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"Bühlmann's Economic Premium Principle
in The Presence of Transaction Costs"

Masaaki Kijima and Akihisa Tamura

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BÜHLMANN'S ECONOMIC PREMIUM PRINCIPLE IN THE PRESENCE OF TRANSACTION COSTS

MASAAKI KIJIMA AND AKIHISA TAMURA

ABSTRACT. This paper examines the Bühlmann's equilibrium pricing model (1980) in the presence of transaction cost and derives the (multivariate) Esscher transform within the framework under some assumptions. The result reveals that the Esscher transform is an appropriate probability transform for the pricing of insurance risks even in the market with transaction costs.

Keywords: Equilibrium pricing, Equilibrium allocation, Incomplete market, Esscher transform, Transaction cost

1. INTRODUCTION

In the finance literature, the theory of asset pricing has been studied for the long time; the theory is well-developed for the so-called *complete* market while there are still many blanks for *incomplete* markets. When there are transaction costs for trading assets in the market, some asset may not be duplicated by other assets and so the market is incomplete. The insurance market is presumably incomplete; new attempts are necessary for the development of economically sound pricing methods.

In the actuarial literature, there have been developed many probability transforms for the pricing of insurance risks. Such methods include the variance loading, the standard deviation loading, and the exponential principle. Among them, one of the popular pricing methods for actuaries is the *Esscher transform* given by

$$\pi(Y) = \frac{\mathbb{E}[Y\mathrm{e}^{-\theta Y}]}{\mathbb{E}[\mathrm{e}^{-\theta Y}]} \tag{1.1}$$

for random variable Y that represents risk, where θ is a positive constant[1] and $\mathbb{E}$ is an expectation operator under the physical probability measure $\mathbb{P}$. As pointed out by Bühlmann (1980), however, the premiums calculated by these methods depend only on the risk, while in economics premiums are not only depending on the risk but also on market conditions.

Bühlmann (1980) considers a pure risk exchange market in which there are N agents. Each agent is characterized by his/her utility function, initial wealth and potential loss, and is willing to buy/sell a risk exchange so as to maximize the expected utility. An equilibrium price of the risk is obtained under the market clearing condition. Following Bühlmann (1980), equilibrium models of insurance risks have been considered by many authors, including Aase (1993, 2002), Malamud, Trubowitz and Wüthrich (2008), and Tsanakas and Christofides (2006).

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M. Kijima: Graduate School of Social Sciences, Tokyo Metropolitan University. Address: 1-1 Minami-Ohsawa, Hachiohji, Tokyo 192-0397, Japan. E-mail: `kijima@tmu.ac.jp`

A. Tamura: Department of Mathematics, Keio University. Address: 3-14-1 Hiyoshi, Kohoku-ku, Yokohama-shi, Kanagawa 223-8522, Japan. E-mail: `aki-tamura@math.keio.ac.jp`

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[1]This paper treats risk as an asset. A liability with loss variable X can be viewed as a negative asset with gain $Y = -X$. See Wang (2002) for details.

Bühlmann (1980) demonstrates that the Esscher transform (1.1) can be derived from the equilibrium price, when exponential utilities are assumed and the risk Y is sufficiently small compared to the whole aggregated risk. Hence, the Esscher transform is not just an exponential tilting (or exponential change of measure), but has a sound economic interpretation. See also Wang (2002) and Kijima (2006) for further discussions on the Esscher transform and its economic interpretations. In particular, Kijima (2006) extends the Esscher transform (1.1) to the multivariate setting as

$$\pi(Y) = \frac{\mathbb{E}[Y\mathrm{e}^{-\theta Z}]}{\mathbb{E}[\mathrm{e}^{-\theta Z}]}, \qquad Z = \sum_{j=1}^{N} Y_j, \tag{1.2}$$

where $Y = h(Y_1, \ldots, Y_N)$ for some function h, called the *multivariate* Esscher transform, and shows that the transform (1.2) possesses many desirable properties as a pricing method.[2]

Although not mentioned explicitly, the risk exchange market considered in Bühlmann (1980) is *complete*, while actual insurance markets are presumably *incomplete*. In particular, there are transaction costs for trading risks (and/or assets) in the market. Recall that a market is complete if and only if any asset is duplicated by other existing assets in the market (see, e.g., Kijima [2013]). In other words, agents can use any asset in order to maximize their expected utilities in the case of complete markets. The market in the presence of transaction costs is a typical example of incomplete markets. The aim of this paper is to extend the Bühlmann's result (1980) to the market with transaction cost, thereby giving a further justification to the Esscher transform (1.1) and its variants.

In the finance literature, many papers have considered the pricing of derivatives in the presence of transaction costs for trading the underlying assets. When the market is complete and there are no transaction costs, any derivative can be duplicated by trading underlying assets continuously (i.e., the perfect hedge) and the price of derivative is given by the initial cost of the duplication. When there are transaction costs, this paradigm no longer holds and elaborated mathematical arguments are required to determine a super-hedging portfolio. See Kabanov and Safarian (2009) and references therein for detailed discussions on this topic. However, in these studies, the underlying asset prices are given *exogenously* and the asset demand to duplicate the derivative has no impact on the prices of both the derivative and the underlying assets. In other words, no attention has been paid to the equilibrium of asset prices in the market with transaction costs.

In the economics literature, on the other hand, there are many papers that investigate the equilibrium of asset prices. Recently, Buss, Uppal and Vilkov (2011) and Hara (2013) consider the problem of asset prices in the general equilibrium with proportional transaction costs . In particular, Hara (2013) studies a single-period model in which there are multiple agents with general utility functions and two assets, one riskfree and one risky, and determines the equilibrium asset prices for each level of transaction costs to show, among others, that an increase in transaction costs will increase buying prices and decrease selling prices under some conditions. Buss, Uppal and Vilkov (2011) investigate a multi-period model in which there are only two agents with recursive utilities. See these papers and references therein for the general equilibrium of asset prices with and without transaction costs.

Finally, in the actuarial literature, there are also many papers that consider the effect of transaction costs. For example, among others, He and Liang (2009) consider an optimal financing and dividend control of the insurance company with transaction costs. Højgaard and Taksar

[2]Another popular pricing method for actuaries is the Wang transform developed by Wang (2002), which is further extended by Kijima (2006) to the multivariate setting, based on the Bühlmann's premium principle (1980). In particular, Kijima (2006) shows that, when risks are normally distributed, the (multivariate) Esscher transform is the same as the (multivariate) Wang transform. See Kijima and Muromachi (2008) for further discussions on the relationship between the Bühlmann's result and the Wang transform.

(1998) study a similar problem for reinsurance policies with transaction costs. However, as in the finance literature, the underlying processes are given exogenously and no aspect of equilibrium is investigated. In this paper, following Hara (2013), we consider a single-period equilibrium model with multiple agents and multiple risky assets. However, because our main goal is to extend the multivariate Esscher transform (1.2) to the market with transaction costs, we focus on the case of exponential utilities.

The present paper is organized as follows. In the next section, we setup the model of asset prices in the general equilibrium with proportional transaction costs. In Section 3, we first review the Bühlmann's result (1980) by solving the equilibrium model for the case of complete market, and then examine the case of incomplete market without transaction costs. It is shown that the problem can be solved under some conditions and the multivariate Esscher transform (1.2) is derived. Section 4 is devoted to the existence of the general equilibrium for the general problem. Some special case of exponential utilities and normally distributed assets (i.e., the CARA-normal case) is also considered. In section 5, we investigate the case that the transaction costs are so small. In particular, when the rates of return of all the assets are normally distributed, it is shown that the asset prices are given by the multivariate Esscher transform (1.2) with the mean rates of return being adjusted by transaction costs. Finally, Section 6 concludes this paper.

2. Model Setup

Consider an agent i with initial risk X_i and utility function $u_i(x)$. The risk X_i may be a portfolio of assets traded in the market or other types of nontradable assets. As usual, we consider a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that $u_i' > 0$ and $u_i'' < 0$. Let us denote by $\mathcal{M}$ the class of traded assets in the market under consideration.

Suppose that there are I agents characterized by the pair (X_i, u_i), $i = 1, 2, \ldots, I$, in the market. We want to derive an equilibrium price $\pi(Y)$, $Y \in \mathcal{M}$, satisfying

$$\text{(2.1)} \quad \begin{cases} \widetilde{Y}_i = \underset{Y_i \in \mathcal{M}}{\operatorname{argmax}} \ \mathbb{E}[u_i(X_i + Y_i)], & i = 1, 2, \ldots, I, \\ \quad \text{subject to} \ \ \pi(Y_i) + tc(Y_i) = 0, & i = 1, 2, \ldots, I, \qquad \text{(budget constraint)} \\ \qquad\qquad \sum_{i=1}^{I} Y_i = 0, & \qquad\qquad\qquad\qquad \text{(market clearing)} \end{cases}$$

where $tc(Y)$ denotes the transaction cost associated with exchange Y. The optimal $\widetilde{\mathbf{Y}} = (\widetilde{Y}_1, \widetilde{Y}_2, \ldots, \widetilde{Y}_I)$ is called an *equilibrium risk exchange* and $\mathbf{X} + \widetilde{\mathbf{Y}}$ an *equilibrium risk allocation*, where $\mathbf{X} = (X_1, X_2, \ldots, X_I)$. In this paper, for the sake of simplicity, the riskfree interest rate is assumed to be zero.[3]

In order to formulate transaction costs explicitly, we assume that only $(N + 1)$ assets are traded in the market. The time-1 (future) value of asset j, $j = 0, 1, \ldots, N$, is denoted by S_j and its time-0 (present) value by $\pi_j = \pi(S_j)$. In this setting, any traded portfolio for agent i is written as

$$\text{(2.2)} \qquad Y_i = \sum_{j=0}^{N} y_j^i S_j, \qquad i = 1, 2, \ldots, I,$$

where the quantity y_j^i represents the number of asset j traded by agent i at time 0. Of course, $y_j^i > 0$ implies that agent i purchases asset j, whereas $y_j^i = 0$ and $y_j^i < 0$ mean no trade and a sell of asset j, respectively. Throughout this paper, we assume that the holdings are real numbers.

[3]Alternatively, we assume that the risks are enumerated by the riskfree money-market account.

The initial risks X_i consist of traded assets and nontradable risks. More specifically, we assume that the initial risk of agent i is given by

$$X_i = \sum_{j=0}^{N} x_j^i S_j + \varepsilon_i, \qquad i = 1, 2, \ldots, I, \tag{2.3}$$

where the quantity x_j^i represents the number of asset j held by agent i at time 0 and $\varepsilon_i \notin \mathcal{M}$ denotes the residual risk. The total number of asset j issued in the market is denoted by

$$A_j \equiv \sum_{i=1}^{I} x_j^i, \qquad j = 0, 1, \ldots, N, \tag{2.4}$$

which are assumed to be positive constants.

Asset 0 is the riskfree discount bond (so that $S_0 = 1$), while the other assets are risky (so that S_j, $j > 0$, are random variables). Denote by c_j the transaction cost of buying and selling one unit of asset j. Then, if $y_j^i > 0$ ($y_j^i < 0$, respectively), agent i must pay the proportional cost $c_j y_j^i \pi_j > 0$ ($c_j(-y_j^i)\pi_j > 0$). It is assumed that the transaction costs disappear from the economy. Throughout this paper, we shall denote

$$\gamma_j(y) = c_j \operatorname{sgn}(y) \equiv \begin{cases} +c_j & \text{if } y > 0, \\ -c_j & \text{if } y < 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Then, the total trading cost (including the transaction cost) is given by

$$\pi(Y) + tc(Y) = \sum_{j=0}^{N} y_j \pi_j (1 + \gamma_j(y_j)), \tag{2.5}$$

where $Y = \sum_{j=0}^{N} y_j S_j$. Note that, in the case of no transaction costs, we have $\gamma_j(y) = 0$ so that $\pi(Y) + tc(Y) = \sum_{j=0}^{N} y_j \pi_j$.

We deal with allocation variables $\theta_j^i = x_j^i + y_j^i$ instead of exchange variables y_j^i for all i and j. Then, from (2.2)–(2.5), the problem (2.1) can be restated as follows: For given transaction costs $c_j > 0$, we want to derive equilibrium prices $\pi_j = \pi(S_j)$ satisfying

$$\begin{cases} \widetilde{\theta}_j^i = \underset{\theta_j \in \mathcal{R}}{\operatorname{argmax}} \ \mathbb{E}\left[u_i\left(\varepsilon_i + \sum_{j=0}^{N} \theta_j S_j\right)\right], & i = 1, 2, \ldots, I, \\ \qquad \text{subject to} \ \sum_{j=0}^{N} (\theta_j^i - x_j^i)\pi_j(1 + \gamma_j(\theta_j^i - x_j^i)) \le 0, & i = 1, 2, \ldots, I, \\ & \text{(budget constraint)} \\ \qquad \sum_{i=1}^{I} \theta_j^i = A_j, \qquad j = 1, 2, \ldots, N, & \text{(market clearing)} \end{cases} \tag{2.6}$$

where $\mathcal{R}$ denotes the set of real numbers. Note that we relax the budget constraints so as to have inequality. Also, the market clearing condition does not apply for the riskfree asset S_0.

3. The Case of No Transaction Cost

Before proceeding, we consider the case of no transaction costs in order to make clear how the transaction costs affect the results in equilibrium. In this section, we first examine the case of complete market and then the incomplete case follows. As we shall see soon, even in the incomplete case, we can obtain similar results to the complete case under some conditions.

3.1. **Complete market.** If the market is complete, then it is well know that there exists a state price density $\eta > 0$ such that $\pi(Y) = \mathbb{E}[\eta Y]$ and $\mathbb{E}[\eta] = 1$. Thus, in order to solve the problem (2.1), we consider the Lagurange equations defined by

$$L_i = \mathbb{E}[u_i(X_i + Y)] - \ell_i \mathbb{E}[\eta Y], \qquad i = 1, 2, \ldots, I. \tag{3.1}$$

The first order condition (FOC for short) of (3.1) with respect to $Y(\omega)$, $\omega \in \Omega$, is given by

$$u_i'(X_i(\omega) + \widetilde{Y}_i(\omega)) - \ell_i \eta(\omega) = 0. \tag{3.2}$$

Let us denote the inverse function of u_i' by $I_i = (u_i')^{-1}$. Then, from the FOC (3.2), we have

$$X_i + \widetilde{Y}_i = I_i(\ell_i \eta), \qquad i = 1, 2, \ldots, I. \tag{3.3}$$

Summing over i and utilizing the market clearing condition in (2.1), we obtain

$$\sum_{i=1}^{I} X_i = \sum_{i=1}^{I} I_i(\ell_i \eta). \tag{3.4}$$

Define Z and $I(x)$ by

$$Z \equiv \sum_{i=1}^{I} X_i, \qquad I(\eta C) \equiv \sum_{i=1}^{I} I_i(\ell_i \eta) \tag{3.5}$$

for some C. Also, denote the inverse function of $I(x)$ by $u'(x)$.[4] It follows from (3.3) and (3.4) that $\eta = u'(Z)/C$. Since $\mathbb{E}[\eta] = 1$ so that $C = \mathbb{E}[u'(Z)]$, we finally obtain the equilibrium price as

$$\pi(Y) = \frac{\mathbb{E}[Y u'(Z)]}{\mathbb{E}[u'(Z)]}; \quad Z = \sum_{i=1}^{I} X_i, \qquad \forall Y \in \mathcal{M}. \tag{3.6}$$

The equilibrium risk allocation is given by (3.3). Note that the expressions (3.6) and (3.3) are not explicit, because they involve the unknown Lagurange multipliers ℓ_i, $i = 1, 2, \ldots, I$.

3.1.1. *Special case: Exponential utility.* When all the agents have exponential utility functions, the above problem can be solved explicitly. Suppose that

$$u_i(x) = -\frac{1}{\lambda_i} \mathrm{e}^{-\lambda_i x}; \quad \lambda_i > 0, \qquad i = 1, 2, \ldots, I. \tag{3.7}$$

Then, since $u_i'(x) = \mathrm{e}^{-\lambda_i x}$, the FOC (3.3) can be written as

$$X_i + \widetilde{Y}_i = \frac{-1}{\lambda_i}(\log \eta + \log \ell_i), \qquad i = 1, 2, \ldots, I. \tag{3.8}$$

Summing over i and utilizing the market clearing condition, we have

$$Z = -\frac{1}{\lambda}(\log \eta + \log C) \tag{3.9}$$

for some C, where we put

$$\frac{1}{\lambda} = \sum_{i=1}^{I} \frac{1}{\lambda_i}. \tag{3.10}$$

[4]The inverse function exists under the condition $u_i'' < 0$ for all i. The function $u'(x)$ can be seen as the marginal utility function of a *representative agent* in the market.

It is readily checked from (3.9) that we have $C = \mathbb{E}[\mathrm{e}^{-\lambda Z}]$ since $\mathbb{E}[\eta] = 1$. Therefore, the equilibrium price (3.6) is given by

$$\pi(Y) = \frac{\mathbb{E}[Y\mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}; \quad Z = \sum_{i=1}^{I} X_i, \qquad \forall Y \in \mathcal{M}, \tag{3.11}$$

where λ is defined by (3.10). The equilibrium pricing formula (3.11) is explicit, because Z and λ are defined only through the given quantities X_i and λ_i, respectively.

The equilibrium risk allocation (3.3) can be also obtained explicitly. Namely, we have

$$X_i + \widetilde{Y}_i = \frac{\lambda}{\lambda_i} Z, \qquad i = 1, 2, \ldots, I. \tag{3.12}$$

Note that the allocation $X_i + \widetilde{Y}_i$ is proportional to the aggregated risk Z with weight $\lambda/\lambda_i > 0$, where $\sum_{i=1}^{I} \lambda/\lambda_i = 1$, for the exponential utility case.

Finally, note that, when $Z = Y + \xi$ with Y and ξ being mutually independent, the equilibrium price (3.11) coincides with the Esscher transform (1.1) for risk Y, as claimed by Bühlmann (1980).

3.2. **Incomplete market.** In this subsection, we consider the problem (2.6) without transaction costs. Because some asset $Y \in \mathcal{M}$ may not be duplicated by tradable assets S_j, $j = 0, 1, \ldots, N$, the market is *incomplete*.

Suppose that the budget constraint in (2.6) is given by

$$\sum_{j=0}^{N} (\theta_j^i - x_j^i)\pi_j = 0, \qquad i = 1, 2, \ldots, I. \tag{3.13}$$

Then, we can assume $\pi_0 = 1$ without loss of generality. Consider the Lagurange equations

$$L_i = \mathbb{E}\left[u_i\left(\varepsilon_i + \sum_{j=0}^{N} \theta_j^i S_j\right)\right] - \ell_i \sum_{j=0}^{N} (\theta_j^i - x_j^i)\pi_j, \qquad i = 1, 2, \ldots, I. \tag{3.14}$$

The FOC of (3.14) with respect to θ_j^i is given by

$$\mathbb{E}\left[S_j u_i'\left(\varepsilon_i + \sum_{k=0}^{N} \widetilde{\theta}_k^i S_k\right)\right] - \ell_i \pi_j = 0, \qquad \forall i,\ j.$$

In particular, for $j = 0$, we have

$$\ell_i = \mathbb{E}\left[u_i'\left(\varepsilon_i + \sum_{k=0}^{N} \widetilde{\theta}_k^i S_k\right)\right],$$

since $\pi_0 = S_0 = 1$. It follows that

$$\pi_j = \frac{\mathbb{E}\left[S_j u_i'\left(\varepsilon_i + \sum_{k=0}^{N} \widetilde{\theta}_k^i S_k\right)\right]}{\mathbb{E}\left[u_i'\left(\varepsilon_i + \sum_{k=0}^{N} \widetilde{\theta}_k^i S_k\right)\right]}, \qquad \forall i,\ j. \tag{3.15}$$

In other words, the equilibrium prices π_j are determined by the system of equations (3.15) by choosing the optimal allocations $\widetilde{\theta}_j^i$ so that the right-hand side of (3.15) becomes independent of i. Note that the prices π_j depend on the joint distribution of $(S_1, \ldots, S_N, \varepsilon_1, \ldots, \varepsilon_N)$. Therefore, the problem (3.15) is much more difficult to solve than the complete case.

3.2.1. *Special case: Exponential utility.* Suppose that all the agents have exponential utilities (3.7). Then, from (3.15), we derive the following system of simultaneous equations:

$$\pi_j = \frac{\mathbb{E}\left[S_j \mathrm{e}^{-\lambda_i\left(\varepsilon_i + \sum_{k=1}^N \widetilde{\theta}_k^i S_k\right)}\right]}{\mathbb{E}\left[\mathrm{e}^{-\lambda_i\left(\varepsilon_i + \sum_{k=1}^N \widetilde{\theta}_k^i S_k\right)}\right]}, \qquad \forall i,\ j > 0. \tag{3.16}$$

Note that the constant term $\lambda_i \widetilde{\theta}_0^i S_0$ is canceled out on both numerator and denominator. This system of equations has $IN + N = N(I+1)$ unknowns (i.e., $\widetilde{\theta}_j^i$ and π_j) and IN equations in (3.16). Together with the market clearing condition, we have $IN + N = N(I+1)$ equations. Hence, we can solve the simultaneous equation (3.16), although the solution may not be unique. Since the constant term $\lambda_i \widetilde{\theta}_0^i S_0$ does not matter in (3.16), the budget constraint is adjusted by $\widetilde{\theta}_0^i$ so as to satisfy (3.13). The equilibrium risk exchanges are determined by $\widetilde{y}_j^i = \widetilde{\theta}_j^i - x_j^i$.

In order to solve the problem, define the moment generating functions (MGFs)

$$m_i(\theta_1, \theta_2, \ldots, \theta_N) = \mathbb{E}\left[\mathrm{e}^{-\lambda_i\left(\varepsilon_i + \sum_{j=1}^N \theta_j S_j\right)}\right], \qquad i = 1, 2, \ldots, I,$$

for which the MGFs m_i exist.[5] The equilibrium prices π_j and the solutions $\widetilde{\theta}_j^i$ are determined by the following simultaneous equations:

$$\pi_j = \frac{\partial}{\partial \theta_j} \log m_i(\widetilde{\theta}_1^i, \widetilde{\theta}_2^i, \ldots, \widetilde{\theta}_N^i), \qquad \forall i,\ j > 0.$$

If in particular there are no residual risks ε_i, then we can solve the problem explicitly. Namely, let

$$m(\rho_1, \rho_2, \ldots, \rho_N) = \mathbb{E}\left[\mathrm{e}^{-\sum_{j=1}^N \rho_j S_j}\right], \qquad i = 1, 2, \ldots, I, \tag{3.17}$$

and consider the system of simultaneous equations

$$\pi_j = -\frac{\partial}{\partial \rho_j} \log m(\rho_1^i, \rho_2^i, \ldots, \rho_N^i), \qquad \forall i,\ j > 0, \tag{3.18}$$

where we put $\rho_j^i = \lambda_i \widetilde{\theta}_j^i$.

But, since the MGF m does not depend on i, the solutions $\rho_j^i = \lambda_i(x_j^i + \widetilde{y}_j^i)$ are also independent of i. That is, we have $\lambda_i(x_j^i + \widetilde{y}_j^i) = \rho_j$ for some ρ_j. It follows that

$$x_j^i + \widetilde{y}_j^i = \frac{\rho_j}{\lambda_i}, \qquad \forall i,\ j > 0,$$

and the market clearing condition together with (2.4) implies that

$$A_j = \sum_i x_j^i = \frac{\rho_j}{\lambda}; \quad \frac{1}{\lambda} = \sum_i \frac{1}{\lambda_i}, \qquad \forall i,\ j > 0.$$

Therefore, we obtain $\rho_j = \lambda A_j$ so that $(x_j^i + \widetilde{y}_j^i) S_j = \frac{\lambda}{\lambda_i} A_j S_j$. Hence, the equilibrium allocation is given by

$$X_i + \widetilde{Y}_i = \sum_{j=1}^N \left(x_j^i + \widetilde{y}_j^i\right) S_j = \frac{\lambda}{\lambda_i} Z; \quad Z = \sum_{j=1}^N A_j S_j, \qquad i = 1, 2, \ldots, I. \tag{3.19}$$

The equilibrium prices are then expressed from (3.16) as

$$\pi_j = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}; \quad Z = \sum_{j=1}^N A_j S_j, \qquad j = 1, 2, \ldots N, \tag{3.20}$$

[5] We need to assume that the MGFs exist in order for the equilibrium prices to exist in the exponential utility case. Note that this excludes the log-normally distributed assets.

the multivariate Esscher transform of S_j; see (1.2). Note the resemblance of the solutions (3.11) and (3.20). However, this does not mean the existence of the state price density η. That is, we cannot price any asset other than S_j, $j = 1, 2, \ldots, N$. Also, the risk allocations (3.12) and (3.19) are similar, because $\sum_{j=1}^N A_j S_j = \sum_{i=1}^I X_i$ if $x_0^i = 0$ for all i.

It should be noted that the equilibrium price (3.20) as well as the equilibrium allocation (3.19) depends on the initial risks X_i only through the aggregated risk Z. However, when there are transaction costs, the allocation (3.19) is no longer optimal and the equilibrium price certainly depends on the initial risks X_i even in the exponential utility case, as we shall see later.

4. Existence of Equilibrium

In this section, we prove that there exists an equilibrium to the problem (2.6) even in the presence of proportional transaction costs.

To this end, we first assume that only asset 1 is traded with proportional transaction costs, for the sake of simplicity. The other assets are traded with no transaction costs (or may be negligibly small). The treatment of general case is similar with exponential growth of combinations. Throughout this section, we shall denote $c = c_1$ and $\gamma(y) = c\,\mathrm{sgn}(y)$.

According to Remark 1 in Hara (2013), we can assume without any loss of generality that $\pi_0 = 1$ and $c_0 = 0$ for the riskfree bond. Hence, the budget constraint in (2.6) is rewritten as

$$\sum_{j=1}^N (\theta_j^i - x_j^i)\pi_j + (\theta_1^i - x_1^i)\pi_1\gamma(\theta_1^i - x_1^i) + (\theta_0^i - x_0^i) \le 0, \qquad i = 1, 2, \ldots, I, \tag{4.1}$$

in this setting.

First, we fix the prices π_j and, in order to solve the optimization problem, we ignore the market clearing condition, which enables us to divide the problem into the following individual optimization problem:

$$\max_{\theta_j^i} \mathbb{E}\left[u_i\left(\varepsilon_i + \sum_{j=0}^N \theta_j^i S_j\right)\right] \text{ subject to (4.1) for each agent } i. \tag{4.2}$$

The problem (4.2) satisfies the Slater constraint qualification. That is, the budget constraint is convex on θ_j^i for fixed prices π_j, and there exists $(\theta_0, \theta_1, \ldots, \theta_J)$ satisfying

$$\sum_{j=1}^N (\theta_j - x_j^i)\pi_j + (\theta_1 - x_1^i)\pi_1\gamma(\theta_1 - x_1^i) + (\theta_0^i - x_0^i) < 0,$$

where inequality in (4.1) is replaced by strict inequality. Thus, a feasible solution $(\widetilde{\theta}_0, \widetilde{\theta}_1, \ldots, \widetilde{\theta}_N)$ of (4.2) is optimal if and only if there exists a Lagrange multiplier ℓ_i such that

$$\ell_i \ge 0, \qquad \ell_i\left(\sum_{j=1}^N (\widetilde{\theta}_j - x_j^i)\pi_j + (\widetilde{\theta}_1 - x_1^i)\pi_1\gamma(\widetilde{\theta}_1 - x_1^i) + (\widetilde{\theta}_0 - x_0^i)\right) = 0, \tag{4.3}$$

$$\mathbb{E}\left[S_j u_i'\left(\varepsilon_i + \sum_{j=0}^N \widetilde{\theta}_j S_j\right)\right] - \ell_i \pi_j = 0, \qquad j \ne 1, \tag{4.4}$$

$$\begin{cases} \mathbb{E}\left[S_1 u_i'\left(\varepsilon_i + \sum_{j=0}^N \widetilde{\theta}_j S_j\right)\right] - \ell_i(1+c)\pi_1 = 0, & \text{if } \widetilde{\theta}_1 > x_1^i, \\ \mathbb{E}\left[S_1 u_i'\left(\varepsilon_i + \sum_{j=0}^N \widetilde{\theta}_j S_j\right)\right] - \ell_i(1-c)\pi_1 = 0, & \text{if } \widetilde{\theta}_1 < x_1^i, \\ \ell_i(1-c)\pi_1 \le \mathbb{E}\left[S_1 u_i'\left(\varepsilon_i + \sum_{j=0}^N \widetilde{\theta}_j S_j\right)\right] \le \ell_i(1+c)\pi_1, & \text{if } \widetilde{\theta}_1 = x_1^i. \end{cases} \tag{4.5}$$

See Borwein and Levis (2000) for detailed discussions.

Next, since $\pi_0 = S_0 = 1$, we have from (4.4) for $j = 0$ that

$$\ell_i = \mathbb{E}\left[u_i'\left(\sum_{k=0}^{N} \widetilde{\theta}_k S_k\right)\right] > 0, \qquad i = 1, 2, \ldots, I,$$

where the strict inequality follows from the assumption $u_i' > 0$. Thus, (4.3) implies that the budget constraint (4.1) must hold with equality.

Let us define

$$\phi_j^i(\theta_0, \theta_1, \ldots, \theta_N) = \frac{\mathbb{E}[S_j u_i'(\varepsilon_i + \theta_0 + \sum_{k=1}^N \theta_k S_k)]}{\mathbb{E}[u_i'(\varepsilon_i + \theta_0 + \sum_{k=1}^N \theta_k S_k)]}, \qquad \forall i,\ j \tag{4.6}$$

for which the functions ϕ_j^i exist. Note that, since $S_0 = 1$, we have $\phi_0^i = 1$ for all i.

Given a transaction cost $c \geq 0$, we divide I agents into three groups:

$$\mathcal{B}_c = \{i : \widetilde{\theta}_1^i > x_1^i\}, \qquad \mathcal{S}_c = \{i : \widetilde{\theta}_1^i < x_1^i\}, \qquad \mathcal{N}_c = \{i : \widetilde{\theta}_1^i = x_1^i\},$$

where $\widetilde{\theta}_1^i$ is the component for asset 1 of an optimal solution of (4.2) for agent i in the presence of transaction cost c.

In this setting, the necessary and sufficient condition (4.3)–(4.5) is written as follows: For $i \in \mathcal{B}_c$, solve

$$\begin{cases} \phi_j^i(\theta_0, \theta_1, \ldots, \theta_N) = \pi_j, \qquad j = 2, \ldots N, \\ \phi_1^i(\theta_0, \theta_1, \ldots, \theta_N) = (1+c)\pi_1, \\ \sum_{j=1}^N (\theta_j - x_j^i)\pi_j + c\pi_1(\theta_1 - x_1^i) + (\theta_0 - x_0^i) = 0, \end{cases} \tag{4.7}$$

to obtain $(\widetilde{\theta}_0^i, \widetilde{\theta}_1^i, \ldots, \widetilde{\theta}_N^i)$. Similarly, for $i \in \mathcal{S}_c$, solve

$$\begin{cases} \phi_j^i(\theta_0, \theta_1, \ldots, \theta_N) = \pi_j, \qquad j = 2, \ldots N, \\ \phi_1^i(\theta_0, \theta_1, \ldots, \theta_N) = (1-c)\pi_1, \\ \sum_{j=1}^N (\theta_j - x_j^i)\pi_j - c\pi_1(\theta_1 - x_1^i) + (\theta_0 - x_0^i) = 0, \end{cases} \tag{4.8}$$

to obtain $(\widetilde{\theta}_0^i, \widetilde{\theta}_1^i, \ldots, \widetilde{\theta}_N^i)$. Finally, for $i \in \mathcal{N}_c$, solve

$$\begin{cases} \phi_j^i(\theta_0, x_1^i, \theta_2, \ldots, \theta_N) = \pi_j, \qquad j = 2, \ldots N, \\ (1-c)\pi_1 \leq \phi_1^i(\theta_0, x_1^i, \theta_2, \ldots, \theta_N) \leq (1+c)\pi_1, \\ \sum_{j \neq 1} (\theta_j - x_j^i)\pi_j = 0, \end{cases} \tag{4.9}$$

to obtain $(\widetilde{\theta}_0^i, x_1^i, \widetilde{\theta}_2^i, \ldots, \widetilde{\theta}_N^i)$. The equilibrium prices π_j are obtained by the market clearing condition in (2.6).

So far, we have shown that the equilibrium prices π_j as well as equilibrium allocations $\widetilde{\theta}_j^i$ in the problem (2.6) with the budget constraint being replaced by (4.1) are obtained by solving (4.7)–(4.9), if the types of agents are known. Hence, we need to solve the problem (4.7)–(4.9) for all possible combinations of agent types in order to find the feasible and optimal solution. The general case can be proved similarly by considering all the possible combinations, although the number of possible combinations grows exponentially fast. We thus have proved the following.

Theorem 4.1. *In the problem (2.6), there exists an equilibrium.*

In the general setup, this is a very difficult problem to solve because of the exponentially growing combinations, and it seems impossible to investigate the effect of the transaction cost on the equilibrium allocations and prices. Hence, in the rest of this paper, we shall impose some additional assumptions either on the utility function or on the joint distribution of risky assets.

4.1. **Exponential Utilities.** As in Subsection 3.2.1, suppose that $\varepsilon_i = 0$ and $u_i'(x) = \mathrm{e}^{-\lambda_i x}$ for all i. In this case, the function ϕ_j^i defined in (4.6) is given by

$$\phi_j^i(\theta_1, \dots, \theta_N) = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda_i P}]}{\mathbb{E}[\mathrm{e}^{-\lambda_i P}]}, \qquad P = \sum_{j=1}^N \theta_j S_j. \tag{4.10}$$

Let $\rho_j^i = \lambda_i \theta_j$, as before, and denote the MGF (moment generating function) of $(S_1, S_2, \dots, S_N)$ by (3.17), i.e.,

$$m(\rho_1, \rho_2, \dots, \rho_N) = \mathbb{E}\left[\mathrm{e}^{-\sum_{j=1}^N \rho_j S_j}\right]$$

for which the MGF exists. It is easy to verify that

$$\phi_j^i(\theta_0, \theta_1, \dots, \theta_N) = -\frac{\partial}{\partial \rho_j} \log m(\rho_1^i, \dots, \rho_N^i).$$

Hence, it is enough to define the function

$$\phi_j(\rho_1, \rho_2, \dots, \rho_N) = -\frac{\partial}{\partial \rho_j} \log m(\rho_1, \rho_2, \dots, \rho_N),$$

which makes the exponential case simpler.

Now, the problem (4.7)–(4.9) is reduced to the following: For $i \in \mathcal{B}_c$, solve

$$\begin{cases} \phi_j(\rho_1, \rho_2, \dots, \rho_N) = \pi_j, & j = 2, \dots N, \\ \phi_1(\rho_1, \rho_2, \dots, \rho_N) = (1+c)\pi_1. \end{cases} \tag{4.11}$$

The solution is denoted by $\widetilde{\rho}_j^+(c)$, $j = 1, 2, \dots, N$. Similarly, for $i \in \mathcal{S}_c$, solve

$$\begin{cases} \phi_j(\rho_1, \rho_2, \dots, \rho_N) = \pi_j, & j = 2, \dots N, \\ \phi_1(\rho_1, \rho_2, \dots, \rho_N) = (1-c)\pi_1. \end{cases} \tag{4.12}$$

The solution is denoted by $\widetilde{\rho}_j^-(c)$, $j = 1, 2, \dots, N$. Finally, for $i \in \mathcal{N}_c$, solve

$$\begin{cases} \phi_j(\lambda_i x_1^i, \rho_2, \dots, \rho_N) = \pi_j, & j = 2, \dots, N, \\ (1-c)\pi_1 \le \phi_1(\lambda_i x_1^i, \rho_2, \dots, \rho_N) \le (1+c)\pi_1. \end{cases} \tag{4.13}$$

The solution is denoted by $\widetilde{\rho}_j^{0i}(c)$, $j = 2, \dots, N$, which may be dependent on i. In either cases, $\widetilde{\rho}_0$ is obtained by the budget constraint.

Given these solutions, the equilibrium allocation of agent i is obtained as follows: For asset 1, we have

$$\widetilde{\theta}_1^i = \begin{cases} \widetilde{\rho}_1^+(c)/\lambda_i, & i \in \mathcal{B}_c, \\ \widetilde{\rho}_1^-(c)/\lambda_i, & i \in \mathcal{S}_c, \\ x_1^i, & i \in \mathcal{N}_c. \end{cases}$$

Summing over i, the market clearing condition is obtained as

$$A_1 = \widetilde{\rho}_1^+(c) \sum_{i \in \mathcal{B}_c} \frac{1}{\lambda_i} + \widetilde{\rho}_1^-(c) \sum_{i \in \mathcal{S}_c} \frac{1}{\lambda_i} + \sum_{i \in \mathcal{N}_c} x_1^i. \tag{4.14}$$

Similarly, for the other asset j, $j \ge 2$, we have

$$\widetilde{\theta}_j^i = \begin{cases} \widetilde{\rho}_j^+(c)/\lambda_i, & i \in \mathcal{B}_c, \\ \widetilde{\rho}_j^-(c)/\lambda_i, & i \in \mathcal{S}_c, \\ \widetilde{\rho}_j^{0i}(c)/\lambda_i, & i \in \mathcal{N}_c. \end{cases}$$

Summing over i, the market clearing condition is obtained as

$$A_j = \widetilde{\rho}_j^+(c) \sum_{i \in \mathcal{B}_c} \frac{1}{\lambda_i} + \widetilde{\rho}_j^-(c) \sum_{i \in \mathcal{S}_c} \frac{1}{\lambda_i} + \sum_{i \in \mathcal{N}_c} \frac{\widetilde{\rho}_j^{0i}(c)}{\lambda_i}, \qquad j \ge 2. \tag{4.15}$$

Note that we have $3N + N_0(N-1)$ unknowns ($\widetilde{\rho}_j^+(c)$, $\widetilde{\rho}_j^-(c)$ and π_j for $j \geq 1$, and $\widetilde{\rho}_j^{0i}(c)$ for $i \in \mathcal{N}_c$ and $j \geq 2$) and the same number of equations (N in (4.11) and (4.12), $N_0(N-1)$ in (4.13), and N in total in (4.14) and (4.15) together), where $N_0 = |\mathcal{N}_c|$ denotes the number of agents in the class $\mathcal{N}_c$. Hence, in principle, we can solve the simultaneous equations, if the division of agents were known.

The allocations $\widetilde{\theta}_0^i$ are determined by the budget constraint as follows. For $i \in \mathcal{B}_c$, we have

$$\widetilde{\theta}_0^i = x_0^i - \sum_{j=1}^N \left(\frac{\widetilde{\rho}_j^+(c)}{\lambda_i} - x_j^i \right) \pi_j - c\pi_1 \left(\frac{\widetilde{\rho}_1^+(c)}{\lambda_i} - x_1^i \right). \tag{4.16}$$

Similarly, for $i \in \mathcal{S}_c$,

$$\widetilde{\theta}_0^i = x_0^i - \sum_{j=1}^N \left(\frac{\widetilde{\rho}_j^-(c)}{\lambda_i} - x_j^i \right) \pi_j + c\pi_1 \left(\frac{\widetilde{\rho}_1^-(c)}{\lambda_i} - x_1^i \right), \tag{4.17}$$

and for $i \in \mathcal{N}_c$,

$$\widetilde{\theta}_0^i = x_0^i - \sum_{j=2}^N \left(\frac{\widetilde{\rho}_j^{0i}(c)}{\lambda_i} - x_j^i \right) \pi_j. \tag{4.18}$$

From (4.16)–(4.18) together with (4.14) and (4.15), we obtain the market clearing condition for asset 0 as

$$\sum_{i=1}^I \widetilde{\theta}_0^i = A_0 - 2c\pi_1 \sum_{i \in \mathcal{B}_c} \left(\frac{\widetilde{\rho}_1^+(c)}{\lambda_i} - x_1^i \right) \leq A_0,$$

which means that the riskfree asset may be left in the market.[6]

Example 4.1. Suppose that the transaction cost c is sufficiently large, so that no agents want to trade asset 1, i.e., $\mathcal{B}_c = \mathcal{S}_c = \emptyset$. In this case, the problem (4.13) is only relevant, and we have $\widetilde{\theta}_j^i = \widetilde{\rho}_j^{0i}(1)/\lambda_i$, $i = 1, 2, \ldots, I$, for every asset j. Recall that, in general, the quantity $\widetilde{\rho}_j^{0i}(1)$ depends on i. The equilibrium price is given from (4.10) as

$$\pi_j = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z_i}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z_i}]}, \quad Z_i = \frac{1}{\lambda} \left(\lambda_i x_1^i S_1 + \sum_{j=2}^N \widetilde{\rho}_j^{0i}(1) S_j \right); \qquad j \geq 2, \tag{4.19}$$

for all i, where the market clearing condition is given by

$$A_j = \sum_{i=1}^I \frac{\widetilde{\rho}_j^{0i}(1)}{\lambda_i}, \qquad j \geq 2.$$

Note that the prices in (4.19) are affected by the non-traded asset 1. When S_1 is independent of the other assets, it is readily shown that the prices are given by (3.20) with $A_1 = 0$.

4.2. **Risks Are Normally Distributed.** Next, we consider the case that $(S_1, S_2, \ldots, S_N)$ is normally distributed. In this subsection, we assume that $0 \leq c \leq 1$. The mean vector and covariance matrix are denoted by $\boldsymbol{\mu} = (\mu_j)$ and $\boldsymbol{\Sigma} = (\sigma_{ij})$, respectively. It is readily obtained that

$$\log m(\rho_1, \ldots, \rho_N) = -\sum_{j=1}^N \rho_j \mu_j + \frac{1}{2} \sum_{j,k} \sigma_{jk} \rho_j \rho_k.$$

[6]This does not cause any problem, since the riskfree bond is traded with no transaction costs and does not contribute to the function ϕ_j for the exponential case.

Thus, we have

$$\phi_j(\rho_1, \ldots, \rho_N) = \mu_j - \sum_{k=1}^{N} \sigma_{jk}\rho_k. \tag{4.20}$$

For the convenience of description, we use the vectors $\boldsymbol{\pi} = (\pi_j)$, $\boldsymbol{a} = (A_j)$ and $\boldsymbol{\rho}^i = (\rho_j^i)$, and divide the covariance matrix $\boldsymbol{\Sigma}$ as

$$\boldsymbol{\Sigma} = \left(\begin{array}{c|c} \sigma_{11} & \hat{\boldsymbol{\sigma}}^\top \\ \hline \hat{\boldsymbol{\sigma}} & \hat{\boldsymbol{\Sigma}} \end{array}\right), \qquad \hat{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_{21} \\ \vdots \\ \sigma_{N1} \end{pmatrix}, \qquad \hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \ddots & \vdots \\ \sigma_{N2} & \cdots & \sigma_{NN} \end{pmatrix},$$

where $\hat{\boldsymbol{\sigma}}^\top$ denotes the transpose of $\hat{\boldsymbol{\sigma}}$. For an N-dimensional vector $\boldsymbol{z} = (z_1, z_2, \ldots, z_N)^\top$, we denote the $(N-1)$-dimensional vector $(z_2, \ldots, z_N)^\top$ by $\hat{\boldsymbol{z}}$. For example, $\hat{\boldsymbol{\pi}} = (\pi_2, \ldots, \pi_N)^\top$ for $\boldsymbol{\pi}$.

From (4.20), the equations for $j = 2, \ldots, N$ of (4.11)−(4.13) are written in matrix form as

$$\hat{\boldsymbol{\mu}} - \rho_1^i \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\rho}}^i = \hat{\boldsymbol{\pi}}, \qquad i = 1, 2, \ldots, I,$$

where $\rho_1^i = \lambda_i x_1^i$ if $i \in \mathcal{N}_c$. Since $\hat{\boldsymbol{\Sigma}}$ is positive definite (i.e., nonsingular), we have

$$\hat{\boldsymbol{\rho}}^i = \hat{\boldsymbol{\Sigma}}^{-1}\left(\hat{\boldsymbol{\mu}} - \rho_1^i \hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\pi}}\right), \qquad i = 1, 2, \ldots, I. \tag{4.21}$$

The market clearing conditions (4.14) and (4.15) are rewritten as

$$A_1 = \sum_{i=1}^{I} \frac{\rho_1^i}{\lambda_i} \tag{4.22}$$

and

$$\hat{\boldsymbol{a}} = \sum_{i=1}^{I} \frac{1}{\lambda_i}\hat{\boldsymbol{\rho}}^i = \hat{\boldsymbol{\Sigma}}^{-1}\left(\frac{1}{\lambda}\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\sigma}}\sum_{i=1}^{I}\frac{\rho_1^i}{\lambda_i} - \frac{1}{\lambda}\hat{\boldsymbol{\pi}}\right), \tag{4.23}$$

respectively, where $\frac{1}{\lambda} = \sum_{i=1}^{I} \frac{1}{\lambda_i}$ as in (3.10). It follows from (4.22) and (4.23) that

$$\hat{\boldsymbol{\pi}} = \hat{\boldsymbol{\mu}} - \lambda\left(A_1\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{a}}\right). \tag{4.24}$$

Hence, when risks are normally distributed, the prices π_j, $j \geq 2$, are not affected from the transaction cost c of asset 1. Moreover, the prices π_j, $j \geq 2$, are given as the multivariate Esscher transform (3.20). To see this, we need the following lemma. See Kijima and Muromachi (2001) for the proof.

Lemma 4.1. *Suppose that (X, Z) is normally distributed. Then,*

$$\mathbb{E}[f(X)\mathrm{e}^{-\lambda Z}] = \mathbb{E}[f(X - \lambda\mathrm{cov}(X, Z))]\mathbb{E}[\mathrm{e}^{-\lambda Z}]$$

for any $f(x)$ for which the expectations exist, where cov *denotes the covariance operator.*

Theorem 4.2. *When risks are normally distributed, the prices π_j, $j \geq 2$, are given by the multivariate Esscher transform*

$$\pi_j = \frac{\mathbb{E}[S_j\mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}; \quad Z = \sum_{k=1}^{N} A_k S_k, \qquad j \geq 2,$$

that are independent of the transaction cost c of asset 1.

Proof. From (4.24), we have

$$\pi_j = \mu_j - \lambda \sum_{k=1}^{N} \sigma_{jk} A_k, \qquad j = 2, 3, \ldots, N,$$

which is just the multivariate Esscher transform (3.20) from Lemma 4.1. □

In contrast, the equilibrium allocation depends on the cost, as we show next. From (4.21) and (4.24), $\hat{\boldsymbol{\rho}}^i$ can be represented as a function of ρ_1^i and given by

$$\hat{\boldsymbol{\rho}}^i(\rho_1^i) = \lambda \left(A_1 \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{a}} \right) - \rho_1^i \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}}. \tag{4.25}$$

Under the condition (4.25), $\phi_1(\boldsymbol{\rho}^i)$ is also a function of ρ_1^i and given by

$$\phi_1(\rho_1^i, \hat{\boldsymbol{\rho}}^i(\rho_1^i)) = \mu_1 - B - \rho_1^i r,$$

where we define

$$B = \lambda \left(A_1 \hat{\boldsymbol{\sigma}}^\top \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\sigma}}^\top \hat{\boldsymbol{a}} \right), \qquad r = \sigma_{11} - \hat{\boldsymbol{\sigma}}^\top \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}}.$$

Here, note that, since $r > 0$,

$$\boldsymbol{\Sigma}^{-1} = \left(\begin{array}{c|c} \frac{1}{r} & -\frac{1}{r} \left(\hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} \right)^\top \\ \hline -\frac{1}{r} \left(\hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} \right) & \hat{\boldsymbol{\Sigma}}^{-1} + \frac{1}{r} \left(\hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} \right) \left(\hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\sigma}} \right)^\top \end{array} \right)$$

must be positive definite.

Recalling that $\rho_j^i = \lambda_i (x_j^i + y_j^i)$, we define

$$v_i(y) = s_i - \lambda_i r y, \qquad s_i = \mu_1 - B - \lambda_i r x_1^i,$$

for each $i = 1, \ldots, I$. In order to find equilibrium prices and equilibrium allocations, it is enough to determine $y_1^1, y_1^2, \ldots, y_1^I$ and π_1 satisfying

$$\begin{cases} v_i(y_1^i) = (1+c)\pi_1, & y_1^i > 0, \\ v_i(y_1^i) = (1-c)\pi_1, & y_1^i < 0, \\ (1-c)\pi_1 \le s_i \le (1+c)\pi_1, & y_1^i = 0, \end{cases} \tag{4.26}$$

under the market clearing condition for asset 1, i.e.,

$$\sum_{i=1}^{I} y_1^i = 0. \tag{4.27}$$

Condition (4.26) can be reformulated as

$$y_1^i = \begin{cases} (s_i - (1+c)\pi_1)/r\lambda_i, & s_i > (1+c)\pi_1, \\ (s_i - (1-c)\pi_1)/r\lambda_i, & s_i < (1-c)\pi_1, \\ 0, & \text{otherwise}, \end{cases} \tag{4.28}$$

and the partition $(\mathcal{B}_c, \mathcal{S}_c, \mathcal{N}_c)$ of agents is rewritten as

$$\begin{cases} \mathcal{B}_c &= \{i : s_i > (1+c)\pi_1\}, \\ \mathcal{S}_c &= \{i : s_i < (1-c)\pi_1\}, \\ \mathcal{N}_c &= \{i : (1-c)\pi_1 \le s_i \le (1+c)\pi_1\}. \end{cases}$$

From (4.28), equation (4.27) is restated as

$$\sum_{i \in \mathcal{B}_c} \frac{1}{\lambda_i} (s_i - (1+c)\pi_1) + \sum_{i \in \mathcal{S}_c} \frac{1}{\lambda_i} (s_i - (1-c)\pi_1) = 0. \tag{4.29}$$

We first consider the case $c = 0$. From (4.28), we have

$$y_1^i = \frac{s_i - \pi_1}{\lambda_i r}, \qquad i = 1, \ldots, I.$$

This together with (4.27) implies

$$\frac{\pi_1}{\lambda} = \sum_{i=1}^{I} \frac{\pi_1}{\lambda_i} = \sum_{i=1}^{I} \frac{s_i}{\lambda_i} = \frac{\mu_1 - B}{\lambda} - rA_1.$$

Hence, we obtain the equilibrium price π_1^0 for $c = 0$ as

$$\pi_1^0 = \mu_1 - B - \lambda r A_1 = \mu_1 - \lambda \left(\sigma_{11} A_1 + \hat{\boldsymbol{\sigma}}^\top \hat{\boldsymbol{a}}\right),$$

which is given by the multivariate Esscher transform by the proof of Theorem 4.2.

We assume that $\pi_1^0 > 0$ in the sequel. We will show that i) an equilibrium price π_1^c for $c \in [0, 1]$ exists and it is uniquely determined by (4.29) if $\mathcal{B}_c, \mathcal{S}_c \neq \emptyset$, ii) $\pi_1^c > 0$, and iii) the buying price $(1 + c)\pi_1^c$ is increasing in c and the selling price $(1 - c)\pi_1^c$ is decreasing in c. Then, from (4.28), the trading volumes $|y_1^i|$ are decreasing as the cost c gets large and, from the definition of $\mathcal{B}_c$ and $\mathcal{S}_c$, once some agent stops trading, he/she will never return for trading.

Suppose that c with $\pi_1^c > 0$ (e.g., $c = 0$) is given. If $y_1^1 = y_1^2 = \cdots = y_1^I = 0$ for the c, then, for all $c' \in [c, 1]$, $\pi_1^{c'} = \pi_1^c$ satisfies (4.26) and (4.27), and the results i), ii) and iii) hold. Hence, we assume that

$$\text{there exists } y_1^i \neq 0 \text{ for } c. \tag{4.30}$$

Equation (4.29) implies that π_1^c is uniquely determined by

$$\pi_1^c = \frac{\sum_{i \in \mathcal{B}_c \cup \mathcal{S}_c} \frac{s_i}{\lambda_i}}{\frac{1}{\lambda_B} + \frac{1}{\lambda_S} + c\left(\frac{1}{\lambda_B} - \frac{1}{\lambda_S}\right)}, \qquad \frac{1}{\lambda_B} = \sum_{i \in \mathcal{B}_c} \frac{1}{\lambda_i}, \qquad \frac{1}{\lambda_S} = \sum_{i \in \mathcal{S}_c} \frac{1}{\lambda_i}. \tag{4.31}$$

We note that (4.29) and (4.30) guarantee the existence of λ_B and λ_S, and that $c \in [0, 1]$ together with $\lambda_B, \lambda_S > 0$ also guarantees $\frac{1}{\lambda_B} + \frac{1}{\lambda_S} + c\left(\frac{1}{\lambda_B} - \frac{1}{\lambda_S}\right) \neq 0$. In a range including c for which $\mathcal{B}_c$ and $\mathcal{S}_c$ are unchanged, π_1^c is a continuous function of c, and hence, $(1 + c)\pi_1^c$ and $(1 - c)\pi_1^c$ are also continuous functions of c.

For simplicity, we assume that $(1 - c)\pi_1^c < s_i < (1 + c)\pi_1^c$ for all $i \in \mathcal{N}_c$. In this case, conversely, we can slightly change c preserving the partition $(\mathcal{B}_c, \mathcal{S}_c, \mathcal{N}_c)$. Now, we slightly increase c to c'. Then, (4.29) for c' and $\pi_1^{c'}$ holds for the same $\mathcal{B}_c$ and $\mathcal{S}_c$. That is,

$$\frac{1}{\lambda_B}(1 + c)\pi_1^c + \frac{1}{\lambda_S}(1 - c)\pi_1^c = \frac{1}{\lambda_B}(1 + c')\pi_1^{c'} + \frac{1}{\lambda_S}(1 - c')\pi_1^{c'} = \sum_{i \in \mathcal{B}_c \cup \mathcal{S}_c} \frac{s_i}{\lambda_i}. \tag{4.32}$$

Suppose to the contrary that $(1 + c)\pi_1^c \geq (1 + c')\pi_1^{c'}$, which implies $\pi_1^c > \pi_1^{c'}$, because $c < c'$ and $\pi_1^c > 0$. From (4.32), we have $0 \leq (1 - c)\pi_1^c \leq (1 - c')\pi_1^{c'}$. This is, however, impossible because $(1 - c) > (1 - c') \geq 0$ and $\pi_1^c > \pi_1^{c'}$. Hence, we obtain $(1 + c)\pi_1^c < (1 + c')\pi_1^{c'}$, which implies that $(1 - c)\pi_1^c > (1 - c')\pi_1^{c'}$ and $\pi_1^{c'} > 0$.

If there is $i \in \mathcal{N}_c$ such that $(1 - c)\pi_1^c = s_i$ or $s_i = (1 + c)\pi_1^c$, then we can show the same results by redefining the partition $(\mathcal{B}_c, \mathcal{S}_c, \mathcal{N}_c)$ by

$$\begin{cases} \mathcal{B}_c &= \{i : s_i \geq (1 + c)\pi_1^c\}, \\ \mathcal{S}_c &= \{i : s_i \leq (1 - c)\pi_1^c\}, \\ \mathcal{N}_c &= \{i : (1 - c)\pi_1^c < s_i < (1 + c)\pi_1^c\}. \end{cases}$$

By repeating the same argument and modifying the partition $(\mathcal{B}_c, \mathcal{S}_c, \mathcal{N}_c)$ until $c = 1$ or (4.30) is violated, we can show i), ii) and iii), as desired. We thus have the following results.

Theorem 4.3. *Suppose that all the risky assets are normally distributed. Then, for any cost $c \in [0,1]$, an equilibrium price π_1^c of asset 1 exists, and is a unique solution of* (4.29) *if $\mathcal{B}_c, \mathcal{S}_c \neq \emptyset$. The equilibrium allocation is $x_1^i + \tilde{y}_1^i$, where $\tilde{y}_1^i$ is given by* (4.28)*, for asset 1 and $x_j^i + \tilde{y}_j^i = \tilde{\rho}_j^i / \lambda_i$, where $\tilde{\rho}_j^i$ are given by* (4.25)*, for asset j, $j \geq 2$.*

The next seemingly plausible results have been proved in Hara (2013) for the single risky-asset case with general distribution.

Corollary 4.1. *When all the risks are normally distributed, the buying price $(1+c)\pi_1^c$ is increasing in c, the selling price $(1-c)\pi_1^c$ is decreasing in c, and the trading volumes $|y_1^i|$ are decreasing in c in equilibrium. Once some agent stops trading, he/she will never return for trading.*

5. The Equilibrium When Transaction Costs are Very Small

Suppose that the transaction costs c_j are so small that all the assets in the market are traded by all the agents, i.e., $\widetilde{y}_j^i \neq 0$ or $\widetilde{\theta}_j^i - x_j^i \neq 0$. Then, as in the case of no transaction costs, we can define the Lagurange equations

$$L_i = \mathbb{E}\left[u_i\left(\varepsilon_i + \sum_{j=0}^N \theta_j^i S_j\right)\right] - \ell_i \sum_{j=0}^N (\theta_j^i - x_j^i)\pi_j(1 + c_j \mathrm{sgn}(\theta_j^i - x_j^i))$$

for all $i = 1, 2, \ldots, I$, and the FOC with respect to θ_j^i is given by

$$\mathbb{E}\left[S_j u_i'\left(\varepsilon_i + \sum_{k=0}^N \widetilde{\theta}_k^i S_k\right)\right] - \ell_i \pi_j (1 + c_j \mathrm{sgn}(\widetilde{\theta}_j^i - x_j^i)) = 0, \qquad \forall i,\ j. \tag{5.1}$$

This is possible, because we assume that $\widetilde{\theta}_j^i - x_j^i \neq 0$ and the sign function $\mathrm{sgn}(y)$ is differentiable except $y = 0$.

5.1. **CARA-Normal Case.** In this subsection, we consider exponential utilities, i.e., $u_i'(x) = \mathrm{e}^{-\lambda_i x}$, $i = 1, 2, \ldots, I$. Then, from (5.1), we derive the following system of simultaneous equations:

$$\mathbb{E}\left[S_j \mathrm{e}^{-\lambda_i\left(\varepsilon_i + \sum_{k=1}^N \widetilde{\theta}_k^i S_k\right)}\right] = \pi_j(1 + c_j \mathrm{sgn}(\widetilde{\theta}_j^i - x_j^i))\mathbb{E}\left[\mathrm{e}^{-\lambda_i\left(\varepsilon_i + \sum_{k=1}^N \widetilde{\theta}_k^i S_k\right)}\right], \qquad \forall i,\ j, \tag{5.2}$$

because as before $\ell_i = \mathbb{E}[\mathrm{e}^{-\lambda_i(\varepsilon_i + \sum_{k=1}^N \widetilde{\theta}_k^i S_k)}]$. Recall that $\mathrm{sgn}(y) = 1$ if $y > 0$ and $\mathrm{sgn}(y) = -1$ if $y < 0$.

In the following, we assume that the prices π_j are strictly positive and denote

$$R_j = \frac{S_j - \pi_j}{\pi_j}, \quad j = 1, 2, \ldots, N; \qquad R_\varepsilon^i = \frac{\varepsilon_i - \pi_i(\varepsilon_i)}{\pi_i(\varepsilon_i)}, \quad i = 1, 2, \ldots, I,$$

where $\pi_i(\varepsilon_i)$ represents the (unobservable) pricing functional of ε_i. In this section, we assume that the random vector $(R_1, \ldots, R_N, R_1^\varepsilon, \ldots, R_I^\varepsilon)$ defined above is normally distributed.

Following the ordinary arguments, we obtain

$$\varepsilon_i + \sum_{j=1}^N \widetilde{\theta}_j^i S_j = \sum_{j=1}^N \pi_j \widetilde{\theta}_j^i R_j + \pi_i(\varepsilon_i) R_i^\varepsilon + \pi_i(\varepsilon_i) + \sum_{j=1}^N \widetilde{\theta}_j^i \pi_j.$$

It follows that the FOC (5.2) can be written as

$$\mathbb{E}\left[(1 + R_j)\mathrm{e}^{-\lambda_i \Gamma^i}\right] = (1 + c_j \mathrm{sgn}(\widetilde{y}_j^i))\mathbb{E}\left[\mathrm{e}^{-\lambda_i \Gamma^i}\right], \qquad \forall i,\ j, \tag{5.3}$$

where

$$\Gamma^i \equiv \sum_{j=1}^{N} \pi_j (x_j^i + \widetilde{y}_j^i) R_j + \pi_i(\varepsilon_i) R_i^{\varepsilon}, \tag{5.4}$$

because $\widetilde{\theta}_j^i = x_j^i + \widetilde{y}_j^i$. A direct application of Lemma 4.1 to (5.3) yields

$$\mu_j - \lambda_i \mathrm{cov}(R_j, \Gamma^i) = c_j \mathrm{sgn}(\widetilde{y}_j^i), \qquad \widetilde{y}_j^i \neq 0,$$

where $\mu_j = \mathbb{E}[R_j]$ denotes the mean rate of return of asset j. It follows from the definition (5.4) of Γ^i that

$$\frac{1}{\lambda_i}(\mu_j - c_j \mathrm{sgn}(\widetilde{y}_j^i)) = \pi_i(\varepsilon_i)\sigma_{ij}^{\varepsilon} + \sum_{k=1}^{N} \pi_k (x_k^i + \widetilde{y}_k^i)\sigma_{kj}, \qquad \forall i,\ j, \tag{5.5}$$

where $\sigma_{ij}^{\varepsilon} = \mathrm{cov}(R_j, R_{\varepsilon}^i)$ and $\sigma_{ij} = \mathrm{cov}(R_i, R_j)$.

Because we have assumed that the equilibrium exchanges $\widetilde{y}_j^i$ are all nonzero, Equation (5.5) holds for all i and j. Summing over i in (5.5) and utilizing the market clearing condition in (2.6), we obtain

$$\frac{1}{\lambda}(\mu_j - c_j \Gamma_j(\widetilde{\mathbf{y}}_j)) = \xi_j + \sum_{k=1}^{N} \pi_k A_k \sigma_{kj}, \qquad j = 1, 2, \ldots, N, \tag{5.6}$$

where $\xi_j = \sum_i \pi_i(\varepsilon_i)\sigma_{ij}^{\varepsilon}$, λ is given in (3.10), and $\Gamma_j(\mathbf{y}_j)$ is defined by

$$\Gamma_j(\mathbf{y}_j) = \sum_{i=1}^{I} \frac{\lambda}{\lambda_i} \mathrm{sgn}(y_j^i), \qquad j = 1, 2, \ldots, N, \tag{5.7}$$

with $\mathbf{y}_j = (y_j^1, y_j^2, \ldots, y_j^I)$. The quantity $c_j \Gamma_j(\widetilde{\mathbf{y}}_j)$ is interpreted as the weighted sum of the (signed) trading costs of asset j in equilibrium. Note that, since $-1 \leq \mathrm{sgn}(y) \leq 1$, we obtain

$$-1 < \Gamma_j(\mathbf{y}_j) < 1, \qquad j = 1, 2, \ldots, N, \tag{5.8}$$

for any $\mathbf{y}_j$. The inequalities in (5.8) are strict, because the market clearing condition cannot hold otherwise.

When $\widetilde{y}_j^i$ are all nonzero, equations in (5.6) can be written in matrix form as

$$\frac{1}{\lambda}(\boldsymbol{\mu} - \boldsymbol{\Gamma}) - \boldsymbol{\xi} = \boldsymbol{\Sigma}\, \mathrm{diag}(A_j)\, \boldsymbol{\pi},$$

where $\boldsymbol{\mu} = (\mu_j)$, $\boldsymbol{\Gamma} = (c_j \Gamma_j(\widetilde{\mathbf{y}}_j))$, $\boldsymbol{\xi} = (\xi_j)$ and $\boldsymbol{\pi} = (\pi_j)$ are N-dimensional vectors, where $\boldsymbol{\Sigma} = (\sigma_{ij})$ is an $N \times N$ symmetric matrix, and where $\mathrm{diag}(A_j)$ denotes the diagonal matrix of order N with diagonal elements A_j. Assuming that the covariance matrix $\boldsymbol{\Sigma}$ is positive definite (hence, it is invertible), the above equation is solved as

$$\boldsymbol{\pi} = \frac{1}{\lambda} \mathrm{diag}(A_j^{-1}) \boldsymbol{\Sigma}^{-1} (\widetilde{\boldsymbol{\mu}} - \lambda \boldsymbol{\xi}), \qquad \widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu} - \boldsymbol{\Gamma}, \tag{5.9}$$

where $\widetilde{\boldsymbol{\mu}} = (\widetilde{\mu}_j)$, $\widetilde{\mu}_j = \mu_j - c_j \Gamma_j(\widetilde{\mathbf{y}}_j)$, denotes the vector of *cost-adjusted* mean rates of return in equilibrium. Hence, the equilibrium prices are written formally as

$$\pi_j = \frac{1}{\lambda A_j} \boldsymbol{\Sigma}_j^{-1} (\widetilde{\boldsymbol{\mu}} - \lambda \boldsymbol{\xi}), \qquad j = 1, 2, \ldots, N, \tag{5.10}$$

where $\boldsymbol{\Sigma}_j^{-1}$ denotes the jth row vector of the inverse matrix $\boldsymbol{\Sigma}^{-1}$.

In the following, we denote the equilibrium price without transaction costs (i.e., $c_j = 0$ for all j) by π_j^{NT} for asset j. In this setting, it is readily proved that, even in the presence of residual risks ε_i, the equilibrium prices are given by

$$\pi_j^{\mathrm{NT}} = \frac{1}{\lambda A_j}\boldsymbol{\Sigma}_j^{-1}(\boldsymbol{\mu} - \lambda\boldsymbol{\xi}) = \frac{\mathbb{E}[S_j \mathrm{e}^{-\lambda Z}]}{\mathbb{E}[\mathrm{e}^{-\lambda Z}]}, \qquad j = 1, 2, \ldots, N, \tag{5.11}$$

the multivariate Esscher transform of S_j, where

$$Z = \sum_{i=1}^{I} X_i = \sum_{j=1}^{N} A_j S_j + \sum_{i=1}^{I} \varepsilon_i \tag{5.12}$$

stands for the aggregated risk.

Note that, in the presence of transaction costs, while the mean rates of return are adjusted, the covariance matrix is unchanged. Hence, comparing (5.10) with (5.11), we conclude the following.

Theorem 5.1. *Suppose that the transaction costs c_j are so small that the optimal exchanges $\widetilde{y}_j^i$ are all nonzero. Then, the equilibrium price of asset j in the presence of transaction costs is given by the multivariate Esscher transform of $\widetilde{S}_j$, i.e.,*

$$\pi_j = \frac{\mathbb{E}[\widetilde{S}_j \mathrm{e}^{-\lambda \widetilde{Z}}]}{\mathbb{E}[\mathrm{e}^{-\lambda \widetilde{Z}}]}, \qquad \widetilde{Z} = \sum_{j=1}^{N} A_j \widetilde{S}_j + \sum_{i=1}^{I} \varepsilon_i, \tag{5.13}$$

where $\widetilde{S}_j$ denotes the asset price with the cost-adjusted mean rate of return, $\widetilde{\mu}_j = \mu_j - c_j \Gamma_j(\widetilde{\mathbf{y}}_j)$.

Note that (5.10) can be written alternatively as

$$\pi_j = \frac{1}{\lambda A_j}\boldsymbol{\Sigma}_j^{-1}(\boldsymbol{\mu} - \lambda\boldsymbol{\xi}) - \frac{1}{\lambda A_j}\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\Gamma} = \pi_j^{\mathrm{NT}} - \frac{1}{\lambda A_j}\boldsymbol{\Sigma}_j^{-1}\boldsymbol{\Gamma}, \qquad j = 1, 2, \ldots, N.$$

Here, the quantities $\Gamma_j(\widetilde{\mathbf{y}}_j)$ can be positive or negative, depending on the optimal risk exchanges $\widetilde{y}_j^i$, which are assumed to be nonzero in Theorem 5.1. Hence, the equilibrium prices in the presence of transaction costs can be higher or lower than those without transaction costs.

Next, the equations in (5.5) can be written in matrix form as

$$\frac{1}{\lambda_i}(\boldsymbol{\mu} - \boldsymbol{\gamma}^i) - \boldsymbol{\xi}^i = \boldsymbol{\Sigma}\,\mathrm{diag}(x_j^i + \widetilde{y}_j^i)\,\boldsymbol{\pi},$$

where $\boldsymbol{\gamma}^i = (c_j \mathrm{sgn}(\widetilde{y}_j^i))$ and $\boldsymbol{\xi}^i = (\pi_i(\varepsilon_i)\sigma_{ij}^{\varepsilon})$ are N-dimensional vectors. Substitution of the equilibrium price (5.9) into the above equation yields

$$\frac{1}{\lambda_i}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\gamma}^i - \lambda_i\boldsymbol{\xi}^i) = \mathrm{diag}(x_j^i + \widetilde{y}_j^i)\frac{1}{\lambda}\mathrm{diag}(A_j^{-1})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\Gamma} - \lambda\boldsymbol{\xi}).$$

It follows that the equilibrium allocation is formally given as

$$(x_j^i + \widetilde{y}_j^i)S_j = \frac{\lambda}{\lambda_i}A_j S_j \frac{\boldsymbol{\Sigma}_j^{-1}(\boldsymbol{\mu} - \boldsymbol{\gamma}^i - \lambda_i\boldsymbol{\xi}^i)}{\boldsymbol{\Sigma}_j^{-1}(\boldsymbol{\mu} - \boldsymbol{\Gamma} - \lambda\boldsymbol{\xi})}, \qquad j = 1, 2, \ldots, N. \tag{5.14}$$

Recall from (3.19) that the part $\frac{\lambda}{\lambda_i}A_j S_j$ corresponds to the equilibrium allocation when there are no transaction costs and residual risks.

5.2. **Pricing of derivative securities.** Suppose that there are derivative securities (written on some traded assets) in the market and that the equilibrium pricing formula (5.13) is valid. In this subsection, we show that the risk-neutral pricing method holds true for the pricing of derivative securities under some conditions.

Consider, as an example, a call option Y with strike price K written on asset S_j. That is, we denote the payoff by

$$Y = (S_j - K)_+ = f(R_j), \qquad f(x) = (\pi_j(1+x) - K)_+,$$

where $(x)_+ = \max\{x, 0\}$ and $R_j = (S_j - \pi_j)/\pi_j$. According to the equilibrium pricing formula (5.13), the price of the call option is given by

$$\pi(Y) = \frac{\mathbb{E}[\widetilde{Y}\mathrm{e}^{-\lambda\widetilde{Z}}]}{\mathbb{E}[\mathrm{e}^{-\lambda\widetilde{Z}}]}, \tag{5.15}$$

where $\widetilde{Y}$ denotes the *cost-adjusted* payoff of the call option.

Suppose that $\widetilde{Y} = f(\widetilde{R}_j)$ and, instead of (5.15), the call option price is given by

$$\pi(Y) = \frac{\mathbb{E}[f(\widetilde{R}_j)\mathrm{e}^{-\lambda\widetilde{Z}}]}{\mathbb{E}[\mathrm{e}^{-\lambda\widetilde{Z}}]}, \tag{5.16}$$

where $\widetilde{R}_j$ denotes the cost-adjusted rate of return of S_j. Note that the transaction costs to trade derivative securities are usually negligible, because agents must pay the option premiums. However, since the transaction costs of other assets affect the option price in equilibrium, the formula $\widetilde{Y} = f(\widetilde{R}_j)$ is merely an assumption in our framework. This assumption states that the transaction costs of other assets do not affect the price of the derivative.

Suppose further that there are so many assets traded in the market, and so the aggregated risk Z can be approximated by a normally distributed random variable.[7] Since the *cost-adjusted* rate of return $\widetilde{R}_j$ is normally distributed by our early assumption, i.e.,

$$\widetilde{R}_j = \widetilde{\mu}_j + \sigma_j w_j,$$

where w_j denotes a standard normal variate, it then follows from (5.16) and Lemma 4.1 that

$$\pi(Y) = \mathbb{E}\left[f(\widetilde{\mu}_j + \sigma_j w_j - \lambda\mathrm{cov}(R_j, Z))\right]. \tag{5.17}$$

However, in this setting, the price of S_j must be given by

$$\pi_j = \mathbb{E}\left[\pi_j(1 + \widetilde{\mu}_j + \sigma_j w_j - \lambda\mathrm{cov}(R_j, Z))\right];$$

hence, we have $\widetilde{\mu}_j = \lambda\mathrm{cov}(R_j, Z)$. It follows that the call option price is given by

$$\pi(Y) = \mathbb{E}\left[(\pi_j(1 + \sigma_j w_j) - K)_+\right]. \tag{5.18}$$

This is so, because the risk premium $\lambda\mathrm{cov}(R_j, R_Z)$ in (5.16) is already reflected in the price π_j of the underlying asset S_j in equilibrium. This result is important for practice, because we do not need to estimate the unknown (unobservable) parameters λ and $\mathrm{cov}(R_j, R_Z)$ for the pricing of derivative securities, provided that the above assumptions hold.

Now, recall that $\mathrm{e}^x \approx 1 + x$ for x small in the magnitude. Hence, if the volatility σ_j is small enough, the following approximation is justified:

$$1 + \sigma_j w_j \;\approx\; \mathrm{e}^{\sigma_j w_j - \sigma_j^2/2}, \tag{5.19}$$

where the term $\sigma_j^2/2$ is subtracted to have the same mean in both sides. In this case, from (5.18), we have

$$\pi(Y) = \mathbb{E}[(\pi_j \mathrm{e}^{\sigma_j w_j - \sigma_j^2/2} - K)_+].$$

[7] See Wang (2003) for the justification of this assumption.

Finally, it is readily shown that the call option price is given by

$$\pi(Y) = \pi_j \Phi(d) - K\Phi(d - \sigma_j), \qquad d = \frac{\log(\pi_j/K)}{\sigma_j} + \frac{\sigma_j}{2},$$

the famous Black–Scholes formula (1973) with $r_f = 0$ and $T = 1$.

6. Cocluding Remarks

In this paper, we examine the Bühlmann's equilibrium pricing model (1980) in the presence of proportional transaction cost. It is shown that an equilibrium exists under some mild conditions and the multivariate Esscher transform (1.2) is an appropriate probability transform for the pricing of insurance risks even in the market with transaction costs.

In the simplest case that only asset 1 is traded with transaction cost (the other assets are traded with no transaction costs), we derive an explicit form of equations to be solved for the equilibrium. In particular, for the CARA-normal case, it is shown that an equilibrium price π_1^c of asset 1 with transaction cost c is a unique solution of a linear equation (4.29) and the prices of the other assets are given by the multivariate Esscher transform. In this case, as the cost c increases, the buying price $(1+c)\pi_1^c$ is increasing, the selling price $(1-c)\pi_1^c$ is decreasing, and the trading volumes $|y_1^i|$ are decreasing in equilibrium.

When the transaction costs are so small, we show that the equilibrium asset prices are given by the multivariate Esscher transform for the CARA-normal case. In this case, while the mean rates of return of the assets are adjusted by the transaction costs, the volatilities of the assets are not affected by them in equilibrium.

When there is a derivative security in the market, we show that the risk-neutral pricing method is possible under the assumption that the transaction costs of other assets do not affect the price of the derivative. However, in our framework, the transaction costs of other assets *do* affect the option price in equilibrium and, hence, the risk-neutral method may not be applicable in the presence of transaction costs. It is of great interest to investigate this problem within the general equilibrium framework as a future research.

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