chi} \end{array}$$

commute. We define a projection $\mathrm{pr}\colon R_{m,N} \longrightarrow \mathbb{Z}/p^N\mathbb{Z}$ by

$$\sum_{g\in \mathrm{Gal}(F_m/\mathbb{Q})} a_g g \longmapsto a_1,$$

where $a_g \in \mathbb{Z}/p^N\mathbb{Z}$ for all $g \in \mathrm{Gal}(F_m/\mathbb{Q})$, and $1 \in \mathrm{Gal}(F_m/\mathbb{Q})$ is the identity element. This projection induces an *isomorphism*

$$P\colon\ \mathrm{Hom}_{R_{m,N}}(\mathcal{W}, R_{m,N}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathcal{W}, \mathbb{Z}/p^N\mathbb{Z})$$

given by $f \longmapsto \mathrm{pr} \circ f$. We define a homomorphism

$$(P\tilde{\psi})_1 \colon \mathcal{W} \longrightarrow \mu_{p^N}$$

of abelian groups by $x \longmapsto (\zeta_{p^N})^{P(\tilde{\psi})(x)}$.

We denote the image of $\mathcal{W}$ in $K(\mu_{p^N})^\times/p^N$ by $\mathcal{W}'$. Let M be the extension field of $K(\mu_{p^N})$ generated by all p^N-th roots of elements of $K(\mu_{p^N})^\times$ whose image in $K(\mu_{p^N})^\times/p^N$ are contained in $\mathcal{W}'$. So, the Kummer pairing induces an isomorphism

$$\mathrm{Kum}\colon \mathrm{Gal}\left(M/K(\mu_{p^N})\right) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{W}', \mu_{p^N}).$$

Note that the complex conjugation c acts on $H^1\left(K(\mu_{p^N})/F_m, \mu_{p^N}\right)$ by -1, and c acts on $F_m^\times/p^N$ trivially. Since we assume $p \neq 2$, the group $H^1\left(K(\mu_{p^N})/F_m, \mu_{p^N}\right)$ vanishes, and the natural homomorphism $F_m^\times/p^N \longrightarrow K(\mu_{p^N})^\times/p^N$ is injective. This implies that the natural homomorphism

$$i \colon \mathcal{W} \twoheadrightarrow \mathcal{W}'$$

is an isomorphism. We take $\lambda \in \mathrm{Gal}(M/K(\mu_{p^N}))$ such that

$$i^* \circ \mathrm{Kum}(\lambda) = (P\tilde{\psi})_1.$$

The third step. Recall that $LM/K(\mu_{p^N})$ is an abelian p-extension, so we regard $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)$ as a $\mathbb{Z}_p[\mathrm{Gal}\left(K(\mu_{p^N})/\mathbb{Q}\right)]$-module. Since we assume $p \neq 2$, we have a natural isomorphism:

$$\mathrm{Gal}\left(LM/K(\mu_{p^N})\right) \xrightarrow{\simeq} \mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_+ \times \mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_-,$$

where $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_+$ (resp. $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_-$) denotes the maximal quotient of $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)$ on which the complex conjugation c acts trivially (resp. by -1). We put $\widetilde{\Delta} := \mathrm{Gal}\left(\mathbb{Q}(\mu_p)/\mathbb{Q}\right)$, and regard $\widetilde{\Delta}$ as a subgroup of $\mathrm{Gal}\left(\mathbb{Q}(\mu_{p^N})/\mathbb{Q}\right)$. Note that on the one hand, $\mathrm{Gal}\left(L/K(\mu_{p^N})\right)$ is a quotient of $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_+$ since $\widetilde{\Delta}$ acts trivially on $\mathrm{Gal}\left(L/K(\mu_{p^N})\right)$. On the other hand, the complex conjugation c acts on $\mathrm{Gal}\left(M/K(\mu_{p^N})\right)$ by -1 since $\widetilde{\Delta}$ acts on $\mathrm{Gal}\left(M/K(\mu_{p^N})\right)$ via the character $\chi^{-1}\omega$. This implies $\mathrm{Gal}(M/K(\mu_{p^N}))$ is a quotient of $\mathrm{Gal}\left(LM/K(\mu_{p^N})\right)_-$. Therefore, a natural injective homomorphism

$$\mathrm{Gal}\left(LM/K(\mu_{p^N})\right) \hookrightarrow \mathrm{Gal}\left(L/K(\mu_{p^N})\right) \times \mathrm{Gal}\left(M/K(\mu_{p^N})\right)$$

is also surjective. By the condition (Chb), there exist infinitely many prime numbers q' such that

$$\begin{cases} (q'_L, L/K(\mu_{p^N})) = \sigma \\ (q'_M, M/K(\mu_{p^N})) = \lambda^{-1}. \end{cases} \tag{6}$$

The fourth step. Let q' be a prime number satisfying the condition (6), and assume that q' is unramified in $L/\mathbb{Q}$. Here, let us show that such q' satisfies the conditions (i)–(iii) of Proposition 6.1. First, we show q' satisfies conditions (i) and (ii). Let $\alpha = (\alpha_v)_v \in \mathbb{A}_{F_m}^\times$ be an idele whose q'_{F_m}-component is a uniformizer of $F_{m,q'_{F_m}}$, and other components are 1. We define $\beta = (\beta_v)_v \in \mathbb{A}_{F_m}^{\times}$ be an element as follows. The components above q are given by

$$(\beta_v)_{v|q} \in \prod_{v|q} F_{m,v}^{\times},$$

where $\beta_{q_{F_m}}$ is the $|\Delta_p|$-th power of a uniformizer of $F_{m,q_{F_m}}$, and $\beta_v = 1$ otherwise. For all places v of F_m not above q, we put $\beta_v = 1$.

By definition, the ideles α and β have the same image in the group

$$\left(\frac{(\prod_{v|n} F_{m,v}^{\times}/\mathcal{O}_{F_{m,v}}^{1}) \times (\bigoplus_{u\nmid n} F_{m,u}^{\times}/\mathcal{O}_{F_{m,u}}^{\times})}{F_m^{\times}} \otimes \mathbb{Z}_p \right)_{\chi} \simeq \mathrm{Gal}\left(F_m\{n\}_{\chi}/F_m\right).$$

This implies there exists an element $z \in (F_m^{\times} \otimes \mathbb{Z}_p)_{\chi}$ such that

$$\alpha = z\beta \ \text{ in } \left(((\prod_{v|n} F_{m,v}^{\times}/\mathcal{O}_{F_{m,v}}^{1}) \times (\bigoplus_{u\nmid n} F_{m,u}^{\times}/\mathcal{O}_{F_{m,u}}^{\times})) \otimes \mathbb{Z}_p \right)_{\chi}.$$

Hence, we have $(z)_{F_{m,\chi}} = (q'_{F_m} - |\Delta_p| \cdot q_{F_m})_{\chi}$, and $\phi_{m,N,\chi}^{\ell_i}(z) = 0$ for each $i = 1, \dots, r$. Obviously, the prime number q' satisfies the conditions (i) and (ii).

Next, we shall prove q' also satisfies the condition (iii). Recall that we have

$$(q'_M, M/K(\mu_{p^N})) = \lambda^{-1}.$$

So, by the definition of λ, we have

$$(\zeta_{p^N})^{P(\tilde{\psi})(x)} = (x^{1/p^N})^{1-\mathrm{Fr}_{q'}},$$

for any $x \in W$, where we put

$$\mathrm{Fr}_{q'} := (q'_M, M/K(\mu_{p^N})) \in \mathrm{Gal}\left(M/K(\mu_{p^N})\right),$$

and $x^{1/p^N} \in L$ is a p^N-th root of x. Since q' is unramified in $M/\mathbb{Q}$, the group W is contained in the kernel of $[\cdot]_{m,N,\chi}^{q'}$. So, we obtain

$$(\zeta_{p^N})^{P(\tilde{\psi})(x)} \equiv x^{(q'-1)/p^N} \pmod{q'}.$$

We take an intermediate field F of $F_m(\mu_{q'})/F_m$ whose degree over F_m is p^N. Since $q' \equiv 1 \pmod{p^N}$, such a field F exists, and it is unique. We denote the image of $\sigma_{q'} \in H_{q'}$ in

$$H_{q'} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{Gal}(F/F_m)$$

by $\bar{\sigma}_{q'}$. Let π be a uniformizer of $F_{q'_F}$. By the definition of $\sigma_{q'}$, we have

$$\pi^{\bar{\sigma}_{q'}-1} \equiv \zeta_{p^N} \pmod{\mathfrak{m}_{q'}},$$

where $\mathfrak{m}_{q'}$ is the maximal ideal of $F_{q'_F}$. Recall that W is contained in the kernel of $[\cdot]_{m,N,\chi}^{q'}$. We put

$$\phi(x) := \bar{\sigma}_{q'}^{P(\bar{\phi}_{m,N}^{q'})(x)} \in \mathrm{Gal}(F/F_m).$$

By [Se] Chapter XIV Proposition 6, we have

$$(\zeta_{p^N})^{P(\bar{\phi}_{m,N}^{q'})(x)} = \pi^{\phi(x)-1} \equiv x^{(1-q')/p^N} \pmod{\mathfrak{m}_{q'}}$$

for all $x \in W$. Hence, we obtain

$$(\zeta_{p^N})^{P(\tilde{\psi})(x)} = (\zeta_{p^N})^{P(\bar{\phi}^{q'}_{m,N})(x)}$$

for all $x \in W$. Therefore q' satisfies the condition (iii). □

7. Euler system arguments via Kurihara's elements

In this section, we prove the assertions of Theorem 1.1 on the upper bounds of the higher Fitting ideals by using Kurihara's Euler system arguments ([Ku]). Let us state Theorem 7.1, which is the goal of this section. Let the ideals $I_{P^F_\chi}$ and $J_{P^E_\chi}$ of Λ_χ be as in Proposition 3.6, the ideal I_C as in §4.3, and the Λ_χ-submodule Y of X_χ as in Proposition 3.8. We denote the ideal of Λ_χ generated by i-th power of elements of $\mathrm{ann}_{\Lambda_\chi}(Y/(\gamma-1)X)_\chi$ by I_i for each $i \in \mathbb{Z}_{\geq 0}$.

Theorem 7.1. *Let $\chi \in \widehat{\Delta}$ be a non-trivial character. Then, the following hold.*

(i) $\mathfrak{C}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
(ii-0) $(\gamma-1)^{a_\chi}\,|\Delta_p|^4\, I_C I_{P^F_\chi} J_{P^E_\chi}\,\mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi) \subseteq \mathfrak{C}_{0,\chi}$.
(ii-i) $(\gamma-1)^{a_\chi}\,|\Delta_p|^{6+4i}\, I_i I_C I_{P^F_\chi} J_{P^E_\chi}\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$ *for any* $i \in \mathbb{Z}_{\geq 1}$.

In particular, we have the following estimates.

(i) $\mathfrak{C}_{0,\chi} \prec \mathrm{Fitt}_{\Lambda_\chi,0}(X_\chi)$.
(ii-0) $(\gamma-1)^{a_\chi}\,|\Delta_p|^4\,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$.
(ii-i) $(\gamma-1)^{a_\chi}\,|\Delta_p|^{6+4i}\,\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \prec \mathfrak{C}_{i,\chi}$ *for any* $i \in \mathbb{Z}_{\geq 1}$.

Note that we assume neither $p \nmid [K:\mathbb{Q}]$ nor $\chi(p) \neq 1$. But assuming these conditions, we get the simpler statments. The assertions of Theorem 1.1 on the upper bounds of the higher Fitting ideals are special cases of Theorem 7.1.

Corollary 7.2. *Assume that the extension degree of $K/\mathbb{Q}$ is prime to p, and $\chi \in \widehat{\Delta}$ is a character satisfying $\chi(p) \neq 1$. Then, we have the following.*

(i) $\mathfrak{C}_{0,\chi} \subseteq \mathrm{Fitt}_{\Lambda_\chi,0}(X'_\chi)$.
(ii) $\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})\,\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \subseteq \mathfrak{C}_{i,\chi}$ *for any* $i \in \mathbb{Z}_{\geq 0}$.

Proof. Here, let us deduce Corollary 7.2 from Theorem 7.1. Under the assumption of Corollary 7.2, we have $|\Delta_p| = 1$, and we can take $I_i = I_C = I_{P^F_\chi} = \Lambda_\chi$ and $J_{P^E_\chi} = \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$ by Proposition 3.7, Proposition 3.9 and Proposition 4.5. Corollary 7.2 follows from these results. □

7.1. We spend this subsection on the setting of notations. We assume that $\chi \in \widehat{\Delta}$ is non-trivial. Let $\mathfrak{m}_\chi$ be the maximal ideal of Λ_χ. Recall that we denote by $X_{\chi,\mathrm{fin}}$ the maximal pseudo-null submodule of X_χ, and put $X'_\chi := X_\chi/X_{\chi,\mathrm{fin}}$. Since X'_χ has no non-trivial pseudo-null submodules, we have an exact sequence

$$0 \longrightarrow \Lambda_\chi^h \xrightarrow{f} \Lambda_\chi^h \xrightarrow{g} X'_\chi \longrightarrow 0, \tag{7}$$

by Lemma 2.6. Let M be the matrix corresponding to f with respect to the standard basis $(\mathbf{e}_i)_{i=1}^h$ of Λ_χ^h. We may assume that all entries of M are contained in $\mathfrak{m}_\chi$. In particular, we have

$$\mathbf{e}_i - a\mathbf{e}_j \notin \operatorname{Ker} g \tag{8}$$

for all $i, j \in \mathbb{Z}$ with $1 \leq i \neq j \leq h$ and all $a \in \Lambda_\chi$.

Let $\{m_1, ..., m_h\}$ and $\{n_1, ..., n_h\}$ be permutations of $\{1, ..., h\}$, and let i be an integer satisfying $1 \leq i \leq h-1$. Let us consider the matrix M_i which is obtained from M by eliminating the n_j-th rows $(j = 1, ..., i)$ and the m_k-th columns $(k = 1, ..., i)$. If $\det(M_i) = 0$, this is trivial, so we assume that $\det(M_i) \neq 0$. If necessary, we permute $\{m_1, ..., m_i\}$, and assume $\det(M_r) \neq 0$ for all integers r satisfying $0 \leq r \leq i$.

We fix a sequence $\{N_m\}_{m \in \mathbb{Z}_{\geq 0}}$ of positive integers satisfying $N_{m+1} > N_m > m+1$ and $p^{N_m} > |A_m|$ for any $m \in \mathbb{Z}_{\geq 0}$. First, we fix a sufficiently large integer m. For simplicity, we put $F := F_m$, $R := \mathbb{Z}_p[\mathrm{Gal}(F_m/\mathbb{Q})]_\chi$, $N := N_m$ and $R_N := R_{m,N,\chi} = \mathbb{Z}/p^N[\mathrm{Gal}(F_m/\mathbb{Q})]_\chi$. By taking the Γ_m-coinvariants of the exact sequence (7), we obtain the exact sequence

$$0 \longrightarrow R^h \xrightarrow{\bar{f}} R^h \xrightarrow{\bar{g}} X'_{\chi,\Gamma_m} \longrightarrow 0.$$

Note that the map $\bar{f}$ is injective since X'_{χ,Γ_m} has finite order. Let $A_{m,\chi,\mathrm{fin}}$ be the image of $X_{\chi,\mathrm{fin}}$ in $A_{m,\chi}$ by the natural homomorphism

$$X_\chi \twoheadrightarrow X_{\Gamma_m,\chi} \twoheadrightarrow A_m.$$

We put $A'_{m,\chi} := A_{m,\chi}/A_{\mathrm{fin},\chi}$. The image of $\mathbf{e}_r$ in R^h is denoted by $\mathbf{e}_r^{(m)}$. For each $i \in \mathbb{Z}$ with $1 \leq i \leq h$, we denote by $\mathbf{c}_r^{(m)} \in A'_{m,\chi}$ the image of $\mathbf{e}_r^{(m)}$ by the homomorphism

$$R^h \overset{\bar{g}}{\twoheadrightarrow} X'_{\chi,\Gamma_m} \twoheadrightarrow A'_m.$$

We fix a lift $\tilde{\mathbf{c}}_r^{(m)} \in A_{m,\chi}$ of $\mathbf{c}_r^{(m)}$. The condition (8) and Proposition 3.8 imply that if necessary, we replace m with larger one, and we may assume $\tilde{\mathbf{c}}_r^{(m)} \neq \tilde{\mathbf{c}}_s^{(m)}$ for any $r, s \in \mathbb{Z}$ with $1 \leq r, s \leq h$ and $r \neq s$. If the extension degree of $K/\mathbb{Q}$ is divisible by p, then we additionally assume that $\tilde{\mathbf{c}}_r^{(m)} \neq |\Delta_p| \cdot \tilde{\mathbf{c}}_s^{(m)}$ for any $r, s \in \mathbb{Z}$ satisfying $1 \leq r, s \leq h$. We define

$$P_r := \{\ell \in \mathcal{S}_N | [\ell_F]_\chi = \tilde{\mathbf{c}}_r^{(m)}\};$$
$$P'_r := \{\ell' \in \mathcal{S}_N | [\ell'_F]_\chi = |\Delta_p| \cdot \tilde{\mathbf{c}}_r^{(m)}\},$$

where $[\ell_F]_\chi$ is the ideal class of ℓ_F in $A_{m,\chi}$. By the hypothesis (Chb), note that P_r and P'_r are not empty for all r. We define a set P of prime numbers by the union

$$P := \coprod_{r=1}^{i} (P_r \cup P'_r),$$

and we denote by P_F the set of all prime ideals of $\mathcal{O}_F$ above primes contained in P. By definition, we have $P_r = P'_r$ if $\Delta_p = 0$. Otherwise $P_r \cap P'_r = \emptyset$.

Let J be the subgroup of $\mathcal{I}_F$ generated by P_F, and $\mathscr{J}$ the image of $(J \otimes \mathbb{Z}_p)_\chi$ in $(\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi$. We denote by $\mathcal{F}$ the inverse image of $\mathscr{J}$ by the homomorphism

$$(\cdot)_{F,\chi} \colon (F^\times \otimes \mathbb{Z}_p)_\chi \longrightarrow (\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi.$$

We define a surjective homomorphism

$$\alpha \colon \mathscr{J} \longrightarrow R^h$$

by $\ell_F \mapsto \mathbf{e}_r$ (resp. $\ell'_F \mapsto |\Delta_p| \cdot \mathbf{e}_r$) for each integer r with $1 \leq r \leq h$ and each $\ell \in P_r$ (resp. $\ell' \in P'_r$). We put the composite map

$$\alpha_r := \mathrm{pr}_r \circ \alpha \colon \mathscr{J} \xrightarrow{\alpha} R^h \xrightarrow{\mathrm{pr}_r} R,$$

where pr_r is the r-th projection. We consider the following diagram

$$\begin{tikzcd}
 & \mathcal{F} \arrow[r, "(\cdot)_{F,\chi}"] & \mathscr{J} \arrow[r, two heads] \arrow[d, "\alpha"] & A'_{m,\chi} & \\
0 \arrow[r] & R^h \arrow[r, "\bar{f}"] & R^h \arrow[r, "\bar{g}"] & X'_{\chi,\Gamma_m} \arrow[r] \arrow[u, "\iota_A"'] & 0,
\end{tikzcd}$$

where ι_A is induced by the canonical homomorphism. We fix a non-zero element $\varepsilon \in \mathrm{ann}_{\Lambda_\chi}(Y/(\gamma - 1)X)$. (Note that by Proposition 3.9, if $\Delta_p = 0$ and $\chi(p) \neq 1$, then we may assume $\varepsilon = 1$.) By Lemma 3.8, we can define the homomorphism $\beta \colon \mathcal{F} \longrightarrow R^h$ which makes the diagram

(9)
$$\begin{tikzcd}
 & \mathcal{F} \arrow[r, "(\cdot)_{F,\chi}"] \arrow[d, dashed, "\beta"] & \mathscr{J} \arrow[r, "\varepsilon\cdot\pi'_A"] \arrow[d, "\varepsilon\cdot\alpha"] & A'_{m,\chi} & \\
0 \arrow[r] & R^h \arrow[r, "\bar{f}"] & R^h \arrow[r, "\bar{g}"] & X'_{\chi,\Gamma_m} \arrow[r] \arrow[u, "\iota_A"'] & 0
\end{tikzcd}$$

commute, where π'_A is the natural homomorphism. Note that since the second row of the diagram is exact, β is well-defined. We put the composite map

$$\beta_r := \mathrm{pr}_r \circ \beta \colon \mathcal{F} \xrightarrow{\beta} R^h \xrightarrow{\mathrm{pr}_r} R.$$

Let us consider the diagram (9) by taking $(- \otimes \mathbb{Z}/p\mathbb{Z})$. We regard $(F^\times/p^N)_\chi$ as a Λ_χ-module. For any element $x \in (F^\times/p^N)_\chi$ and $\delta \in \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$, we denote the scaler multiple of x by $\delta \in \Lambda_\chi$ by x^δ. We need the following two lemmas, namely Lemma 7.3 and 7.4.

Lemma 7.3. *The kernel of the natural homomorphism*

$$\iota_{\mathcal{F},N} \colon \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$$

is annihilated by $|\Delta_p|$.

Proof. Let x be an element in the kernel of the homomorphism

$$\iota_{\mathcal{F},N} \colon \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$$

and $\tilde{x}$ a lift of x in $\mathcal{F}$. Then, there exists $y \in F^\times \otimes \mathbb{Z}_p$ such that $\tilde{x} = y_\chi{}^{p^N}$. The $\mathbb{Z}_p$-module $(\mathcal{I}_F \otimes \mathbb{Z}_p)/(J \otimes \mathbb{Z}_p)$ is torsion free, so Corollary 3.2 implies that all $\mathbb{Z}_p$-torsion elements of $(\mathcal{I}_F \otimes \mathbb{Z}_p)_\chi/\mathcal{J}$ are annihilated by $|\Delta_p|$. Since $(\tilde{x})_{F,\chi} \in \mathcal{J}$, we have $(y^{|\Delta_p|})_{F,\chi} \in \mathcal{J}$. Therefore we have $y^{|\Delta_p|} \in \mathcal{F}$, and we obtain $x^{|\Delta_p|} = 1$. $\square$

We denote the image of the natural homomorphism $\iota_{\mathcal{F},N} \colon \mathcal{F}/p^N \longrightarrow (F^\times/p^N)_\chi$ by $\bar{\mathcal{F}}_N$. By Lemma 7.3, there exists an R_N-homomorphism

$$\tilde{\beta}_{r,N} \colon \bar{\mathcal{F}}_N \longrightarrow R_N$$

which makes the diagram

$$\begin{tikzcd} \mathcal{F}/p^N \arrow[r, two heads, "\iota_{\mathcal{F},N}"] \arrow[dr, "|\Delta_p|\cdot\bar{\beta}_{r,N}"'] & \bar{\mathcal{F}}_N \arrow[d, dashed, "\tilde{\beta}_{r,N}"] \\ & R_N \end{tikzcd}$$

for each integer r with $1 \leq r \leq h$, where $\bar{\beta}_{r,N} \colon \mathcal{F}/p^N \longrightarrow R_N$ is the homomorphism induced by β_r.

Lemma 7.4. *Let $[\cdot]_{F,N,\chi} \colon (F^\times/p^N)_\chi \longrightarrow (\mathcal{I}_F/p^N)_\chi$ be the homomorphism induced by $(\cdot)_F \colon F^\times \longrightarrow \mathcal{I}_F$. Let x be an element of $(F^\times/p^N)_\chi$ such that $[x]_{F,N,\chi} \in \mathcal{J}/p^N$. Then, for any $\delta \in \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$, the element $x^{\delta|\Delta_p|^2}$ is contained in $\bar{\mathcal{F}}_N$.*

Proof. Recall the natural exact sequence:

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I}_F \otimes \mathbb{Z}_p \longrightarrow A_m \longrightarrow 0,$$

where $\mathcal{P}$ is defined by $\mathcal{P} := (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$. By the snake lemma for the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P} \arrow[r] \arrow[d, "\times p^N"] & \mathcal{I}_F \otimes \mathbb{Z}_p \arrow[r] \arrow[d, "\times p^N"] & A_m \arrow[r] \arrow[d, "\times p^N"] & 0 \\ 0 \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{I}_F \otimes \mathbb{Z}_p \arrow[r] & A_m \arrow[r] & 0, \end{tikzcd}$$

we obtain the following exact sequence

$$0 \longrightarrow A_m \longrightarrow \mathcal{P}/p^N \longrightarrow \mathcal{I}_F/p^N \longrightarrow A_m \longrightarrow 0.$$

(Recall we take $p^{N_m} > |A_m|$.)

Let B_m be the image of $J \otimes \mathbb{Z}_p$ in A_m, and $\mathcal{P}_0 \subset \mathcal{P}$ the inverse image of $J \otimes \mathbb{Z}_p$. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{P}_0 \longrightarrow J \otimes \mathbb{Z}_p \longrightarrow B_m \longrightarrow 0,$$

and by a similar argument as above, we obtain the exact sequence

$$0 \longrightarrow B_m \longrightarrow \mathcal{P}_0/p^N \longrightarrow J/p^N \longrightarrow B_m \longrightarrow 0.$$

Now, we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & B_m \arrow[r] \arrow[d, hook] & \mathcal{P}_0/p^N \arrow[r] \arrow[d, hook] & J/p^N \arrow[r] \arrow[d, hook] & B_m \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & A_m \arrow[r] & \mathcal{P}/p^N \arrow[r] & \mathcal{I}_F/p^N \arrow[r] & A_m \arrow[r] & 0 \end{tikzcd} \tag{10}$$

whose two rows are exact, and the vertical arrows are injective. Let $\tilde{\delta}$ be an arbitrary element of $\mathrm{ann}_\Lambda(A_m/B_m)$. Let x be an arbitrary element of $\mathcal{P}/p^N$ satisfying $[x]_N \in J/p^N$. Let us show that x^δ is contained in $\mathcal{P}_0/p^N$. By the diagram (10), there exists an element $y \in \mathcal{P}_0/p^N$ satisfying $[x]_N = [y]_N$. Since $[xy^{-1}]_N = 0$, the element xy^{-1} is contained in the image of A_m. Since δA_m is contained in B_m, $(xy^{-1})^\delta$ is contained in the image of B_m. In particular, we have $(xy^{-1})^\delta \in \mathcal{P}_0/p^N$, and we obtain $x^\delta \in \mathcal{P}_0/p^N$. Combining this result with Lemma 3.1, we obtain the lemma. □

We define the R_N-submodule $\bar{\mathcal{F}}'_N$ of $(F^\times/p^N)_\chi$ by

$$\bar{\mathcal{F}}'_N := \left\{ x^{|\Delta_p|^6} \mid x \in \bar{\mathcal{F}}_N \right\}.$$

Corollary 7.5. *The order of the kernel of* $[\cdot]_{m,N,\chi} \colon \mathcal{F}'_N \longrightarrow \mathcal{J}/p^N$ *is finite.*

Proof. If we assume $\Delta_p = 0$, this corollary immediately follows from the first row of the diagram (10). When $\Delta_p \neq 0$, we have to treat a little more carefully. Let $\mathcal{P}_0 \subseteq (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$ be as in the proof of Lemma 7.4. We denote the inverse image of $\mathcal{P}_0$ by the natural homomorphism

$$F^\times \otimes \mathbb{Z}_p \longrightarrow \mathcal{P} = (F^\times/\mathcal{O}_F^\times) \otimes \mathbb{Z}_p$$

by $\widetilde{\mathcal{P}}_0$. Note that the kernel of the natural homomorphism

$$\mathrm{pr}_{\mathcal{P}_0,\mathrm{N}} \colon \widetilde{\mathcal{P}}_0/p^N \longrightarrow \mathcal{P}_0/p^N$$

coincides with the image of $\mathcal{O}_F^\times/p^N$ in $\widetilde{\mathcal{P}}_0/p^N$. So, the order of $\mathrm{Ker}\,\mathrm{pr}_{\mathcal{P}_0}$ is finite. The top row of the diagram (10) implies that the kernel of the homomorphism

$$[\cdot]'_{m,N} \colon \mathcal{P}_0/p^N \longrightarrow (J/p^N)_\chi$$

induced by the natural homomorphism $[\cdot]_{m,N} \colon F^\times/p^N \longrightarrow (J/p^N)_\chi$ has finite order. Then, the kernel of the composite map

$$[\cdot]'_{m,N} \circ \mathrm{pr}_{\mathcal{P}_0,N} \colon \widetilde{\mathcal{P}}_0/p^N \longrightarrow (J/p^N)_\chi$$

is finite.

Let us consider the commutative diagram

$$\begin{tikzcd} (\widetilde{\mathcal{P}}_0/p^N)_\chi \arrow[rr, two heads, "([\cdot]'_{m,N}\circ \mathrm{pr}_{\mathcal{P}_0,N})_\chi"] \arrow[d, "\iota_1"'] & & (J/p^N)_\chi \arrow[d, "\iota_2"] \\ \mathcal{F}/p^N \arrow[r, two heads, "\iota_{\mathcal{F},N}"] & \bar{\mathcal{F}}_N \arrow[r, "{[\cdot]_{m,N,\chi}}"] & \mathcal{J}/p^N \end{tikzcd} \tag{11}$$

of natural homomorphisms. Lemma 3.1 implies that the $\mathrm{Coker}\,\iota_1$ is annihilated by $|\Delta_p|^2$. By Corollary 3.3, it follows that $\mathrm{Ker}\,\iota_2$ is annihilated by $|\Delta_p|^2$. The finiteness of the kernel of $[\cdot]'_{m,N} \circ \mathrm{pr}_{\mathcal{P}_0,N}$ and Corollary 3.3 imply that the order of $|\Delta_p|^2 \operatorname{Ker}([\cdot]'_{m,N} \circ \mathrm{pr}_{\mathcal{P}_0,N})_\chi$ is finite. Hence we obtain the corollary by chasing the commutative diagram (11). □

Let n be an element of $\mathcal{N}_N$ whose prime divisors are in P. We define P_F^n to be the set of all elements of P dividing n. Let J_n be the subgroup of J generated by P_F^n, and the submodule $\mathcal{J}_{n,N}$ of $\mathcal{J}/p^N$ the image of $J_n \otimes \mathbb{Z}_p$ in $\mathcal{J}/p^N$. We denote by $\bar{\mathcal{F}}_{n,N}$ the inverse image of $\mathcal{J}_{n,N}$ by $[\cdot]_{m,\chi} \colon (F^\times/p^N)_\chi \longrightarrow \mathcal{J}/p^N$. Then, we define the R_N-submodule $\bar{\mathcal{F}}'_{n,N}$ of $(F^\times/p^N)_\chi$

$$\bar{\mathcal{F}}'_{n,N} := \bar{\mathcal{F}}_{n,N} \cap \bar{\mathcal{F}}'_N.$$

Note that $\bar{\mathcal{F}}'_{n,N}$ is a *finite* R_N-submodule of $(F^\times/p^N)_\chi$ by Corollary 7.5.

For each integer r with $1 \leq r \leq h$, let

$$\bar{\alpha}_{r,N} \colon \mathcal{J}_{n,N} \longrightarrow R_N$$

be the R_N-homomorphism induced by α_r. We put

$$\bar{\alpha}_N := (\bar{\alpha}_{s,N})_{s=1}^h \colon \mathcal{J}_{n,N} \longrightarrow R_N^h,$$
$$\tilde{\beta}_N := (\tilde{\beta}_{s,N})_{s=1}^h \colon \bar{\mathcal{F}}_N \longrightarrow R_N^h.$$

Then, we obtain the commutative diagram

$$\begin{tikzcd} \bar{\mathcal{F}}'_{n,N} \arrow[r, "{[\cdot]_{F,\chi}}"] \arrow[d, "\tilde{\beta}_N"'] & \mathcal{J}_{n,N} \arrow[d, "|\Delta_p|\varepsilon\cdot\bar{\alpha}_N"] \\ R_N^h \arrow[r, "\bar{f}"] & R_N^h \end{tikzcd} \tag{12}$$

of R_N-modules.

7.2. Let δ_A be a non-zero element of $\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$. In the present and the next subsections, we write $\bar{\phi}^\ell$ in place of $\bar{\phi}^\ell_{m,N,\chi}$ for simplicity. Here, as in [Ku], we use the elements $x_{\nu,q} \in (F^\times/p^N)_\chi$ introduced in Definition 5.5, in order to translate β_r to homomorphisms of the type $\bar{\phi}^\ell$. Recall the element $x_{\nu,q} \in (F^\times/p^N)_\chi$ is determined by η, q, ν, and $\{w_\ell\}_{\ell|\nu}$. We shall take them as follows.

First, let us take a prime number q as follows. For each integer r with $1 \leq r \leq h$, we fix a prime number $q_r \in P_{n_r}$. We put $Q := \prod_{r=1}^h q_r \in \mathcal{N}_N$. We fix a homomorphism $\varphi \colon E_{\infty,\chi} \longrightarrow \Lambda_\chi$ of Λ_χ-modules with pseudo-null cokernel. By the Iwasawa main conjecture, we have

$$\varphi(\bar{C}_{\infty,\chi}) = \det(M_0) \cdot I_\varphi(E; C),$$

where $\bar{C}_{\infty,\chi}$ is the image of $C_{\infty,\chi}$ in $E_{\infty,\chi}$, and $I_\varphi(E;C)$ is an ideal of Λ_χ of finite index. We fix an element $\delta_\varphi \in I_\varphi(E;C)$. (By Proposition 4.5, if $\Delta_p = 0$ and $\chi(p) = 1$, then we may assume $\varphi(\bar{C}_{\infty,\chi}) = \det(M_0)\Lambda_\chi$ and $\delta_\varphi = 1$.)

We fix a family $(\eta_m)_{m\geq 0} \in C_{\infty,\chi}$ of circular units which is defined by Λ_χ-linear combination of basic circular units, and satisfies $\varphi\big((\eta_m)_{m\geq 0}\big) = \delta_\varphi \det(M_0)$. We write

$$\eta_m = \eta_m(1) := \prod_{d \mid \mathfrak{f}_K} \eta_m^d(1)^{u_d} \times \prod_{j=1}^{r} \eta_m^{1,a_j}(1)^{v_j} \in F_m^\times$$

where r is a positive integer, $a_1, ..., a_r$ are integers prime to p, and u_d and v_j are elements of Λ_χ for each positive integers d and i with $d \mid \mathfrak{f}_K$ and $1 \leq i \leq r$. Here, we assume that r, $a_1, ..., a_r$, u_d's and v_j's are constant independent of m. As in the previous section, we fix m in the present and the next sections, and put $\eta := \eta_m$ for simplicity.

Let $\bar{\varphi}_{F,N,\chi} \colon (E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow R_N$ be the induced homomorphism by $\bar{\varphi}$. As in the proof of Proposition 4.17, we denote by $\mathcal{NO} := \mathcal{NO}_{m,N,\chi}$ the image of the natural homomorphism

$$(E_{\infty,\chi})_{\Gamma_m}/p^N \longrightarrow \big(\mathcal{O}_F^\times/p^N\big)_\chi \longrightarrow \big(F_m^\times/p^N\big)_\chi.$$

We fix non-zero elements $\delta_I \in I_{P_\chi^F}$ and $\delta_J \in J_{P_\chi^E}$. (Note that we may assume $\delta_I = 1$ and $\delta_J \in J_{P_\chi^E} = \mathrm{ann}_{\Lambda_\chi}(X_{\mathrm{fin},\chi})$ if we treat the case when $\Delta_p = 0$ and $\chi(p) = 1$.) By the same argument as in Lemma 4.18, there exists a homomorphism $\psi \colon \mathcal{NO} \longrightarrow R_N$ which makes the diagram

$$\begin{tikzcd}
(C_{\infty,\chi})_{\Gamma_m}/p^N \arrow[r] \arrow[d] & (E_{\infty,\chi})_{\Gamma_m}/p_\chi^N \arrow[r, "(\gamma-1)^{a_\chi}\cdot\delta_I\delta_J|\Delta_p|^4\cdot\bar{\varphi}_{F,N,\chi}"] \arrow[d] & R_N \\
\mathcal{W}_{m,N,\chi}(1) \arrow[r, hook] & \mathcal{NO} \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}$$

commute. By Proposition 6.1, we can take a prime number $q \in \mathcal{S}_N$ satisfying the following two conditions:

(q1) $q \in P'_{n_1} \setminus \{q_1\}$;
(q2) $\mathcal{NO}_{m,N,\chi}$ is contained in the kernel of $[\cdot]^q_{m,N,\chi}$, and for all $x \in \mathcal{NO}_{m,N,\chi}$,

$$\bar{\phi}^q(x) = |\Delta_p|^2\, \psi(x).$$

In particular, we have

$$\begin{aligned}
\bar{\phi}^q(\eta) &= |\Delta_p|^2\, \psi(\eta) \\
&= (\gamma-1)^{a_\chi}|\Delta_p|^6\, \delta_I\delta_J\bar{\varphi}_{m,N,\chi}\big((\eta_m)_{m\geq 0}\big) \\
&= |\Delta_p|^6\, \delta_I\delta_J\delta_\varphi(\gamma-1)^{a_\chi/2}\cdot\det(M_0).
\end{aligned}$$

Next, let us take ν and $\{w_\ell\}_{\ell\mid\nu}$. First, we consider $\tilde{\beta}_{m_1,N} \colon \bar{\mathcal{F}}'_{Qq,N} \longrightarrow R_N$. By Proposition 6.1, we can take $\ell_2 \in \mathcal{S}_N$ which splits completely in $F(\mu_q)/\mathbb{Q}$, and satisfies $\ell_2 \in P'_{n_2}$, $\ell \neq q_2$ and

$$\bar{\phi}^{\ell_2}(x) = |\Delta_p|^2\cdot\tilde{\beta}_{m_1,N}(x)$$

for all $x \in \bar{\mathcal{F}}'_{Qq,N}$. We put $\nu_1 := 1$.

In the case $i = 1$, we put $\nu := \nu_1 = 1$, and $x_{\nu,q} = x_{1,q} = \kappa_{m,N}(\eta; q) = \kappa(q)$. It follows from Proposition 5.6 (i) and Lemma 7.4 that $x_{1,q}^{\delta_A|\Delta_p|^8}$ is an element of $\mathcal{F}'_{Qq,N}$.

Suppose $i \geq 2$, and let r be an integer satisfying $2 < r \leq i+1$. We shall choose prime numbers ℓ_r by induction on r as follows, and put $\nu_{r-1} := \prod_{s=2}^{r-1} \ell_s$. Suppose that we have chosen prime numbers $\ell_1, \dots, \ell_{r-1}$ satisfying the following conditions:

- For any integers s satisfying $2 \leq s \leq r-1$, it holds that $\ell_s \in \mathcal{S}_N$.
- For any integers s satisfying $2 \leq s \leq r-1$, the prime number ℓ_s splits completely in $F(\mu_{q\nu_{s-1}})/\mathbb{Q}$.

We consider the R_N-linear homomorphism

$$\tilde{\beta}_{m_{r-1},N} \colon \bar{\mathcal{F}}'_{Qq\nu_{r-1},N} \longrightarrow R_N.$$

Applying Proposition 6.1, we can take $\ell_r \in \mathcal{S}_N$ which splits completely in $F(\mu_{q\nu_{r-1}})/\mathbb{Q}$, and satisfies the following conditions:

(x1) $\ell_r \in P'_{n_r} \setminus \{q_r\}$;
(x2) there exists $b_r \in (F^\times \otimes \mathbb{Z}_p)_\chi$ such that $(b_r)_{F,\chi} = (\ell_{r,F} - |\Delta_p| \cdot q_{r,F})_\chi$ and $\bar{\phi}^{\ell_s}(b_r) = 0$ for any s with $2 \leq s < r$;
(x3) $\bar{\phi}^{\ell_r}(x) = |\Delta_p|^2 \cdot \tilde{\beta}_{m_{r-1},N}(x)$ for any $x \in \bar{\mathcal{F}}'_{Qq\nu_{r-1},N}$.

Thus, we have taken $\ell_2, \dots, \ell_{i+1}$, and we put $\nu := \nu_i = \prod_{r=2}^{i} \ell_r \in \mathcal{N}_N$. For each r with $2 \leq r \leq i$, we put

$$w_{\ell_r} := -\phi^{\ell_r}(b_r) \in R_N \otimes H_{\ell_r},$$

and we obtain $x_{\nu,q} \in (F^\times/p^N)_\chi$. It follows from Proposition 5.6 (1) and Lemma 7.4 that $x_{\nu,q}^{\delta_A|\Delta_p|^8}$ is an element of $\bar{\mathcal{F}}'_{Qq\nu,N}$. Note that $q\nu$ is *well-ordered.*

7.3. In this subsection, we observe two homomorphism α and β by using $x_{\nu,q}$, and describe $\det(M_i)$ in R_N. Recall that we fix non-zero elements $\delta_A \in \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$, $\varepsilon \in \mathrm{ann}_{\Lambda_\chi}(Y/(\gamma-1)X)$, $\delta_\varphi \in I_\varphi(E; C)$, $\delta_I \in I_{P^F_\chi}$ and $\delta_J \in J_{P^E_\chi}$. Note that if $\Delta_p = 0$ and $\chi(p) \neq 1$, then we may assume $\varepsilon = \delta_\varphi = \delta_\varphi = 1$ and $\delta_J \in J_{P^E_\chi} = \mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}})$.

We need the following lemma.

Lemma 7.6 (cf. [Ku] Lemma 10.2)**.** *Suppose $i \geq 2$. Then,*

(i) $\tilde{\beta}_{m_{r-1},N}(x_{\nu,q}^{\delta_A|\Delta_p|^{10}}) = 0$ *for all r with $2 \leq r \leq i$;*
(ii) $\bar{\alpha}_{j,N}([x_{\nu,q}]_{m,N,\chi}) = 0$ *for any $j \neq n_1, \dots, n_i$.*

Proof. The second assertion (ii) of Lemma 7.6 is clear by Proposition 5.6 (i).

Let us prove the first assertion. We define an element $y_r \in (F^\times/p^N)_\chi$ by

$$y_r = x_{\nu,q} \prod_{s=r}^{i} b_s^{\bar{\phi}^{\ell_s}(x_{\nu/\ell_s,q})}$$

for any r satisfying $2 \le r \le i$. Note that we have $\bar{\alpha}_N([b_r]_{m,N,\chi}) = 0$ for any r satisfying $2 \le r \le i$ since we have $(b_r)_{F,\chi} = (\ell_{r,F} - |\Delta_p| \cdot q_{r,F})_\chi$ and $\ell_r \in P'_{n_r}$. By the definition of β, we have $\beta(b_r) = 0$ for any r satisfying $2 \le r \le i$. So, we have

$$\tilde{\beta}(x_{\nu,q}^{\delta_A |\Delta_p|^8}) = \tilde{\beta}(y_r^{\delta_A |\Delta_p|^8})$$

for any r with $2 \le r \le i$. Let us show $\tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8}) = 0$ for any integer r satisfying $2 \le r \le i$. By Proposition 5.6 (i), we have $[y_r]_{F,N,\chi} \in J_{Qq\nu_{r-1}}$. Then, by Lemma 7.4, we have $y_r^{\delta_A|\Delta_p|^8} \in \mathcal{F}_{Qq\nu_{r-1},N}$. Therefore, we obtain

$$\delta_A \, |\Delta_p|^8 \bar{\phi}^{\ell_r}(y_r) = |\Delta_p|^2 \, \tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8})$$

by the condition (x3). Note that by the condition (x2), we have $\bar{\phi}^{\ell_r}(b_s) = 0$ for all integers s satisfying $r+1 \le s \le i$. So, it holds that

$$\bar{\phi}^{\ell_r}(y_r) = \bar{\phi}^{\ell_r}(x_{\nu,q} b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}).$$

By Proposition 5.6 (iii), we have

$$\begin{aligned}
\bar{\phi}^{\ell_r}(x_{\nu,q} b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}) &= \bar{\phi}^{\ell_r}(x_{\nu,q}) + \bar{\phi}^{\ell_r}(b_r^{\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})}) \\
&= -\bar{\phi}^{\ell_r}(b_r)\bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q}) + \bar{\phi}^{\ell_r}(x_{\nu/\ell_r,q})\bar{\phi}^{\ell_r}(b_r) \\
&= 0.
\end{aligned}$$

Therefore, we obtain

$$\tilde{\beta}_{m_{r-1},N}(x_{\nu,q}^{\delta_A|\Delta_p|^{10}}) = |\Delta_p|^2 \, \tilde{\beta}_{m_{r-1},N}(y_r^{\delta_A|\Delta_p|^8}) = \delta_A \, |\Delta_p|^8 \, \bar{\phi}^{\ell_r}(y_r) = 0.$$

□

The goal of this subsection is the following proposition.

Proposition 7.7 (cf. [Ku] pp.763–764)**.** *The following equalities of elements in R_N holds.*

(i) *We have*

$$\begin{aligned}
&\delta_A \, |\Delta_p|^{10} \det(M)\bar{\phi}^{\ell_2}(x_{1,q}) \\
&\qquad = \pm \, |\Delta_p|^{20} \, (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \det(M_1) \bar{\varphi}_{F,N,\chi}\big((\eta_m)_{m \ge 0}\big).
\end{aligned}$$

(ii) *We have*

$$\delta_A \, |\Delta_p|^{10} \det(M_{r-1})\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q}) = \pm \, |\Delta_p|^{14} \, \delta_A \varepsilon \det(M_r)\bar{\phi}^{\ell_r}(x_{\nu_{r-1},q})$$

for any r with $2 \le r \le i$.

The signs $\pm$ in (i) *and* (ii) *do not depend on m.*

***Proof*.** For each r satisfying $1 \le r \le i$ we put

$$\begin{aligned}
\mathbf{x}^{(r)} &:= \tilde{\beta}_N({x_{\nu_r,q}}^{|\Delta_p|^{10}\delta_A}) \in R_N^h; \\
\mathbf{y}^{(r)} &:= |\Delta_p| \, \varepsilon \bar{\alpha}_N({x_{\nu_r,q}}^{|\Delta_p|^{10}\delta_A}) \in R_N^h,
\end{aligned}$$

and regard them as column vectors. Then, by the commutative diagram (12), we have $\mathbf{y}^{(r)} = M\mathbf{x}^{(r)}$ in R_N^h.

We first prove the assertion (i) of Proposition 7.7. Note that $\delta_A |\Delta_p|^2$ times of $x_{1,q} = \kappa(q)$ is an element of $\mathcal{F}_{q,N}$, and we have

$$\begin{aligned}
\mathbf{y}^{(1)} &= |\Delta_p|^{12} \delta_A \varepsilon [\kappa(q)_\chi]^q_{F,N,\chi} \mathbf{e}^{(m)}_{n_1} \\
&= |\Delta_p|^{12} \delta_A \varepsilon \bar{\phi}^q(\eta_m) \mathbf{e}^{(m)}_{n_1} \\
&= |\Delta_p|^{14} \delta_A \varepsilon \psi(\eta_m) \mathbf{e}^{(m)}_{n_1} \\
&= |\Delta_p|^{18} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \cdot \bar{\varphi}_{F,N,\chi}\big((\eta_m)_{m \geq 0}\big) \mathbf{e}^{(m)}_{n_1}.
\end{aligned}$$

Note that the first equation follows from the condition (q1) and definition of α, the second one follows from Proposition 5.3 (ii), and the third one follows from the condition (q2). Let $\widetilde{M}$ be the matrix of cofactors of M. Multiplying the both sides of $\mathbf{y}^{(1)} = M\mathbf{x}^{(1)}$ by $\widetilde{M}$, and comparing the m_1-st components, we obtain

$$\begin{aligned}
&(-1)^{n_1+m_1} |\Delta_p|^{18} (\gamma - 1)^{a_\chi} \delta_A \delta_I \delta_J \varepsilon \det(M_1) \bar{\varphi}_{F,N,\chi}\big((\eta_m)_{m \geq 0}\big) \\
&= \det(M) \tilde{\beta}_{m_1,N}({x_{1,q}}^{|\Delta_p|^{10}\delta_A}).
\end{aligned}$$

By condition (x3) for ℓ_2, we have

$$|\Delta_p|^2 \tilde{\beta}_{m_1,N}({x_{1,q}}^{|\Delta_p|^{10}\delta_A}) = |\Delta_p|^{10} \delta_A \bar{\phi}^{\ell_2}(x_{1,q}),$$

and the assertion (i) follows.

Next, we assume $i \geq 2$, and we shall prove Proposition 7.7 (ii). The proof is essentially the same as the proof of assertion (i). It is sufficient to prove the assertion when $r = i$. We write $\mathbf{x} = \mathbf{x}^{(i)}$ and $\mathbf{y} = \mathbf{y}^{(i)}$. Let $\mathbf{x}' \in R_N^{h-i+1}$ be the vector obtained from $\mathbf{x}$ by eliminating the m_j-th rows for $j = 1, ..., i-1$, and $\mathbf{y}'$ the vector obtained from $\mathbf{y}$ by eliminating the n_k-th rows for $k = 1, ..., i-1$. Since the m_r-th rows of $\mathbf{x}$ are 0 for all r with $1 \leq r \leq i-1$ by Lemma 7.6 (i), we have $\mathbf{y}' = M_{i-1}\mathbf{x}'$. We assume the m'_i-th component of $\mathbf{x}'$ corresponds to the m_i-th component of $\mathbf{x}$, and the n'_i-th component of $\mathbf{y}'$ corresponds to the n_i-th component of $\mathbf{y}$. By Lemma 7.6 (ii) and Proposition 5.6 (ii), we have

$$\mathbf{y}' = |\Delta_p|^{12} \delta_A \varepsilon \bar{\phi}^{\ell_i}(x_{\nu_{i-1},q}) \mathbf{e}'^{(m)}_{n'_i},$$

where $(\mathbf{e}'^{(m)}_i)_{i=1}^{h-i+1}$ denotes the standard basis of R_N^{h-i+1}.

Let $\widetilde{M}_{i-1}$ be the matrix of cofactors of M_{i-1}. Multiplying the both sides of $\mathbf{y}' = M_{i-1}\mathbf{x}'$ by $\widetilde{M}_{i-1}$, and comparing the m'_i-th components, we obtain

$$(-1)^{n'_i+m'_i} |\Delta_p|^{12} \delta_A \varepsilon \det(M_i) \bar{\phi}^{\ell_i}(x_{\nu_{i-1},q}) = \det(M_{i-1}) \tilde{\beta}_{m_i,N}({x_{\nu,q}}^{|\Delta_p|^{10}\delta_A}).$$

Since ${x_{\nu,q}}^{|\Delta_p|^{10}\delta_A}$ belongs to $\mathcal{F}_{Qq\nu,N}$, the condition (x3) for ℓ_{i+1} implies

$$|\Delta_p|^2 \tilde{\beta}_{m_i,N}({x_{\nu,q}}^{|\Delta_p|^{10}\delta_A}) = |\Delta_p|^{10} \delta_A \bar{\phi}^{\ell_{i+1}}(x_{\nu,q}).$$

Here, the proof of Proposition 7.7 is complete. □

7.4. Now we shall prove Theorem 7.1. It is convenient to use the following notion of convergence.

Definition 7.8. A sequence $(a_m)_{m\geq 0} \in \prod_{m\geq 0} R_{F_m,N_m,\chi}$ is said to *converge* to

$$b = (b_m)_{m\geq 0} \in \varprojlim_{m\geq 0} R_{F_m,N_m,\chi} = \Lambda_\chi$$

if for each m, there exists an integer L_m such that the image of $a_{m'}$ in $R_{F_m,N_m,\chi}$ coincides with $b_m \in R_{F_m,N_m,\chi}$ for any $m' \geq L_m$.

Proof of Theorem 7.1 . Here, we vary m. In this subsection, we denote the element

$$\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q}) \in R_N = (\mathbb{Z}/p^{N_m})[\mathrm{Gal}(F_m/F_0)]_\chi$$

defined in §6.2 by $\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q})_m$. By induction on r, let us prove that the sequence $\left(\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q})_m\right)_{m\geq 0}$ converges to

$$\pm\left|\Delta_p\right|^{6+4r}(\gamma-1)^{a_\chi}\delta_I\delta_J\delta_\varphi\varepsilon^r\det(M_r) \in \Lambda_\chi$$

in the sense of Definition 7.8 for any integer r satisfying $0 \leq r \leq i$.

First, we consider the equality

$$\begin{aligned}&\left|\Delta_p\right|^{10}\delta_A\det(M)\cdot\bar{\phi}^{\ell_2}(x_{1,q})\\&\qquad=\pm\left|\Delta_p\right|^{20}(\gamma-1)^{a_\chi}\delta_A\delta_I\delta_J\varepsilon\det(M_1)\bar{\varphi}_{F,N,\chi}\big((\eta_m)_{m\geq 0}\big).\end{aligned}$$

Since the right hand side converges to

$$\pm\left|\Delta_p\right|^{20}(\gamma-1)^{a_\chi}\delta_A\delta_I\delta_J\delta_\varphi\varepsilon\det(M_1)\det(M)$$

and $\left|\Delta_p\right|^{10}\delta_A\det(M)$ is non-zero, it follows that $\left(\bar{\phi}^{\ell_2}(x_{1,q})_m\right)_{m\geq 0}$ converges to

$$\pm\left|\Delta_p\right|^{10}(\gamma-1)^{a_\chi}\delta_I\delta_J\delta_\varphi\varepsilon\det(M_1).$$

(Note the sign $\pm$ does not depend on m, see Proposition 7.7).

Next, we assume that the sequence $\left(\bar{\phi}^{\ell_r}(x_{\nu_{r-1},q})_m\right)_{m\geq 0}$ converges to

$$\pm\left|\Delta_p\right|^{2+4r}(\gamma-1)^{a_\chi}\delta_I\delta_J\delta_\varphi\varepsilon^{r-1}\det(M_{r-1}).$$

Then, the right hand side of

$$\left|\Delta_p\right|^{10}\delta_A\det(M_{r-1})\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q}) = \pm\left|\Delta_p\right|^{14}\delta_A\varepsilon\det(M_r)\bar{\phi}^{\ell_r}(x_{\nu_{r-1},q})$$

converges to

$$\pm\left|\Delta_p\right|^{16+4r}(\gamma-1)^{a_\chi}\delta_A\delta_I\delta_J\delta_\varphi\varepsilon^r\det(M_r)\det(M_{r-1})$$

Since we take $\det(M_{r-1}) \neq 0$, the sequence $\left(\bar{\phi}^{\ell_{r+1}}(x_{\nu_r,q})_m\right)_{m\geq 0}$ converges to

$$\pm\left|\Delta_p\right|^{6+4r}(\gamma-1)^{a_\chi}\delta_I\delta_J\delta_\varphi\varepsilon^r\det(M_r).$$

By induction, in particular, we conclude $\left(\bar{\phi}^{\ell_{i+1}}(x_{\nu,q})_m\right)$ converges to

$$\pm\left|\Delta_p\right|^{6+4i}(\gamma-1)^{a_\chi}\delta_I\delta_J\delta_\varphi\varepsilon^i\det(M_i).$$

Since $(x_{\nu,q})_m$ is contained in an R_N-submodule of $(F^\times/p^N)_\chi$ generated by the set $\bigcup_{e|q\nu} \mathcal{W}_{m,N,\chi}(e)$ with $\epsilon(q\nu) = i$, we have $\bar{\phi}^{\ell_{i+1}}(x_{\nu,q})_m \in \mathfrak{C}_{i,F_m,N\chi}$ for all $m \in \mathbb{Z}_{\geq 0}$. Hence we have

$$\pm |\Delta_p|^{6+4i} (\gamma - 1)^{a_\chi} \delta_I \delta_J \delta_\varphi \varepsilon^i \det(M_i) \in \mathfrak{C}_{i,\chi}.$$

This completes the proof of theorem. □

8. The higher cyclotomic ideals and Mazur–Rubin theory

In this section, we assume that the extension degree of $K/\mathbb{Q}$ is prime to p, and fix a character $\chi \in \widehat{\Delta}$ satisfying $\chi(p) \neq 1$. We denote by $\mathcal{O}$ the $\mathbb{Z}_p$-algebra isomorphic to $\mathcal{O}_\chi$ with trivial $G_\mathbb{Q}$-action. We identify the ring $\mathcal{O}$ (resp. $\mathcal{O}[[\Gamma]]$) with $\mathcal{O}_\chi$ (resp. Λ_χ) when we ignore the action of $G_\mathbb{Q}$. In particular, we sometimes regard $\mathfrak{C}_{i,\chi}$ as an ideal of $\mathcal{O}[[\Gamma]]$, and $R_{m,N,\chi}$ as a quotient ring of $\mathcal{O}[[\Gamma]]$. In this section, we complete the proof of Theorem 1.1. In the previous section, we have already proved

$$\mathrm{ann}_{\Lambda_\chi}(X_{\chi,\mathrm{fin}}) \,\mathrm{Fitt}_{\Lambda_\chi,i}(X'_\chi) \subseteq \mathfrak{C}_{i,\chi}$$

for any $i \in \mathbb{Z}_{\geq 0}$. In order to complete the proof of Theorem 1.1, we have to show

$$\mathfrak{C}_{i,\chi} \prec \mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi)$$

for any $i \in \mathbb{Z}_{\geq 0}$. Since this statement is already proved for $i = 0$ (see Theorem 4.16), it is sufficient to show the following theorem.

Theorem 8.1. *Let $i \geq 1$ be a positive integer, and $\mathfrak{P} \subset \mathcal{O}[[\Gamma]] = \Lambda_\chi$ a prime ideal of height one containing $\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi)$. We define two integers $\alpha_i(\mathfrak{P})$ and $\beta_i(\mathfrak{P})$ by*

$$\mathrm{Fitt}_{\Lambda_{\chi,\mathfrak{P}},i}(X_\chi \otimes_{\mathcal{O}[[\Gamma]]} \Lambda_{\chi,\mathfrak{P}}) = \mathfrak{P}^{\alpha_i(\mathfrak{P})}\Lambda_{\chi,\mathfrak{P}},$$
$$\mathfrak{C}_{i,\chi}\Lambda_{\chi,\mathfrak{P}} = \mathfrak{P}^{\beta_i(\mathfrak{P})}\Lambda_{\chi,\mathfrak{P}}.$$

Then, we have $\beta_i(\mathfrak{P}) \geq \alpha_i(\mathfrak{P})$.

In the rest of this section, we prove Theorem 8.1. The key of the proof is a comparison between the higher cyclotomic ideals $\mathfrak{C}_{i,\chi}$ defined in this paper and the theory of Kolyvagin systems, which is established by Mazur and Rubin ([MR]).

8.1. In this subsection, we recall some results (specialized for our purposes) on Kolyvagin systems proved in [MR] §§5.2. (See also Appendix §A.) Note that here, we use some notation introduced in Appendix §A.

We define an $\mathcal{O}$-module T_χ by $T_\chi := \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\chi^{-1}}$. A set Σ of places of $\mathbb{Q}$ is defined by

$$\Sigma := \{p, \infty\} \cup \{\ell \mid \ell \text{ ramifies in } K/\mathbb{Q}\}.$$

Then, let us consider the Selmer structure $(T_\chi, \mathcal{F}_{\mathrm{can}}, \Sigma)$. (See Appendix §A.1 for the definition of the local condition $\mathcal{F}_{\mathrm{can}}$.) By Kummer theory, we have

$$H^1(\mathbb{Q}, T/p^N T) = (F_0^\times/p^N)_\chi,$$

and global class field theory implies

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*_\chi) = \mathrm{Hom}(A_{F_0,\chi}, \mathbb{Q}_p/\mathbb{Z}_p).$$

(See also Appendix §A.4.)

We define

$$\mathbf{T}_\chi := T_\chi \otimes_{\mathcal{O}} \mathcal{O}[[\Gamma]] = \Lambda_{\chi^{-1}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1),$$

and the local condition $\mathcal{F}_\Lambda$ on $\mathbf{T}_\chi$ by

$$H^1_{\mathcal{F}_\Lambda}(\mathbb{Q}_v, \mathbf{T}_\chi) := H^1(\mathbb{Q}_v, \mathbf{T}_\chi).$$

By standard arguments, we have

$$H^1(\mathbb{Q}, \mathbf{T}_\chi) = H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbf{T}_\chi) \simeq \varprojlim H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_n, T_\chi),$$

where the projective limit in the rightmost term is defined by the projective system with respect to corestriction maps. (See, for example [MR] Lemma 5.3.1.) In particular, the triple $(\mathbf{T}, \mathcal{F}_\Lambda, \Sigma)$ is a Selmer structure in the sense of §A.1. By local duality, we have $H^2(\mathbb{Q}_p, T^*_\chi) = 0$, so (the limit of) Poitou-Tate exact sequence implies

$$X_\chi = \mathrm{Hom}(H^1_{\mathcal{F}^*_\Lambda}(\mathbb{Q}, \mathbf{T}^*_\chi), \mathbb{Q}_p/\mathbb{Z}_p) = H^2(\mathbb{Q}_\Sigma/\mathbb{Q}, \mathbf{T}_\chi).$$

We define a set Σ_Λ of prime ideals of height one of the ring $\mathcal{O}[[\Gamma]] = \Lambda_\chi$ by

$$\Sigma_\Lambda := \left\{\mathfrak{P} \in \mathrm{Spec}(\Lambda_\chi) \mid \mathrm{ht}\mathfrak{P} = 1 \text{ and } \mathrm{char}_{\Lambda_\chi}(X_\chi) \subseteq \mathfrak{P}\right\} \cup \{p\Lambda_\chi\}.$$

Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]] = \Lambda_\chi$ be a prime ideal of height one not contained in Σ_Λ. We denote the normalization of $\mathcal{O}[[\Gamma]]/\mathfrak{Q}$ by $S_\mathfrak{Q}$. Note that $S_\mathfrak{Q}$ is a complete discrete valuation ring of mixed characteristic $(0, p)$ whose residue field is finite. We fix a uniformizer π of $S_\mathfrak{P}$. Here, let us consider the Selmer triple

$$\left(\mathbf{T}_\chi \otimes_{\mathcal{O}[[\Gamma]]} S_\mathfrak{Q} = T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}, \mathcal{F}_{\mathrm{can}}, \mathcal{P} := \mathcal{P}_1\right).$$

Note that $T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}$ satisfies conditions (T1)–(T4) in Appendix §A.1, so we can apply results on Kolyvagin systems mentioned in the appendix.

For each positive integer N, we define

$$\mathcal{S}_N(\mathfrak{Q}) := \left\{\ell \in \mathcal{S}_N \mid \mathrm{Fr}_\ell \text{ acts trivially on } T \otimes_{\mathcal{O}} S_\mathfrak{Q}/p^N S_\mathfrak{Q}\right\},$$

and we denote the set of all well-ordered square-free products of $\mathcal{S}_N(\mathfrak{Q})$ by $\mathcal{N}_N^{\mathrm{w.o.}}(\mathfrak{Q})$. We also set $\mathcal{P} := \mathcal{P}_1(T \otimes_{\mathcal{O}} S_\mathfrak{Q}, \Sigma)$ and $\mathcal{N}(\mathcal{P})$ as in Appendix A.2. Let M be any $\mathcal{S}_\mathfrak{Q}/p^N\mathcal{S}_\mathfrak{Q}$-module, and $n \in \mathcal{S}_N(\mathfrak{Q})$ any element. Then, throughout this section, we identify $M \otimes_{\mathbb{Z}} \bigotimes_{\ell \mid n} H_\ell$ with M by the isomorphism

$$M \otimes_{\mathbb{Z}} \bigotimes_{\ell \mid n} H_\ell \xrightarrow{\simeq} M;\ m \otimes \bigotimes_{\ell \mid n} \sigma_\ell \longmapsto m.$$

Note that we have $\mathcal{S}_{N'} \subseteq \mathcal{S}_N(\mathfrak{Q})$ for any sufficiently large N'. Under the assumption in Proposition A.6, there exits a natural R-linear map

$$\mathrm{ES}(T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}, \Sigma) \longrightarrow \mathrm{KS}(T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}, \Sigma);\ \mathbf{c} \longmapsto \kappa(\mathbf{c}) = \{\kappa(\mathbf{c})_n\}_n.$$

Proposition 8.2 (See Proposition A.6)**.** *Assume that the action of* $\mathrm{Fr}_\ell^{p^k}$ *on* $T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}$ *is non-trivial for any* $\ell \in \mathcal{P}$ *and* $k \in \mathbb{Z}_{\geq 0}$. *Take arbitrary elements* $\mathbf{c} \in \mathrm{ES}(T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}, \Sigma)$ *and* $n \in \mathcal{N}_N^{\mathrm{w.o.}}(\mathfrak{Q})$. *Let* $\kappa_{0,N}(\mathbf{c}; n)$ *be the Kolyvagin derivative of* $\mathbf{c}$ *at* n. *Then, we have*

$$\kappa(\mathbf{c})_n = \kappa_{0,N}(c; n) \text{ in } H^1(\mathbb{Q}, T_\chi \otimes_{\mathcal{O}} S_\mathfrak{Q}/p^N T)/M(n, \mathbf{c}),$$

where $M(n,\mathbf{c})$ is an $S_{\mathfrak{Q}}$-submodule of $H^1(\mathbb{Q}, T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}/p^N T)$ generated by

$$\left\{ \kappa(\mathbf{c})_d \;\middle|\; 0 < d \mid n,\ d \neq n \right\}.$$

Recall that for each element $n := \ell_1 \times \cdots \times \ell_r \in \mathcal{N}(\mathcal{P})$, we denote the number of prime divisors of n by $\epsilon(n) := r$. For any non-zero Kolyvagin system $\kappa = \{\kappa_n\} \in \mathrm{KS}(T_\chi \otimes S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ and any non-negative integer i, we denote the maximum element of the set

$$\left\{ j \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \kappa_n \in \pi^j H_{\mathcal{F}(n)}(\mathbb{Q}, T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}/I_n) \\ \text{for all } n \in \mathcal{N}(\mathcal{P}) \text{ with } \epsilon(n) = i \end{array} \right\}$$

by $\partial_i(\kappa; \mathfrak{Q}) := \partial_i(\kappa; T_\chi \otimes S_{\mathfrak{Q}})$. (See also Appendix §A.2.) We also define

$$\partial_i(\mathfrak{Q}) := \min\{\partial_i(\kappa; \mathfrak{Q}) \mid \kappa = \{\kappa_n\} \in \mathrm{KS}(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \Sigma)\}.$$

We have the following result by Mazur–Rubin.

Proposition 8.3 (See Proposition A.3 in Appendix)**.** *Let $\mathfrak{Q} \subseteq \mathcal{O}[[\Gamma]]$ be a prime ideal of height one not contained in Σ_Λ. We fix a uniformizer $\pi_{\mathfrak{Q}}$ of $S_{\mathfrak{Q}}$. Then,*

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*)^\vee := \mathrm{Hom}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

is a torsion $S_{\mathfrak{Q}}$-module, and we have

$$\begin{aligned} \mathrm{Fitt}_{S_{\mathfrak{Q}},i}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*)^\vee\right) &= \mathrm{Fitt}_{S_{\mathfrak{Q}},i}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}})^*)\right) \\ &= \pi_{\mathfrak{Q}}^{\partial_i(\mathfrak{Q})} S_{\mathfrak{Q}} \end{aligned}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

8.2. In this section, we study the higher Fitting ideals of $A_{0,\chi}$. We combine Proposition 8.3 with the usual Euler system arguments (without Kurihara's elements), and obtain the following theorem.

Theorem 8.4. *Assume the extension degree of $K/\mathbb{Q}$ is prime to p. Let $\chi \in \widehat{\Delta}$ be a character satisfying $\chi(p) \neq 1$. Then, we have*

$$\mathrm{Fitt}_{\mathcal{O}_\chi/p^N,i}(A_{0,\chi}) = \mathfrak{C}_{i,0,N,\chi}$$

for any non-negative integer i and sufficiently large integer N.

Proof. Recall that we assume that the order of χ is prime to p here. So, p is a prime element of the discrete valuation ring $\mathcal{O}_\chi$. Since $A_{0,\chi}$ is a finitely generated torsion $\mathcal{O}_\chi$-module, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_\chi^r \xrightarrow{f} \mathcal{O}_\chi^r \xrightarrow{g} A_{0,\chi} \longrightarrow 0,$$

of $\mathcal{O}_\chi$-modules, where the matrix M_f associated with f for the standard basis $(\mathbf{e}_j)_{j=1}^r$ of $\mathcal{O}_\chi^r$ is a diagonal matrix

$$M_f := \begin{pmatrix} p^{d_1} & & & \\ & p^{d_2} & & \\ & & \ddots & \\ & & & p^{d_r} \end{pmatrix}$$

satisfying $d_1 \geq d_2 \geq \cdots \geq d_r$.

We fix an integer N satisfying $p^N \geq \# A_{0,\chi}$. First, let us show the inequality $\mathrm{Fitt}_{\mathcal{O}_\chi/p^N,i}(A_{0,\chi}) \supseteq \mathfrak{C}_{i,0,N,\chi}$. Let

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

be an Euler system of circular units defined by a Λ_χ-linear combination of basic circular units, $n \in \mathcal{N}_N^{\mathrm{w.o.}}$ satisfying $\epsilon(n) \leq i$, and

$$f\colon (F_0^\times/p^N)_\chi \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N$$

an arbitrary homomorphism of $R_{0,N,\chi}$-modules. Then, from Proposition 8.3 for the prime ideal $(\gamma-1)\Lambda_\chi$ and Proposition 8.2, it follows that $\kappa_{0,N,\chi}(n,\eta)_\chi$ is a $p^{\sum_{j=i+1}^r d_j}$-power of some element in $(F_0^\times/p^N)_\chi$. This implies

$$f(\kappa_{0,N}(\eta;n)_\chi) \in p^{\sum_{j=i+1}^r d_j}(\mathcal{O}_\chi/p^N) = \mathrm{Fitt}_{\mathcal{O}_\chi,i}(A_{0,\chi})(\mathcal{O}_\chi/p^N),$$

and we obtain $\mathrm{Fitt}_{\mathcal{O}_\chi/p^N,i}(A_{0,\chi}) \supseteq \mathfrak{C}_{i,0,N,\chi}$.

Note that the inequality $\mathrm{Fitt}_{\mathcal{O}_\chi,i}(A_{0,\chi}) \subseteq \mathfrak{C}_{i,0,N,\chi}$ follows from the usual Euler system argument. (See, for example, the arguments in [Ru2] §4.) Here, let us sketch the proof of it briefly. Let N be a sufficiently integer. Note that any circular unit in F_0 extends to an Euler system defined by a Λ_χ-linear combination of basic circular units since we assume $\chi(p) \neq 1$ (cf. Remark 4.6). Fix an Euler system

$$\eta = \{\eta_m(n)_\chi \in (F_m(\mu_n)^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\chi\}_{m,n}$$

of circular units defined by a Λ_χ-linear combination of basic circular units, and assume that the circular unit $\eta_0(1)$ generates the free $\mathcal{O}_\chi$-module $C_{0,\chi} = (C_0 \otimes \mathbb{Z}_p)_\chi$ of rank one. Recall that $E_{0,\chi} := (\mathcal{O}_{F_0}^\times \otimes \mathbb{Z}_p)_\chi$ is a free $\mathcal{O}_\chi$-module of rank one. We fix an isomorphism

$$\psi_0\colon E_{0,\chi}/p^N \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N$$

of $\mathcal{O}_\chi$-modules and a prime number ℓ_1 whose ideal class $[\ell_1]_{F_0,\chi}$ in $A_{0,\chi}$ coincides with $g(\mathbf{e}_1)$ and satisfying

$$\phi^{\ell_1}_{m,N,\chi} \mid_{E_{0,\chi}/p^N} = \psi_0.$$

(Note that Proposition 6.1 ensures the existence of such a prime number ℓ_1.) By the arguments in [Ru2] §4 combined with Proposition 6.1, we can inductively take prime numbers $\ell_1, ..., \ell_{r+1} \in \mathcal{S}_N$ and homomorphisms

$$\psi_j\colon (F_0^\times/p^N)_\chi \longrightarrow R_{0,N,\chi} = \mathcal{O}_\chi/p^N \quad (j = 1, ..., r)$$

satisfying the following conditions.

- $[\ell_{j,F_0}]_\chi = g(\mathbf{e}_j)$ in $A_{0,\chi}$ for any integer j with $1 \leq j \leq r$.
- The integer $n_j := \prod_{\nu=1}^j \ell_j$ is well-ordered any integer j with $1 \leq j \leq r$.
- $p^{d_{j-1}}\psi_j(\kappa_{0,N}(\eta;n_j)) = \psi_{j-1}(\kappa_{0,N}(\eta;n_{j-1})_\chi)$ for any integer j with $1 \leq j \leq r$. Here, we put $n_0 := 1$.
- The restriction of $\phi^{\ell_j}_{0,N,\chi}$ on $\mathcal{W}_{0,N.\chi}(n_j)$ coincides with ψ_{j-1} any integer j with $1 \leq j \leq r$.

Then, we obtain

$$p^{\sum_{j=1}^{i-1} d_j}\psi_i(\kappa_{0,N}(\eta; n_i)_\chi) = p^{\sum_{j=1}^{i-2} d_j}\psi_{i-1}(\kappa_{0,N}(\eta; n_{i-1})_\chi) = \cdots = \psi_0(\eta).$$

By [Ru2] Theorem 4.2 (see also [MW] Theorem 1.10.1 or [Ru4] Corollary 3.2.4 for general cases), there exists a unit $u \in \mathcal{O}_\chi^\times$ such that

$$\psi_0(\eta) = u\# A_{0,\chi} = up^{\sum_{j=1}^{r} d_j}.$$

Therefore, we obtain

$$\mathfrak{C}_{0,N,\chi} \subseteq \psi_i(\kappa_{0,N}(\eta; n_i)_\chi)R_{0,N,\chi} = p^{\sum_{j=i+1}^{r} d_j}R_{0,N,\chi}.$$

This completes the proof. □

Remark 8.5. Fix a pseudo-isomorphism

$$X_\chi \longrightarrow \bigoplus_{j=1}^{r} \Lambda_\chi/f_i\Lambda_\chi,$$

where $f_1, ..., f_r$ are non-unit and non-zero elements of Λ_χ satisfying $f_r \mid \cdots \mid f_2 \mid f_1$. Then, by similar arguments to that in the proof of the inequality $\mathrm{Fitt}_{\mathcal{O}_\chi,i}(A_{0,\chi}) \subseteq \mathfrak{C}_{i,0,N,\chi}$, we can prove rough estimates

$$\mathrm{Fitt}_{\Lambda_\chi,i}(X_\chi) \prec \mathfrak{C}_{i,\chi} \tag{13}$$

without using Kurihara's elements. In this arguments, we use the arguments of [Ru2] §5 and the Iwasawa main conjecture instead of the arguments of [Ru2] §4 and [Ru2] Theorem 4.2. Note that when we apply such arguments, we have to ignore error factors of "$\prec$" completely. So, Theorem 7.1, which is proved by Euler arguments via Kurihara's elements, is stronger than results obtained by usual arguments without Kurihara's elements.

Note that Theorem 8.4 implies that for any non-negative integer i and any two integers N and N' satisfying $N' \geq N > 0$, the image of $\mathfrak{C}_{0,N',\chi}$ in $R_{0,N,\chi}$ coincides with $\mathfrak{C}_{0,N,\chi}$. Combining this fact and the second assertion of Corollary 4.14, we obtain the following corollary immediately.

Corollary 8.6. *Assume the extension degree of $K/\mathbb{Q}$ is prime to p. Let i be a non-negative integer, and $\chi \in \widehat{\Delta}$ a character satisfying $\chi(p) \neq 1$. Then, the following holds.*

(i) *The image of $\mathfrak{C}_{i,\chi}$ in $R_{0,N,\chi}$ coincides with the ideal $\mathfrak{C}_{i,0,N,\chi}$ for any positive integer N.*
(ii) *The image of $\mathfrak{C}_{i,\chi}$ in $R_{0,\chi} := \mathbb{Z}_p[\mathrm{Gal}(F_0/\mathbb{Q})]_\chi$ coincides with the ideal $\mathfrak{C}_{i,F_0,\chi} := \varprojlim_N \mathfrak{C}_{i,0,N,\chi}$.*

We put $\mathfrak{m} := p\Lambda_\chi + (\gamma - 1)\Lambda_\chi$. Note we have the natural isomorphism

$$X_\chi/\mathfrak{m}X_\chi \simeq A_{0,\chi}/p$$

by Proposition 3.9. So, the least cardinality of generators of the Λ_χ-module X_χ coincides to that of the $\mathcal{O}_\chi/p$-module $A_{0,\chi}$ by Nakayama's lemma. Hence the following corollary follows from Remark 2.3, Theorem 8.4 and Corollary 8.6.

Corollary 8.7. *Let $K/\mathbb{Q}$ and $\chi \in \widehat{\Delta}$ be as in Theorem 1.1. Let r be a non-negative integer. Then, the following two conditions are equivalent.*

(i) *The least cardinality of generators of the Λ_χ-module X_χ is r.*
(ii) *$\mathfrak{C}_{r-1,\chi} \neq \Lambda_\chi$ and $\mathfrak{C}_{r,\chi} = \Lambda_\chi$.*

8.3. In this subsection, we prove Theorem 8.1. Here, we fix a positive integer i and a height one prime ideal $\mathfrak{P}$ of $\mathcal{O}[[\Gamma]]$ containing $\mathrm{Fitt}_{\Lambda_\chi}(X_\chi)$. In particular, we have $\mathfrak{P} \neq (p)$. For simplicity, we put $\alpha := \alpha_i(\mathfrak{P})$ and $\beta := \beta_i(\mathfrak{P})$. We define a non-negative integer s by

$$p^s = (S_{\mathfrak{P}} : \mathcal{O}[[\Gamma]]/\mathfrak{P}).$$

In the proof of Theorem 8.1, we mainly use a method of "deformation of Weierstrass polynomials" which was used in [MR] §5.3. (In [MR] §5.3, they treat only the case of $\mathcal{O}_\chi = \mathbb{Z}_p$, but we can prove similar results for general $\mathcal{O}_\chi$ by similar arguments.) We identify $\mathcal{O}[[\Gamma]]$ with the ring $\mathcal{O}[[T]]$ of formal power series by an isomorphism $\mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]]$ defined by $\gamma \mapsto 1+T$. Let $f(T) \in \mathcal{O}[T]$ be the Weierstrass polynomial generating the fixed prime ideal $\mathfrak{P}$ of $\mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]]$. For any positive integer k, we put

$$f_k(T) := f(T) + p^k$$

and let $\mathfrak{P}_k$ be the principal ideal of $\Lambda_\chi = \mathcal{O}_\chi[[T]]$ generated by $f_k(T)$. We need the following lemma (cf. [MR] p. 66).

Lemma 8.8. *There exists a positive integer $N(\mathfrak{P})$ satisfying the following properties.*

(i) *The ideal $\mathfrak{P}_k$ is a prime ideal for any $k \geq N(\mathfrak{P})$.*
(ii) *The ideal $\mathfrak{P}_k$ is not contained in Σ_Λ for any $k \geq N(\mathfrak{P})$.*
(iii) *The residue ring $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$ is (non-canonically) isomorphic to $\mathcal{O}[[\Gamma]]/\mathfrak{P}$ as $\mathcal{O}$-algebra for any $k \geq N(\mathfrak{P})$.*
(iv) *The action of $\mathrm{Fr}_\ell^{p^m} - 1$ on $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}$ is injective for any $\ell \in \mathcal{P}$, any $m \geq 0$ and any $k \geq N(\mathfrak{P})$.*

***Proof*.** The arguments in [MR] p. 66 implies that there exists an integer $N'(\mathfrak{P})$ such that the conditions (i)–(iii) in the lemma holds for any integer k satisfying $k \geq N'(\mathfrak{P})$. (See loc. cit. for details.) So, it is sufficient to show that the fourth condition holds for any sufficiently large integer k. We denote the cyclotomic character by

$$\chi^{\mathrm{cyc}} \colon \Gamma \longrightarrow 1 + p\mathbb{Z}_p \hookrightarrow \mathcal{O}^\times.$$

Let k be an integer satisfying $k \geq N(\mathfrak{P})$. The natural projection induces a continuous character

$$\rho_k \colon \Gamma \longrightarrow (\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times.$$

Note that the action of $\mathrm{Fr}_\ell^{p^m} - 1$ on $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}$ is *not* injective for some $\ell \in \mathcal{P}$ and some $m \geq 0$ if and only if the order of the character $\chi^{\mathrm{cyc}}\rho_k$ is finite.

We denote the number of p-power torsion elements of $(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times$ by p^ν. Assume that the character $\chi^{\mathrm{cyc}}\rho_k$ has finite order. The image of $\chi^{\mathrm{cyc}}\rho_k$ is contained in

$(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)^\times$, so the character $\chi^{\mathrm{cyc}}\rho_k$ is annihilated by p^ν. In particular, we have

$$\rho_k(\gamma^{p^\nu}) = (\chi^{\mathrm{cyc}})^{-p^\nu}(\gamma)$$

in $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$. This implies the polynomial

$$(1+T)^{p^\nu} - (\chi^{\mathrm{cyc}})^{-p^\nu}(\gamma) \in \mathcal{O}[T]$$

is divisible by the monic polynomial $f_k(T)$. Obviously, such a situation occurs for only finitely many k, so the condition (iv) holds for any sufficiently large integer k. □

Definition 8.9. Let M be an integer, and $\{x_k\}_{k\geq M}$ and $\{y_k\}_{k\geq M}$ sequences of real numbers. We write $x_k \succ y_k$ if $\liminf_{k\to\infty}(x_k - y_k) \neq -\infty$. We write $x_N \sim y_N$ if $x_k \succ y_k$ and $y_k \succ x_k$. In other words, we write $x_k \sim y_k$ if $|x_k - y_k|$ is bounded independent of k.

We denote the ramification index of $\mathrm{Frac}(S_{\mathfrak{P}})/\mathbb{Q}_p$ by $e_{\mathfrak{P}}$, and the extension degree of the residue field of $\mathcal{O}_\chi$ over $\mathbb{F}_p$ by f_χ. Let us recall the following observations in [MR] p. 66. Let d be a non-negative integer. Then, we have

$$\mathfrak{P}_k + \mathfrak{P}^d = \mathfrak{P}_k + {\pi_k}^{de_{\mathfrak{P}}k}\mathcal{O}[[\Gamma]]$$

for any sufficiently large integer k. So, we obtain the natural isomorphism

$$(\mathcal{O}[[\Gamma]]/\mathfrak{P}^d) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \simeq S_{\mathfrak{P}_k}/(\pi_k)^{de_{\mathfrak{P}}k}$$

of $S_{\mathfrak{P}_k}$-algebras for any sufficiently large k. Moreover, we obtain the following lemma from the observations in [MR] p. 66. (See [MR] loc. cit. for the proof.)

Lemma 8.10. *Let M be a finitely generated torsion $\mathcal{O}[[\Gamma]]$-module, and*

$$E := \bigoplus_{j=0}^{r} \mathcal{O}[[\Gamma]]/\mathfrak{P}^{d_j} \oplus \bigoplus_{j'=0}^{r'} \mathcal{O}[[\Gamma]]/(g_{j'}(T)^{e_{j'}})$$

an elementary $\mathcal{O}[[\Gamma]]$-module, where d_j and $e_{j'}$ are positive integers, and $g_{j'}(T)$ is a Weierstrass polynomial in $\mathcal{O}_\chi[T]$ prime to $f(T)$ for any integer j and j'. Suppose that the $\mathcal{O}[[\Gamma]]$-module M is pseudo-isomorphic to E. Then, there exists a sequence

$$\left\{ \iota_k \colon M \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow \bigoplus_{j=0}^{r} S_{\mathfrak{P}_k}/(\pi_k)^{d_j e_{\mathfrak{P}} k}; S_{\mathfrak{P}_k}\text{-linear} \right\}_{k>N(\mathfrak{P})}$$

of homomorphisms such that the orders of the kernel and cokernel of ι_k are finite for any $k \geq N(\mathfrak{P})$, and bounded by a constant independent of k.

Then, we immediately obtain the following Corollary 8.11 of Lemma 8.10 combined with Lemma 2.7. This corollary plays an important role in this section.

Corollary 8.11. *Let M be a finitely generated torsion $\mathcal{O}[[\Gamma]]$-module. We define a non-negative integer C by*

$$\mathrm{Fitt}_{\mathcal{O}[[\Gamma]],i}(M \otimes_{\mathcal{O}[[\Gamma]]} \Lambda_{\chi,\mathfrak{P}}) = \mathfrak{P}^C \Lambda_{\chi,\mathfrak{P}}.$$

For each positive integer k with $k \geq N(\mathfrak{P})$, fix a uniformizer π_k of $S_{\mathfrak{P}_k}$, and define a non-negative integer c_k by

$$\mathrm{Fitt}_{S_{\mathfrak{P}_k},i}(M \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}) = \pi_k^{c_k} S_{\mathfrak{P}_k}.$$

Then, we have $c_k \sim Ce_{\mathfrak{P}}k$.

Definition 8.12. Let k be a non-negative integer with $k \geq N(\mathfrak{P})$. We define non-negative integers a_k and b_k by

$$\pi_k^{a_k} S_{\mathfrak{P}_k} = \mathrm{Fitt}_{S_{\mathfrak{P}_k},i}(X_\chi \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}),$$
$$b_k = \mathrm{length}_{S_{\mathfrak{P}_k}}\big((\mathcal{O}[[\Gamma]]/\mathfrak{C}_{i,\chi}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big).$$

By Corollary 8.11, we have $a_k \sim \alpha e_{\mathfrak{P}}k$ and $b_k \sim \beta e_{\mathfrak{P}}k$.

Proposition 8.13 ([MR] Proposition 5.3.14). *Let k be an integer satisfying $k \geq N(\mathfrak{P})$, and*

$$\pi_k \colon X_\chi \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow \mathrm{Hom}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})^*), \mathbb{Q}_p/\mathbb{Z}_p\right)$$

a natural homomorphism. Then, the kernel and cokernel of π_k are both finite, and the orders of kernel and cokernel of π_k are bounded by a constant independent of k.

Combining Proposition 8.13 with Proposition 8.3, we obtain the following corollary.

Corollary 8.14. *We have* $a_k \sim \partial_i(\mathfrak{P}_k)$.

For an integer k with $k \geq N(\mathfrak{P})$, we take an integer N'_k satisfying

- $N'_k \geq \partial_i(\mathfrak{P}_k)$, and
- $p^{N'_k} \in \mathfrak{C}_{i,\chi} + \mathfrak{P}_k$.

Note that there exist such an N'_k since the ideal $\mathfrak{C}_{i,\chi} + \mathfrak{P}_k$ has finite index in $\mathcal{O}[[\Gamma]]$. Then, we take $N''_k \geq N'_k$ satisfying

- $\gamma^{p^{N''_k - 1}} - 1 \in \mathfrak{P}_k + p^{N'_k}\mathcal{O}[[\Gamma]]$, and
- $\mathcal{S}_{N''_k} \subseteq \mathcal{S}_{N'_k}(\mathfrak{P}_k)$.

We put $m_k := N''_k - 1$.

Proof of Theorem 8.1. Now, we shall prove Theorem 8.1. It is sufficient to show $\beta e_{\mathfrak{P}}k \succ \alpha e_{\mathfrak{P}}k$. Let k be any positive integer satisfying $k \geq N(\mathfrak{P})$. Then, we have

$$\begin{aligned}
\beta e_{\mathfrak{P}}k \sim b_k &= \mathrm{length}_{S_{\mathfrak{P}_k}}\big((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k)) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big) \\
&= \mathrm{length}_{S_{\mathfrak{P}_k}}\big((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k + p^{N'_k}\mathcal{O}[[\Gamma]])) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big) \\
&= \mathrm{length}_{S_{\mathfrak{P}_k}}\big((\mathcal{O}[[\Gamma]]/(\mathfrak{C}_{i,\chi} + \mathfrak{P}_k + (p^{N'_k}, \gamma^{p^{m_k}} - 1))) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big) \\
&= \mathrm{length}_{S_{\mathfrak{P}_k}}\big((R_{m_k,N'_k,\chi}/(\text{the image of } \mathfrak{C}_{i,\chi})) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big) \\
&\geq \mathrm{length}_{S_{\mathfrak{P}_k}}\big((R_{m_k,N'_k,\chi}/(\text{the image of } \mathfrak{C}_{i,m_k,N''_k,\chi})) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}\big)
\end{aligned}$$

Since the ring $R_{m_k,N'_k,\chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ is a quotient of the discrete valuation ring $S_{\mathfrak{P}_k}$, the image of $\mathfrak{C}_{i,m_k,N''_k,\chi}$ in $R_{m_k,N'_k,\chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ is a principal ideal. So, there exist

- a circular unit

$$\eta(n_k) = \eta^{(k)}_{m_k}(n_k) := \prod_{d \mid \mathfrak{f}_K} \eta^d_{m_k}(n)^{u_d} \times \prod_{i=1}^{r} \eta^{1,a_i}_{m_k}(n)^{v_i} \in F_{m_k}(\mu_{n_k})^\times,$$

  where $r \in \mathbb{Z}_{>0}$, u_d and v_i are elements of $\mathbb{Z}[\mathrm{Gal}(F_m/\mathbb{Q})]$ for each positive integers d and i with $d \mid \mathfrak{f}_K$ and $1 \le i \le r$, and $a_1, ..., a_r$ are integers prime to p,
- an element $n_k \in \mathcal{N}^{\mathrm{w.o.}}_{N''_k}$,
- a homomorphism $h_k \colon F^\times_{m_k}/p^{N''_k} \longrightarrow R_{m_k,N''_k,\chi}$,

such that the ideal of $R_{m_k,N'_k,\chi} \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$ generated by the image of $\mathfrak{C}_{i,m_k,N''_k,\chi}$ is a principal ideal generated by the image of $h_k(\kappa_{m_k,N''_k}(\eta; n_k))$. Therefore, we obtain

$$\beta e_{\mathfrak{P}} k \succ \mathrm{length}_{S_{\mathfrak{P}_k}} \left(\left(R_{m_k,N'_k,\chi}/h_k(\kappa_{m_k,N''_k}(\eta; n_k)) R_{m_k,N'_k,\chi} \right) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \right). \tag{14}$$

We denote by $\bar{h}_k \colon F^\times_m/p^{N'_k} \longrightarrow R_{m_k,N'_k,\chi}$, the $R_{m_k,N'_k,\chi}$-linear homomorphism induced by h_k.

For a moment, we fix an integer $k \ge N(\mathfrak{P})$, and put $N' := N_k$, $N'' := N''_k$, $m := m_k$, $n = n_k$ and $\bar{h}_k := \bar{h}$ for simplicity. We put

$$N_{H_n} := \sum_{\sigma \in H_n} \sigma \in \mathbb{Z}[H_n].$$

Let $\nu_{H_n} \colon R_{m,N',\chi} \longrightarrow R_{m,N,\chi}[H_n]^{H_n}$ be an isomorphism of $R_{m,N',\chi}[H_n]$-module defined by $1 \mapsto N_{H_n}$. Note that $R_{m,N',\chi}[H_n]$ is an injective $R_{m_k,N'_k,\chi}$-module, so there exist an $R_{m_k,N'_k,\chi}$-linear homomorphism $\tilde{h} \colon (F_m(\mu_n)^\times/p^{N'})_\chi \longrightarrow R_{m,N',\chi}[H_n]$ which makes the diagram

$$\begin{tikzcd}
(F^\times_m/p^{N'})_\chi \arrow[r, "\bar{h}"] \arrow[d, hook] & R_{m,N',\chi} \arrow[d, "\nu_{H_n}"] \\
(F_m(\mu_n)^\times/p^{N'})_\chi \arrow[r, "\tilde{h}"] & R_{m,N',\chi}[H_n]
\end{tikzcd}$$

commute.

Note that by Shapiro's lemma and limit arguments (cf. [MR] Lemma 5.3.1), we have a natural isomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \xrightarrow{\simeq} \varprojlim_{m'} (F_{m'}(\mu_n)^\times/p^{N'})_\chi.$$

Then, by Lemma 4.13 (ii), we obtain the following lemma.

Lemma 8.15. *There exists a homomorphism*

$$\tilde{h}_\infty \colon H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \longrightarrow \mathcal{O}[[\Gamma]][H_n]/p^{N'} = \varprojlim_{m'} R_{m',N',\chi}[H_n]$$

of $\mathcal{O}[[\Gamma]][H_n]/p^{N'}$-modules which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \arrow[r, "\tilde{h}_\infty"] \arrow[d] & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \arrow[d, "\mathrm{mod}\ (\gamma^{p^m}-1)"] \\ (F_m(\mu_n)^\times/p^{N'})_\chi \arrow[r, "\tilde{h}"] & R_{m,N',\chi}[H_n] \end{tikzcd}$$

commute.

Recall that we define a non-negative integer s by

$$p^s = (S_{\mathfrak{P}} : \mathcal{O}[[\Gamma]]/\mathfrak{P}).$$

Let us show the following proposition.

Proposition 8.16. *There exists an $S_{\mathfrak{P}_k}[H_n]$-linear homomorphism*

$$\tilde{h}_{\mathfrak{P}_k,N'} \colon H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \arrow[r, "p^{4s}\tilde{h}_\infty"] \arrow[d] & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \arrow[d] \\ H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \arrow[r, dashed, "\tilde{h}_{\mathfrak{P}_k}"] & S_{\mathfrak{P}_k}[H_n]/p^{N'} \end{tikzcd}$$

commute. Here, the vertical maps in this diagram are the natural ones.

Proof. We divide the vertical maps in some short steps, and we will construct suitable homomorphisms step by step. In order to prove Proposition 8.16, we need the following Lemma 8.17 and its corollary. (Note that the following lemma is proved by similar arguments to Lemma 3.1, so we omit the proof.)

Lemma 8.17. *Let M be an $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$-module. Then, the kernel and the cokernel of the natural $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$-linear map*

$$M \longrightarrow M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$$

is annihilated by p^s.

The following corollary follows from Lemma 8.17 by the similar arguments to Corollary 3.3.

Corollary 8.18. *Let $f \colon M \longrightarrow N$ be a homomorphism of $\mathcal{O}[[\Gamma]]/\mathfrak{P}_k$-modules, and*

$$i_M \colon M \longrightarrow M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$$

the natural homomorphism. Consider the $S_{\mathfrak{P}_k}$-linear map

$$f \otimes S_{\mathfrak{P}_k} \colon M \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k} \longrightarrow N \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k}$$

induced by f. Then, $p^{2s}\operatorname{Ker}(f \otimes S_{\mathfrak{P}_k})$ is contained in $i_M(\operatorname{Ker} f)$. In particular, if f is injective, then $\operatorname{Ker}(f \otimes S_{\mathfrak{P}_k})$ is annihilated by p^{2s}.

Here, we return to the proof of Proposition 8.16. From the exact sequence

$$0 \longrightarrow \mathbf{T}_\chi/p^{N'} \xrightarrow{\times f_k(T)} \mathbf{T}_\chi/p^{N'} \longrightarrow \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi) \longrightarrow 0,$$

it follows that the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} (\mathcal{O}[[\Gamma]]/\mathfrak{P}_k) \longrightarrow H^1\big(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi)\big)$$

is injective. So, by Corollary 8.18, the kernel of the $S_{\mathfrak{P}_k}[H_n]$-linear map

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1\big(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi)\big) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}$$

is annihilated by p^{2s}.

The $\mathcal{O}[[\Gamma]][H_n]$-linear homomorphism $\tilde{h}_\infty$ induces an $S_{\mathfrak{P}_k}[H_n]$-linear homomorphism

$$\tilde{h}^{(0)}\colon H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}.$$

For simplicity, we put

$$\overline{\mathbf{T}}_{\chi,k} := \mathbf{T}_\chi/(p^{N'}\mathbf{T}_\chi + \mathfrak{P}_k\mathbf{T}_\chi).$$

Let $\mathrm{Im}_{S_{\mathfrak{P}_k}}$ be the image of $S_{\mathfrak{P}_k}[H_n]$-linear map

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k}.$$

Then, there exists an $S_{\mathfrak{P}_k}[H_n]$-linear homomorphism

$$\tilde{h}^{(1)}\colon \mathrm{Im}_{S_{\mathfrak{P}_k}} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'}$$

which makes the diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \arrow[r, "p^{2s}\tilde{h}^{(0)}"] \arrow[d] & S_{\mathfrak{P}_k}[H_n]/p^{N'} \\
\mathrm{Im}_{S_{\mathfrak{P}_k}} \arrow[ur, "\tilde{h}^{(1)}"'] &
\end{tikzcd}$$

commute. Note that $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ is an injective $S_{\mathfrak{P}_k}[H_n]/p^{N'}$-module since $S_{\mathfrak{P}_k}/p^{N'}$ is a quotient of a complete discrete valuation ring $S_{\mathfrak{P}_k}$ with finite residue field. So, we can extend $\tilde{h}^{(1)}$ to an $S_{\mathfrak{P}_k}[H_n]/p^{N'}$-linear map

$$\tilde{h}^{(1)}\colon H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'},$$

and we obtain the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \arrow[r, "p^{2s}\tilde{h}_\infty"] \arrow[d] & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \arrow[d] \\
H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \arrow[r, dashed, "\tilde{h}^{(1)}"] & S_{\mathfrak{P}_k}[H_n]/p^{N'}.
\end{tikzcd}$$

Since $\mathrm{Tor}_1^{\Lambda_\chi/\mathfrak{P}_k}\big(\overline{\mathbf{T}}_{\chi,k}, S_{\mathfrak{P}_k}/(\mathcal{O}[[\Gamma]]/\mathfrak{P}_k)\big)$ and $(\overline{\mathbf{T}}_{\chi,k} \otimes_{\mathcal{O}[[\Gamma]]/\mathfrak{P}_k} S_{\mathfrak{P}_k})/\overline{\mathbf{T}}_{\chi,k}$ are annihilated by p^s, the kernel of the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \longrightarrow H^1\big(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}\big)$$

is annihilated by p^{2s}. Then, the injectivity of $S_{\mathfrak{P}_k}[H_n]/p^{N'}$ implies that there exists an $S_{\mathfrak{P}_k}[H_n]/p^{N'}$-linear map

$$\tilde{h}_{\mathfrak{P}_k}\colon H^1\big(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}\big) \longrightarrow S_{\mathfrak{P}_k}[H_n]/p^{N'},$$

which makes the diagram

$$\begin{tikzcd} H^1(\mathbb{Q}(\mu_n), \overline{\mathbf{T}}_{\chi,k}) \otimes_{\mathcal{O}[[\Gamma]]} S_{\mathfrak{P}_k} \arrow[r, "p^{2s}\tilde{h}^{(1)}"] \arrow[d] & S_{\mathfrak{P}_k}[H_n]/p^{N'} \\ H^1\big(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}\big) \arrow[ur, dashed, "\tilde{h}_{\mathfrak{P}_k}"'] & \end{tikzcd}$$

commute. The homomorphism $\tilde{h}_{\mathfrak{P}_k}$ is what we want to construct, and this completes the proof of Proposition 8.16. □

We identify $S_{\mathfrak{P}_k}/p^{N'}$ with $S_{\mathfrak{P}_k}/p^{N'}[H_n]^{H_n}$ as an $S_{\mathfrak{P}_k}/p^{N'}[H_n]$-module by the isomorphism

$$\nu_{H_n}\colon S_{\mathfrak{P}_k}/p^{N'} \longrightarrow S_{\mathfrak{P}_k}/p^{N'}[H_n]^{H_n};\ 1 \longmapsto N_{H_n},$$

and let

$$h_{\mathfrak{P}_k}\colon H^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \longrightarrow S_{\mathfrak{P}_k}/p^{N'}$$

be the homomorphism induced by $\tilde{h}_{\mathfrak{P}_k}$. Since we assume the ideal $\mathfrak{P}_k + p^{N'}\mathcal{O}[[\Gamma]]$ contains $\gamma^{p^m} - 1$, the natural homomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \longrightarrow H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'})$$

factors through

$$H^1\big(\mathbb{Q}(\mu_n), \mathbf{T}_\chi \otimes_{\Lambda_\chi} R_{m,N',\chi}\big) \simeq (F_m(\mu_n)^\times/p^{N'})_\chi.$$

We denote the image of $H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'})$ in $(F_m(\mu_n)^\times/p^{N'})_\chi$ by Im_F. Note that by the exact sequence

$$0 \longrightarrow \mathbf{T}_\chi/p^{N'} \xrightarrow{\gamma^{p^m}-1} \mathbf{T}_\chi/p^{N'} \longrightarrow \mathbf{T}_\chi \otimes_{\Lambda_\chi} R_{m,N',\chi} \longrightarrow 0,$$

we have a natural isomorphism

$$H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\Lambda_\chi} R_{m,N',\chi} \xrightarrow{\simeq} \mathrm{Im}_F \subseteq (F_m(\mu_n)^\times/p^{N'})_\chi,$$

and $\tilde{h}|_{\mathrm{Im}_F}$ coincides with the map

$$\tilde{h}_\infty \otimes_{\Lambda_\chi} R_{m,N',\chi}\colon \mathrm{Im}_F \simeq H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \otimes_{\Lambda_\chi} R_{m,N',\chi} \longrightarrow R_{m,N',\chi}[H_n]$$

induced by $\tilde{h}_\infty$. By Proposition 8.16, we obtain the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) \arrow[r, "p^{4s}\tilde{h}_\infty"] \arrow[d] & \mathcal{O}[[\Gamma]][H_n]/p^{N'} \arrow[d] \\
\mathrm{Im}_F \arrow[r, "p^{4s}\tilde{h}"] \arrow[d] & R_{m,N',\chi}[H_n] \arrow[d] \\
H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \arrow[r, "\tilde{h}_{\mathfrak{P}_k}"] & S_{\mathfrak{P}_k}[H_n]/p^{N'} \\
H^1(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}) \arrow[r, "h_{\mathfrak{P}_k}"] \arrow[u, hook] & S_{\mathfrak{P}_k}/p^{N'}. \arrow[u, hook, "\nu_{H_n}"']
\end{tikzcd}$$

By the norm compatibility of circular units, we can define the element

$$\eta_\infty^{D_n} := (\eta_m(n)^{D_n}) \in H^1(\mathbb{Q}(\mu_n), \mathbf{T}_\chi/p^{N'}) = \varprojlim_{m'} (F_{m'}(\mu_n)^\times/p^{N'})_\chi.$$

In particular, we have

$$\eta_m(n)^{D_n} \in \mathrm{Im}_F \subseteq (F_m(\mu_n)^\times/p^{N'})_\chi.$$

Let $\mathcal{O}_S$ be a ring which is isomorphic to $S_{\mathfrak{P}_k}$ as a $\mathcal{O}_\chi$-algebra, and we assume that the Galois group $G_{\mathbb{Q}}$ acts on $\mathcal{O}_S$ trivially. The action of $G_{\mathbb{Q}}$ on $S_{\mathfrak{P}_k}$ defines a continuous character

$$\rho \colon \Gamma \longrightarrow \mathcal{O}_S^\times.$$

We regard both $T_\chi \otimes_{\mathcal{O}} \mathcal{O}_S$ and $T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}$ as free $\mathcal{O}_S$-modules of rank one. Let

$$\eta \otimes \rho := \left\{ (\eta \otimes \rho)_m(n') \in H^1\big(\mathbb{Q}_{m'}(\mu_{n'}), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})\big) \right\}_{m',n'}$$

be the Euler system for $(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k}, \Sigma)$ which is the twist of the Euler system

$$\eta^{(k)} := \left\{ \eta_{m'}^{(k)}(n') \in H^1\big(\mathbb{Q}_{m'}(\mu_{n'}), T_\chi\big) \right\}_{m',n'}$$

for $(T_\chi \otimes_{\mathcal{O}} \mathcal{O}_S, \Sigma)$ by the character ρ in the sense of [Ru4]. Since we assume $n \in \mathcal{N}_{N''}^{\mathrm{w.o.}} \subseteq \mathcal{N}^{\mathrm{w.o.}}(\mathcal{S}_{N'}(\mathfrak{P}_k))$, we can define the Kolyvagin derivative class

$$\kappa_{0,N'}(\eta \otimes \rho; n) \in H^1\big(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}\big)$$

of the Euler system $\eta \otimes \rho$, whose image in $H^1(\mathbb{Q}(\mu_n), (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'})$ coincides with the image of $\eta_m(n)^{D_n}$. By Proposition 8.2, we have

$$\kappa_{0,N'}(\eta \otimes \rho; n) = \kappa(\eta \otimes \rho)_n + \sum_d w_d \cdot \kappa(\eta \otimes \rho)_d \in H^1\big(\mathbb{Q}, (T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{P}_k})/p^{N'}\big)$$

where d runs through all positive divisors of n satisfying $d < n$, w_d's are elements of $S_{\mathfrak{P}_k}$, and $\kappa(\eta \otimes \rho)_n$ is the n-component of the Kolyvagin system corresponding to the

Euler system $\eta \otimes \rho$. Therefore, we obtain

$$\begin{aligned}
\beta e_{\mathfrak{P}} k &\succ \mathrm{length}_{S_{\mathfrak{P}_k}} \left(\frac{S_{\mathfrak{P}_k}/p^{N'}}{\big(\text{the image of } h_k(\kappa_{m_k, N''_k}(\eta; n_k))\big) \cdot (S_{\mathfrak{P}_k}/p^{N'})} \right) \\
&\sim \mathrm{length}_{S_{\mathfrak{P}_k}} \left(\frac{S_{\mathfrak{P}_k}/p^{N'}}{\nu_{H_n}^{-1}\big(p^{4s}\tilde{h}(\eta_{m_k}(n_k)^{D_{n_k}})\big) \cdot (S_{\mathfrak{P}_k}/p^{N'})} \right) \\
&\geq \mathrm{length}_{S_{\mathfrak{P}_k}} \left(\frac{S_{\mathfrak{P}_k}/p^{N'}}{\sum_{d|n} h_{\mathfrak{P}_k}(\kappa(\eta \otimes \rho)_d) \cdot (S_{\mathfrak{P}_k}/p^{N'})} \right) \\
&\geq \min\{\partial_0(\mathfrak{P}_k), \dots, \partial_i(\mathfrak{P}_k), e_{\mathfrak{P}} N'\} = \partial_i(\mathfrak{P}_k) \sim a_k \\
&\sim \alpha e_{\mathfrak{P}} k
\end{aligned}$$

Thus, we obtain $\beta \geq \alpha$, and this completes the proof of Theorem 8.1. □

Appendix A. Review of some results on Kolyvagin systems

In this appendix, we briefly review some basic notions and results on Kolyvagin systems over complete discrete valuation rings of mixed characteristic $(0,p)$ whose residue fields are finite. For details, see [MR]. In our paper, we only use results on Kolyvagin systems for free modules of *rank one*, so we focus on such cases in this appendix. For application to the classical setting to study the structure of $A_{0,\chi}$, see §A.4.

A.1. First, we recall the notion of Selmer structures over complete local ring in a general context. Let $(R, \mathfrak{m})$ be a Noetherian complete local $\mathbb{Z}_p$-algebra whose residue field $R/\mathfrak{m}$ is finite. Here, we may assume R is the integer ring of a finite extension field of $\mathbb{Q}_p$ or the Iwasawa algebra $\mathcal{O}_\chi[[\Gamma]]$ for some $\chi \in \widehat{\Delta}$. Fix a finite set Σ of places of $\mathbb{Q}$ containing p and ∞. In this appendix, we consider a free R-module T of rank *one* on which $G_{\mathbb{Q},\Sigma}$ acts via a continuous character

$$\rho_T \colon G_{\mathbb{Q},\Sigma} \longrightarrow R^\times.$$

We naturally regard the abelian group $T^* := \mathrm{Hom}_{\mathbb{Z}_p}(T, \mu_{p^\infty})$ as an $R[G_\mathbb{Q}]$-module. Throughout this appendix, we always assume the following four conditions (T1)–(T4):

- (T1) Recall that we fix an embedding $p_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ in §1. We denote by $\mathbb{Q}_{p,\infty}$ the composite field of $p_{\overline{\mathbb{Q}}}(\mathbb{Q}_\infty)$ and $\mathbb{Q}_p$. Then, the action of $G_{\mathbb{Q}_{p,\infty}}$ on $T^*[\mathfrak{m}]$ is not trivial.
- (T2) Recall that we fix an embedding $\infty_{\overline{\mathbb{Q}}} \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ in §1. The character ρ_T is not trivial on $G_{\mathbb{Q}_\mathbb{R}}$. (So, the character ρ_T is "odd".)
- (T3) The order of the image of $G_{\mathbb{Q}_\infty}$ by ρ_T is finite, and prime to p.
- (T4) The actions of $G_\mathbb{Q}$ on $T/\mathfrak{m}T$ and $T^*[\mathfrak{m}]$ are non-trivial, and the $\mathbb{F}_p[G_\mathbb{Q}]$-module $T/\mathfrak{m}T$ is not isomorphic to $T^*[\mathfrak{m}]$.

A local condition $\mathcal{F}$ of T is a family

$$\mathcal{F} = \{H^1_\mathcal{F}(\mathbb{Q}_v, T) \subseteq H^1(\mathbb{Q}_v, T)\}_{v:\text{place of } \mathbb{Q}}$$

of R-submodules of local Galois cohomology groups satisfying

$$H^1_{\mathcal{F}}(\mathbb{Q}_v, T) = H^1(\mathbb{Q}_v^{\mathrm{unr}}/\mathbb{Q}_v, T)$$

for all finite places $v \notin \Sigma$. We call a triple $(T, \mathcal{F}, \Sigma)$ a *Selmer structure* over R.

We define the *Selmer group* for a Selmer structure $(T, \mathcal{F}, \Sigma)$ by

$$H^1_{\mathcal{F}}(\mathbb{Q}, T) := \mathrm{Ker}\left(H^1(\mathbb{Q}, T) \longrightarrow \prod_v \frac{H^1(\mathbb{Q}_v, T)}{H^1_{\mathcal{F}}(\mathbb{Q}_v, T)} \right),$$

where in the product, v runs through all places of $\mathbb{Q}$.

Assume that R is the ring of integers of a finite extension field of $\mathbb{Q}_p$. Then, we treat the local condition $\mathcal{F}_{\mathrm{can}}$ for T by

$$H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_v, T) := \begin{cases} H^1_f(\mathbb{Q}_v, T) & \text{if } v \text{ is a finite place prime to } p; \\ H^1(\mathbb{Q}_p, T) & \text{if } v = p; \\ H^1(\mathbb{R}, T) = 0 & \text{if } v = \infty, \end{cases}$$

where $H^1_f(\mathbb{Q}_v, T)$ is defined by

$$H^1_f(\mathbb{Q}_v, T) = \mathrm{Ker}\left(H^1(\mathbb{Q}_v, T) \longrightarrow H^1(\mathbb{Q}_v^{\mathrm{unr}}, T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\right)$$

for finite places v prime to p. We define the dual local condition $\mathcal{F}^*_{\mathrm{can}}$ on T^* by

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_v, T^*) := \begin{cases} H^1_f(\mathbb{Q}_v, T^*) & \text{if } v \text{ is a finite place prime to } p; \\ 0 & \text{if } v = p, \infty. \end{cases}$$

Let I be an ideal of R. Then, for any place v of $\mathbb{Q}$, we denote by $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_v, T/IT)$ the image of $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_v, T)$ in $H^1(\mathbb{Q}_v, T/IT)$, and by $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_v, T^*[I])$ the inverse image of $H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_v, T^*)$ in $H^1(\mathbb{Q}_v, T^*[I])$ respectively.

A.2. Here, we recall the definition of Kolyvagin systems over complete discrete valuation rings. In this subsection, let $(R, \mathfrak{m})$ be the integer ring of a finite extension field of $\mathbb{Q}_p$, and $(T, \mathcal{F}_{\mathrm{can}}, \Sigma)$ be a Selmer structure over R introduced in the previous subsection. Note that the Selmer structure $(T, \mathcal{F}_{\mathrm{can}}, \Sigma)$ satisfies the conditions (H.0)–(H.4) and (H.6) in [MR] §3.5. We also remark that we have $\chi(T, \mathcal{F}_{\mathrm{can}})$=1, where $\chi(T, \mathcal{F}_{\mathrm{can}})$ is the core rank defined in [MR] Definition 4.1.11. (See also [MR] Theorem 5.2.15.)

For each prime number $\ell \notin \Sigma$, we define

$$P_\ell(x) := \det_R(1 - \mathrm{Fr}_\ell x \mid T) = 1 - \rho_T(\mathrm{Fr}_\ell)x \in R[x],$$

where $\mathrm{Fr}_\ell \in G_{\mathbb{Q}}$ is an arithmetic Frobenius element. Note that the polynomial $P_\ell(x)$ is well-defined since the action of $G_{\mathbb{Q}}$ on T is unramified at ℓ. We denote by I_ℓ the ideal of R generated by $\ell - 1$ and $P_\ell(1)$. Take a square-free product $n := \ell_1 \times \cdots \times \ell_r$, where ℓ_i is a prime number not contained in Σ for $i = 1, ..., r$. Then, we define an ideal $I_n := \sum_{i=1}^r I_{\ell_i}$.

Let $\mathcal{P}$ a set of rational primes disjoint from Σ. We denote the set of all square-free products of $\mathcal{P}$ by $\mathcal{N}(\mathcal{P})$. (Note $1 \in \mathcal{N}(\mathcal{P})$.) We call such a triple $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ a *Selmer* *triple.* In this paper, we always take $\mathcal{P} = \mathcal{P}_1$. Here, for any positive N, we denote by $\mathcal{P}_N := \mathcal{P}_N(T, \Sigma)$ the set of all prime numbers ℓ not contained in Σ satisfying $I_\ell \subseteq \mathfrak{m}^N$.

In order to define Kolyvagin systems, we have to introduce new local conditions $\mathcal{F}_{\mathrm{can}}(n)$ on $T/\mathfrak{m}^N T$ for all $n \in \mathcal{N}(\mathcal{P}_N)$, and two homomorphisms of Galois cohomology groups, the "localization" map and the "finite-singular comparison" map.

Let N be a positive integer and $\ell \in \mathcal{P}_N$ an arbitrary element. Then, we define

$$H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) := \mathrm{Ker}\left(H^1(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \longrightarrow H^1(\mathbb{Q}_\ell(\mu_\ell), T/\mathfrak{m}^N T)\right).$$

Then, by [MR] Lemma 1.2.4, we have a direct sum decomposition

$$H^1(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) = H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \oplus H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T),$$

so the natural projection

$$H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \longrightarrow H^1_s(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) := \frac{H^1(\mathbb{Q}_\ell, T/\mathfrak{m}^N T)}{H^1_f(\mathbb{Q}_\ell, T/\mathfrak{m}^N T)}$$

is an isomorphism. For any $n \in \mathcal{N}(\mathcal{P})$, we define a new local condition

$$H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) := \begin{cases} H^1_{\mathcal{F}_{\mathrm{can}}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) & \text{if } \ell \nmid n; \\ H^1_{\mathrm{tr}}(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) & \text{if } \ell \mid n. \end{cases}$$

Let ℓ be a prime number prime to n and not contained in Σ. Recall that we put $H_\ell := \mathrm{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q}) \simeq \mathbb{F}_\ell^\times$ in §4.1. The *localization map*

$$(\cdot)_{\ell,s} \colon H^1_{\mathcal{F}_{\mathrm{can}}(n\ell)}(\mathbb{Q}, T/I_{n\ell}T) \otimes_{\mathbb{Z}} H_{n\ell} \longrightarrow H^1_s(\mathbb{Q}_\ell, T/I_{n\ell}T) \otimes_{\mathbb{Z}} H_{n\ell},$$

is the homomorphism induced by the restriction map of Galois cohomology groups.

In order to define the "finite-singular comparison" map, we need the following lemma.

Lemma A.1 ([Ru2] Lemma 1.4.7)**.** *Let $\ell \in \mathcal{P}_N$ be an arbitrary element. Then, evaluating cocycles on Fr_ℓ and H_ℓ induces isomorphisms*

$$\alpha \colon H^1_f(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \xrightarrow{\simeq} T/\left(\mathfrak{m}^N T + (\mathrm{Fr}_\ell - 1)T\right) = T/\mathfrak{m}^N T$$
$$\beta \colon H^1_s(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbb{Z}}(H_\ell, T/\mathfrak{m}^N T)^{\mathrm{Fr}_\ell = 1} = \mathrm{Hom}_{\mathbb{Z}}(H_\ell, T/\mathfrak{m}^N T)$$

of $R/\mathfrak{m}^N$-modules respectively.

Now we define the "finite-singular comparison" map

$$\phi^{\mathrm{fs}}_\ell \colon H^1_f(\mathbb{Q}_\ell, T/I_n T) \longrightarrow H^1_s(\mathbb{Q}_\ell, T/I_n T) \otimes_{\mathbb{Z}} H_\ell$$

to be the inverse map of the composite map

$$\begin{aligned} H^1_s(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \otimes_{\mathbb{Z}} H_\ell &\xrightarrow[\beta^{-1}]{\simeq} \mathrm{Hom}(H_\ell, T/\mathfrak{m}^N T) \otimes_{\mathbb{Z}} H_\ell \\ &\xrightarrow{\simeq} T/\mathfrak{m}^N T \xrightarrow[-\alpha]{\simeq} H^1_f(\mathbb{Q}_\ell, T/\mathfrak{m}^N T) \end{aligned}$$

for any $n \in \mathcal{N}_N$ and any prime divisor ℓ of n.

Definition A.2. A *Kolyvagin system* for a Selmer triple $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_1)$ is a family of cohomology classes

$$\kappa = \left\{ \kappa_n \in H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, T/I_nT) \otimes_{\mathbb{Z}} \otimes_{\mathbb{Z}} \bigotimes_{\ell \mid n} H_\ell \right\}_{n \in \mathcal{N}(\mathcal{P}_1)}$$

satisfying

$$(\kappa_{n\ell})_{\ell,s} = \phi_\ell^{\mathrm{fs}}(\kappa_n) \quad \text{in} \quad H^1_s(\mathbb{Q}_\ell, T/I_{n\ell}T) \otimes_{\mathbb{Z}} \bigotimes_{\ell \mid n} H_\ell$$

for any $n \in \mathcal{N}(\mathcal{P}_1)$ and any $\ell \in \mathcal{P}_1$ satisfying $n\ell \in \mathcal{N}(\mathcal{P}_1)$. We denote the set of all Kolyvagin systems for $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_1)$ by $\mathrm{KS}(T, \Sigma)$.

As we shall see below Proposition A.3, Kolyvagin systems for the triple $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P})$ describe the structure of the Selmer group

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*) = \mathrm{Ker}\left(H^1(\mathbb{Q}, T^*) \longrightarrow \prod_v \frac{H^1(\mathbb{Q}_v, T^*)}{H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}_v, T^*)} \right),$$

where in the product, v runs through all places of $\mathbb{Q}$.

Recall that for each element $n := \ell_1 \times \cdots \times \ell_r \in \mathcal{N}(\mathcal{P})$, we denote the number of prime divisors of n by $\epsilon(n) := r$. For any non-zero element $\kappa = \{\kappa_n\} \in \mathrm{KS}(T, \Sigma)$ and any non-negative integer i, we denote the maximum (accepting ∞) of the set

$$\left\{ j \in \mathbb{Z}_{\geq 0} \mid \kappa_n \in \mathfrak{m}^j H_{\mathcal{F}(n)}(\mathbb{Q}, T/I_nT) \otimes H_n \text{ for all } n \in \mathcal{N}(\mathcal{P}_1) \text{ with } \epsilon(n) = i \right\}$$

by $\partial_i(\kappa; T)$. We also define

$$\partial_i(T) := \min\{\partial_i(\kappa; T) \mid \kappa = \{\kappa_n\} \in \mathrm{KS}(T, \Sigma)\}.$$

Note that $\partial_i(T) = 0$ for sufficiently large i, and $\partial_j(T) \geq \partial_{j+1}(T)$ for any $j \in \mathbb{Z}_{\geq 0}$. (See [MR] Theorem 5.10 (ii) and Theorem 5.12).

Proposition A.3 (a special case of [MR] Theorem 5.2.12)**.** *Let* $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_1)$ *be as above, and we put*

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*)^\vee := \mathrm{Hom}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*), \mathbb{Q}_p/\mathbb{Z}_p \right).$$

Then, we have

$$\mathrm{Fitt}_{R,i}\left(H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*)^\vee \right) = \mathfrak{m}^{\partial_i(T)}$$

for any $i \in \mathbb{Z}_{\geq 0}$.

A.3. Here, we recall a result with relate Euler systems to Kolyvagin systems. Let $(R, \mathfrak{m})$ and $(T, \mathcal{F}_{\mathrm{can}}, \mathcal{P}_1)$ be as in the previous subsection. First, we recall the definition of Euler systems in the sense of [Ru4] Remark 2.1.4.

Definition A.4. Let $\mathcal{N}(\Sigma^c)$ the set of all positive integers decomposed into square free products of prime numbers not contained in Σ. In this paper, we call a family

$$\mathbf{c} := \left\{ c_m(n) \in H^1(\mathbb{Q}_m(\mu_n), T) \right\}_{m \geq 0, n \in \mathcal{N}(\Sigma^c)}$$

of cohomology classes *an Euler system for* (T, Σ) if $\mathbf{c}$ satisfies the following conditions:

(ES1) For any $n \in \mathcal{N}(\Sigma^c)$ and any non-negative integer m, we have

$$\mathrm{Cor}_{\mathbb{Q}_{m+1}(n)/\mathbb{Q}_m(n)}\left(c_{m+1}(n)\right) = c_m(n).$$

(ES2) Let $n \in \mathcal{N}(\Sigma^c)$ and m a non-negative integer. Then, for any prime divisor ℓ of n, we have

$$\mathrm{Cor}_{\mathbb{Q}_m(n)/\mathbb{Q}_m(n/\ell)}\left(c_{m+1}(n)\right) = \left(1 - \ell^{-1}\rho_T(\mathrm{Fr}_\ell)\cdot \mathrm{Fr}_\ell^{-1}\right)\cdot c_m(n/\ell).$$

We denote the set of all Euler systems for (T, Σ) by $\mathrm{ES}(T, \Sigma)$.

Remark A.5. Note that "Euler factors" in (ES2) are a bit different from that in [MR]. In order to fit our notation into that in [MR], see [Ru4] Lemma 9.6.1. (In particular, see the construction of $\mathbf{c}_F$ in the proof of [Ru4] Lemma 9.6.1.)

Fix an Euler system $\mathbf{c} = \{c_m(n)\}_{m\geq 0, n\in\mathcal{N}(\Sigma^c)} \in \mathrm{ES}(T,\Sigma)$, a positive integer N and any element $n \in \mathcal{N}(\mathcal{P}_N)$. Let $D_n \in \mathbb{Z}[H_n]$ be the element introduced in Definition 4.7. Then, as in Definition 4.9, we can construct the canonical element $\kappa_{0,N}(n;\mathbf{c}) \in H^1(\mathbb{Q}, T/\mathfrak{m}^N T)$ called *Kolyvagin derivative*, whose image in $H^1(\mathbb{Q}(\mu_n), T/\mathfrak{m}^N T)$ coincides with $D_n \cdot c_m(n)$. (For details, see [Ru4] §4.4.) The following result relates Euler systems to Kolyvagin systems.

Proposition A.6. *Assume that the action of $\rho_T(\mathrm{Fr}_\ell)^{p^k} \neq 1$ for any $\ell \in \mathcal{P}$ and any $k \in \mathbb{Z}_{\geq 0}$. Then, there exists an R-linear map*

$$\mathrm{ES}(T,\Sigma) \longrightarrow \mathrm{KS}(T,\Sigma); \quad \mathbf{c} = \{c_m(n)\} \longmapsto \kappa(\mathbf{c}) := \{\kappa(\mathbf{c})_n\}_n$$

satisfying the following property.

(EK) *Let $n \in \mathcal{N}(\mathcal{P}_1)$ be any well-ordered element (cf. Definition 4.10), and put $I_n = \mathfrak{m}^N$. Then, for any $\mathbf{c} \in \mathrm{ES}(T,\Sigma)$, we have*

$$\kappa(\mathbf{c})_n \equiv \kappa(n;\mathbf{c}) \otimes \bigotimes_{\ell \mid n} \sigma_\ell \mod M(n;\mathbf{c}),$$

where $M(n,\mathbf{c})$ is an R-submodule of $H^1(\mathbb{Q}, T/I_nT) \otimes_{\mathbb{Z}} \bigotimes_{\ell\mid n} H_\ell$ generated by

$$\left\{ \kappa(\mathbf{c})_d \;\middle|\; 0 < d \mid n,\ d \neq n \right\}.$$

Here, we put $M(1;\mathbf{c}) := 0$.

Remark A.7. For the construction of map in A.6, see [MR] Appendix A. (See also [MR] Proposition 5.2.9.) The property (EK) follows from the construction of the map.

A.4. In §8.2, we specialize the above general setting to the following case. We assume $p \nmid [K : \mathbb{Q}]$, and fix a character $\chi \in \widehat{\Delta}$ satisfying $\chi(p) \neq 1$. We define a set Σ of places of $\mathbb{Q}$ by

$$\Sigma := \{p, \infty\} \cup \{\ell \mid \ell \text{ ramifies in } K/\mathbb{Q}\},$$

Put $R = \mathcal{O} := \mathbb{Z}_p[\mathrm{Im}\,\chi]$ and $T = T_\chi = \mathcal{O}(1) \otimes \chi^{-1}$. Note that p is a prime element in $\mathcal{O}$ since the order of χ is prime to p. In §8, we consider the Selmer structure $(T_\chi, \mathcal{F}_{\mathrm{can}}, \Sigma)$. (Note that in our paper, we also treat "twisted" Selmer structures

$(T_\chi \otimes_{\mathcal{O}} S_{\mathfrak{Q}}, \mathcal{F}_{\mathrm{can}}, \Sigma)$, but we do not introduce them here.) For any positive integer N, we have

$$H^1(\mathbb{Q}, T/p^N T) = (F_0^\times/p^N)_\chi$$

by Kummer theory, and for any prime number ℓ, local class field theory implies

$$H^1_f(\mathbb{Q}_\ell, T/p^N T) = \left((\mathcal{O}_{F_0} \otimes \mathbb{Z}_\ell)^\times/p^N\right)_\chi.$$

Since we assume $\chi(p) \neq 1$, by global class field theory, we have

$$H^1_{\mathcal{F}^*_{\mathrm{can}}}(\mathbb{Q}, T^*_\chi) = \mathrm{Hom}(A_{F_0,\chi}, \mathbb{Q}_p/\mathbb{Z}_p).$$

By definition, we have $\mathcal{S}_N \subseteq \mathcal{P} := \mathcal{P}_1$ for any $N \in \mathbb{Z}_{>0}$. Let $\ell \in \mathcal{S}_N$ and $n \in \mathcal{N}_N$ be arbitrary elements satisfying $(\ell, n) = 1$. We put $I_n = p^{N'}\mathcal{O}$ and $I_{n\ell} = p^{N''}\mathcal{O}$. Then, the localization map

$$(\cdot)_{\ell,s} \colon H^1_{\mathcal{F}_{\mathrm{can}}(n\ell)}(\mathbb{Q}, T_\chi/I_{n\ell}T_\chi) \otimes_{\mathbb{Z}} H_{n\ell} \longrightarrow H^1_s(\mathbb{Q}_\ell, T_\chi/I_{n\ell}T_\chi) \otimes_{\mathbb{Z}} H_{n\ell},$$

and the composite map

$$\begin{aligned} H^1_{\mathcal{F}_{\mathrm{can}}(n)}(\mathbb{Q}, T_\chi/I_nT_\chi) \otimes_{\mathbb{Z}} H_n &\longrightarrow H^1_f(\mathbb{Q}_\ell, T_\chi/I_{n\ell}T_\chi) \otimes_{\mathbb{Z}} H_n \\ &\xrightarrow{\phi^{\mathrm{fs}}_\ell \otimes 1} H^1_s(\mathbb{Q}_\ell, T_\chi/I_{n\ell}T_\chi) \otimes_{\mathbb{Z}} H_{n\ell}, \end{aligned}$$

coincide with the restriction of the map

$$[\cdot]^\ell_{0,N'',\chi} \otimes 1 \colon (F_0^\times/p^{N''})_\chi \otimes_{\mathbb{Z}} H_{n\ell} \longrightarrow R_{0,N'',\chi} \otimes_{\mathbb{Z}} H_{n\ell}$$

defined in Definition 5.1 and that of the composite map

$$(F_0^\times/p^{N'})_\chi \otimes_{\mathbb{Z}} H_n \longrightarrow (F_0^\times/p^{N''})_\chi \otimes_{\mathbb{Z}} H_n \xrightarrow{\phi^\ell_{0,N',\chi} \otimes 1} R_{0,N'',\chi} \otimes_{\mathbb{Z}} H_{n\ell}$$

defined in Definition 5.2 respectively.

Note that for any Kolyvagin system $\kappa = \{\kappa_n\}_n \in \mathrm{KS}(T_\chi, \Sigma)$, each κ_n is written by an $\mathcal{O}_\chi$-linear combination of Kolyvagin derivatives $\{\kappa^\bullet_{0,N}(d)\}_{\bullet, d|n}$ of basic circular units.

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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: ohshita@kurims.kyoto-u.ac.jp