主論文

Separatrices of competition-diffusion equations

Hirokazu NINOMIYA

DEPARTMENT OF APPLIED PHYSICS
TOKYO INSTITUTE OF TECHNOLOGY
Oh-Okayama, Meguro-ku, Tokyo 152 JAPAN
E-mail: ninomiya@ap.titech.ac.jp

SEPARATRICES OF COMPETITION-DIFFUSION EQUATIONS

By
Hirokazu NINOMIYA

1. Introduction

In mathematical biology, theoretical understanding of the spatio and/or temporal dynamics of biological individuals is one of major subjects. As one example of population dynamics, we meet the situation where two species are strongly competing. It is observed that one species of the two becomes extinct in a habitat by competing, or two species can coexist by avoiding the competition with migration (see, e.g., [10]). The former phenomenon is called the competitive exclusion principle, while the latter means the coexistence of niche-segregation. To understand these phenomena, Lotka-Volterra competition models with diffusion have been often proposed so far.

A simple model in one dimension is described by

$$(1.1)\qquad \begin{cases} u_t = d_1 u_{xx} + u(m_1 - c_{11}u - c_{12}v) & (0 < x < 1,\ t > 0), \\ v_t = d_2 u_{xx} + v(m_2 - c_{21}u - c_{22}v) & (0 < x < 1,\ t > 0) \end{cases}$$

with the Neumann boundary conditions

$$(1.2)\qquad \begin{cases} u_x(0,t) = u_x(1,t) = 0 & (t > 0), \\ v_x(0,t) = v_x(1,t) = 0 & (t > 0), \end{cases}$$

where $u(x,t)$ and $v(x,t)$ usually represent the population densities of two competing species at position $x \in (0,1)$ and time $t > 0$. Thus it is naturally assumed that u and v are nonnegative. The constant m_i is the intrinsic growth rate, c_{ii} the intraspecific competition rate, and c_{ij} $(i \neq j)$ the interspecific competition rate where all constants m_i, c_{ij}, d_i $(i,j = 1,2)$ are positive. By simple rescalings, (1.1) with (1.2) is rewritten as

$$(1.3)\qquad \begin{cases} u_t = du_{xx} + u(a - u - bv) & (0 < x < L,\ t > 0), \\ v_t = v_{xx} + v(1 - cu - v) & (0 < x < L,\ t > 0) \end{cases}$$

with the Neumann boundary conditions

$$(1.4) \qquad \begin{cases} u_x(0,t) = u_x(L,t) = 0 & (t > 0), \\ v_x(0,t) = v_x(L,t) = 0 & (t > 0), \end{cases}$$

where a, b and c are positive constants. The global existence of a solution of the system (1.3) with (1.4) is proved by the maximal principle (see [12]). However, the qualitative property of solutions have not yet been completely revealed. For the first step to do it, the system (1.3) in the absence of diffusion is considered

$$(1.5) \qquad \begin{cases} u_t = u(a - u - bv), \\ v_t = v(1 - cu - v), \end{cases}$$

where both components of initial data are positive. It is known that the asymptotic behavior of solutions to (1.5) consist of four types: (i) ${}^t(a,0)$ is a unique globally stable equilibrium; (ii) ${}^t(0,1)$ is a unique globally stable equilibrium; (iii) ${}^t(\bar{u},\bar{v}) = {}^t((b-a)/(bc-1),(ac-1)/(bc-1))$ is a unique globally stable equilibrium; (iv) there are two stable equilibria ${}^t(a,0)$ and ${}^t(0,1)$. In the first three cases, any solutions generally converge to the unique stable equilibrium, while in the last case, which stable equilibrium the solution converges to depends on the initial state. Therefore, the following question naturally arises: what sort of initial data lead to the specific equilibrium, ecologically speaking, which species of the two becomes extinct depending on the initial state.

In general, the dynamics of solutions depends on the initial data, if multi-stable equilibria coexist. Although there have been many works concerned with the asymptotic behavior of solutions to various systems including (1.3), most of them discuss the existence and the stability of equilibria and/or periodic orbits (c.f. [4]), and do not tell us sufficient information on the dependency of initial data on the dynamics of solutions because we need to investigate the behavior of the solution with given initial data for the full time range. This also motivates us to study the characterization of the basin of attraction for the competition-diffusion system (1.3) as well as (1.5). Hereafter we assume the condition

$$\frac{1}{c} < a < b, \tag{1.6}$$

for the bi-stable case (iv).

For the system (1.5) of ordinary differential equations with the condition (1.6), it is already known that the first quadrant in the (u, v) plane is divided into two basins of attraction by a *separatrix* which makes the boundary between two basins of attraction [8], [7]. The separatrix for (1.5) is represented by the graph of a function h, i.e.,

$$\{{}^t(u,v) \in \mathbb{R}^2 \mid u \geq 0,\ v \geq 0,\ v = h(u)\}$$

(c.f. [7]). That is, if $v(0) > h(u(0))$, then ${}^t(u(t), v(t))$ converges to ${}^t(0,1)$, while if $v(0) < h(u(0))$, then it converges to ${}^t(a, 0)$. For the property of $v = h(u)$, it is shown in [7] that

(i) if $a > 1$, $v = h(u)$ is concave (i.e., $h'' < 0$);

(ii) if $a = 1$, it is a straight line (i.e., $h(u) = (c-1)u/(b-1)$);

(iii) if $a > 1$, it is convex (i.e., $h'' > 0$).

Now, we return to the original system (1.3) with (1.6) under the Neumann conditions (1.4). It is known that stable equilibria are only ${}^t(a, 0)$ and ${}^t(0,1)$, that is, any nonconstant equilibria and periodic solutions are unstable, even if they exist [9], [6]. Therefore, one finds that the problem is to determine the separatrix for the constant equilibria ${}^t(a, 0)$ and ${}^t(0, 1)$.

For the special case where the diffusion coefficients are same ($d = 1$), Iida et al [7] have recently shown that in the case $a > 1$ there exists an initial data ${}^t(u(x, 0), v(x, 0))$ such that even if

$$v(x, 0) > h(u(x, 0)) \quad \text{for every } x \in [0, L],$$

${}^t(u(x,t), v(x,t))$ converges to ${}^t(a, 0)$. In ecological terms, it implies that the species u may wipe out v, even if v is superior to u everywhere at $t = 0$. We call such a phenomenon the diffusion-induced extinction of a superior species. They show that this phenomenon possibly occurs, using the effect of the diffusive migration and the concavity of the separatrix (or $a > 1$). This implies the difference of the structure of separatrix between the systems (1.3) and (1.5). In order to construct the separatrix for ${}^t(a,0)$ and ${}^t(0,1)$ of (1.3), (1.4), and study the dependency of the asymptotic states on the initial data and the parameters, we restrict our discussion to the neighborhood of an unstable constant equilibrium ${}^t(\bar{u},\bar{v})$.

In §2, we construct the local invariant manifold with one codimension which coincides with the separatrix for (1.3) near ${}^t(\bar{u},\bar{v})$ in some sense (see Theorems 2.2 and 2.3). In §3, by using this invariant manifold, we present several results: First, we give some conditions on initial distributions under which one species of the two becomes extinct. For an example, choose $a = 1$, $b = c = 2$, and $d = 1$ in (1.3) which indicates that the system is symmetric with u and v. If the initial data is taken as in Fig. 1, it turns out that the species u survives and v becomes extinct (see §3). Namely, the species u, which distributes more uniformly than v does near the equilibrium at $t = 0$, wipes out the other (see Fig. 2).

Second, we show that even if the images of two different initial states in $\mathbb{R}^2$ coincide together, each solution may converge to the different equilibrium respectively. This means that the asymptotic state of solutions can be never expected by means of the information of initial data in the (u,v) plane.

Third, we consider the dependency of the asymptotic behavior on the parameter d for suitably fixed a. We show that if the diffusion coefficients are different, the diffusion-induced extinction can occur in the absence of the concavity of the separatrix for (1.5). More generally, we investigate the dependency on the parameters a and d. It indicates that one species u tends to be extinct as its diffusion rate d or growth rate a decreases, that is, there is the relation between the diffusion rate and the growth rate such that the two species are equally balanced. It is studied mathematically when (a,d) is close to $(1,1)$ and also numerically when (a,d) is not close to $(1,1)$.

In §4, we give the proof of Theorems 2.2 and 2.3 and Proposition 3.5. If the stable manifold at ${}^t(\bar{u},\bar{v})$ has codimension one, the invariant manifold is uniquely determined. If not, however, the invariant manifold is not unique. Under some conditions specified later, we can construct it uniquely up to the second order (see Theorem 2.3 and Proposition 4.2). We need to know the whole dynamics to prove that the invariant manifold coincides with the separatrix up to the same order. We investigate the local dynamics as long as the solution is close to ${}^t(\bar{u},\bar{v})$. Then we use the comparison theorem to show the convergence of the solution.

2. Local invariant manifolds and separatrices

First we prepare the notation and the spaces. The usual inner product of $\mathbb{R}^2$ is denoted by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \cdot \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} := u_1 u_2 + v_1 v_2$$

for ${}^t(u_1,v_1), {}^t(u_2,v_2) \in \mathbb{R}^2$ and $\langle \cdot,\cdot \rangle$ means an inner product in $L^2(0,L)$, i.e.,

$$\langle u, v \rangle := \int_0^L u(x)v(x)dx \qquad \text{for } u,v \in L^2(0,L).$$

We also introduce Hilbert spaces H and X

$$H := \left\{ \omega(\cdot) = \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} \,\middle|\, u(\cdot) \in L^2(0,L),\ v(\cdot) \in L^2(0,L) \right\},$$
$$X := \left\{ \omega(\cdot) = \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} \in H \,\middle|\, u_x \in L^2(0,L),\ v_x \in L^2(0,L) \right\}$$

with their inner products and their norms respectively

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_H := \int_0^L u_1(x)u_2(x) + v_1(x)v_2(x)dx, \qquad \|\omega\|_H^2 := \langle \omega, \omega \rangle_H,$$
$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_X := \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_H + \left\langle \begin{pmatrix} u_{1x} \\ v_{1x} \end{pmatrix}, \begin{pmatrix} u_{2x} \\ v_{2x} \end{pmatrix} \right\rangle_H, \quad \|\omega\|_X^2 := \langle \omega, \omega \rangle_X.$$

We use a new variable

$$\omega = \begin{pmatrix} u - \bar{u} \\ v - \bar{v} \end{pmatrix}$$

in order to investigate the behavior of solutions near the equilibrium point ${}^t(\bar{u}, \bar{v})$. Let us define a linear operator $\boldsymbol{A}$ and a nonlinear mapping $\boldsymbol{F} : X \to X$ as follows:

$$
\boldsymbol{A} = \begin{pmatrix} -d\dfrac{\partial^2}{\partial x^2} + \bar{u} & b\bar{u} \\ c\bar{v} & -\dfrac{\partial^2}{\partial x^2} + \bar{v} \end{pmatrix} \tag{2.1}
$$

with domain

$$
D(\boldsymbol{A}) = \left\{ \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} \in H \,\middle|\, u_{xx}, v_{xx} \in L^2(0, L), u_x(0) = v_x(0) = u_x(L) = v_x(L) = 0 \right\}
$$

and

$$
\boldsymbol{F}(\boldsymbol{\omega}) = \begin{pmatrix} -\xi(\xi + b\eta) \\ -\eta(c\xi + \eta) \end{pmatrix} \quad \text{where } \boldsymbol{\omega} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in X. \tag{2.2}
$$

The resulting system from (1.3) is rewritten as

$$
\boldsymbol{\omega}_t = -\boldsymbol{A}\boldsymbol{\omega} + \boldsymbol{F}(\boldsymbol{\omega}). \tag{2.3}
$$

It is easily seen that $\boldsymbol{A}$ is a sectorial operator (see [5]). The fractional power of $\boldsymbol{A}$ can be defined in a usual manner.

Let σ_k be the $(k+1)$th eigenvalue of $-d^2/dx^2$ with the Neumann conditions and ζ_k a corresponding eigenfunction, namely,

$$
\sigma_k = \left(\frac{\pi k}{L}\right)^2 \quad (k \geq 0), \quad \zeta_0 = \sqrt{\frac{1}{L}}, \quad \text{and } \zeta_k = \sqrt{\frac{2}{L}} \cos\frac{\pi k x}{L} \quad (k \geq 1). \tag{2.4}
$$

Since

$$
(d\sigma_k + \bar{u} + \sigma_k + \bar{v})^2 - 4((d\sigma_k + \bar{u})(\sigma_k + \bar{v}) - bc\bar{u}\bar{v}) = (d\sigma_k + \bar{u} - \sigma_k - \bar{v})^2 + 4bc\bar{u}\bar{v} > 0,
$$

it is obvious that eigenvalues of the matrix

$$
\boldsymbol{M}_k = \begin{pmatrix} d\sigma_k + \bar{u} & b\bar{u} \\ c\bar{v} & \sigma_k + \bar{v} \end{pmatrix}
$$

are real.

LEMMA 2.1. *Let $\mu_{k,\pm}$ ($\mu_{k,-} < \mu_{k,+}$) be the eigenvalues of $\boldsymbol{M}_k$. The eigenvalues of $\boldsymbol{A}$ are real, which can be denoted by $\{\lambda_j\}_{j\geq 1}$ satisfying*

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \leq \cdots.$$

Precisely there exist functions $j_\pm(k)$ and $k(j)$ such that

$$\begin{cases} \mu_{k,+} = \lambda_{j_+(k)}, \\ \mu_{k,-} = \lambda_{j_-(k)}, \\ \lambda_j = \mu_{k(j),+} \ \text{or}\ \mu_{k(j),-}. \end{cases} \tag{2.5}$$

If $\lambda_{j_1} = \lambda_{j_2}$ ($j_1 \neq j_2$), then $k(j_1) \neq k(j_2)$. Moreover $\lambda_1 = \mu_{0,-} < 0$ and $\mu_{0,+} > 0$.

Proof. We prove the last part only. The remainder is easily shown, because the family of eigenvalues of $\boldsymbol{A}$ consist of $\{\mu_{k,\pm}\}_{k=0}^{\infty}$. The matrix

$$\boldsymbol{M}_0 = \begin{pmatrix} \bar{u} & b\bar{u} \\ c\bar{v} & \bar{v} \end{pmatrix}$$

has two real eigenvalues. The eigenvalues are the roots of the quadratic equation

$$\mu^2 - (\bar{u} + \bar{v})\mu - (bc - 1)\bar{u}\bar{v} = 0. \tag{2.6}$$

Since the last term is negative, we can check $\mu_{0,-} < 0$ and $\mu_{0,+} > 0$. Noting

$$\begin{aligned} &\{\mu^2 - (d\sigma_k + \bar{u} + \sigma_k + \bar{v})\mu + (d\sigma_k + \bar{u})(\sigma_k + \bar{v}) - bc\bar{u}\bar{v}\}|_{\mu=\mu_{0,-}} \\ &\quad = \{-(d+1)\mu_{0,-} + d\sigma_k + \bar{u} + d\bar{v}\}\sigma_k > 0, \\ &\frac{d\sigma_k + \bar{u} + \sigma_k + \bar{v}}{2} > 0 > \mu_{0,-} \quad \text{for } k \geq 1, \end{aligned}$$

we can show that $\mu_{0,-} < \lambda_j$ for $j \geq 2$. □

The corresponding eigenvectors of $\boldsymbol{A}$ are denoted by $\boldsymbol{\varphi}_j$, namely,

$$\boldsymbol{A}\boldsymbol{\varphi}_j = \lambda_j \boldsymbol{\varphi}_j.$$

Especially, we can take

$$\boldsymbol{\varphi}_j = \boldsymbol{e}_{k,\pm}\zeta_k, \quad \text{if } j = j_\pm(k)$$

where $\boldsymbol{e}_{k,\pm}$ are the eigenvectors corresponding to $\mu_{k,\pm}$:

$$\boldsymbol{e}_{k,\pm} = \begin{pmatrix} u_{k,\pm} \\ v_{k,\pm} \end{pmatrix} := \begin{pmatrix} 1 \\ \dfrac{\mu_{k,\pm} - d\sigma_k - \bar{u}}{b\bar{u}} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{c\bar{v}}{\mu_{k,\pm} - \sigma_k - \bar{v}} \end{pmatrix}.$$

The adjoint operator is

$$
(2.7) \qquad \boldsymbol{A}^* := \begin{pmatrix} -d\dfrac{\partial^2}{\partial x^2} + \bar{u} & c\bar{v} \\ b\bar{u} & -\dfrac{\partial^2}{\partial x^2} + \bar{v} \end{pmatrix}
$$

with the same domain as in $\boldsymbol{A}$. Let $\boldsymbol{\varphi}_j^*$ be an eigenfunction corresponding to λ_j. Multiplying appropriate constants, we can take $\boldsymbol{\varphi}_j^*$ satisfying

$$
\left\langle \boldsymbol{\varphi}_{j_1}^*, \boldsymbol{\varphi}_{j_2} \right\rangle = \delta_{j_1, j_2}
$$

where δ_{kl} stands for the Kronecker delta.

It is easily shown that $\{\boldsymbol{\varphi}_j\}_{j=1}^{\infty}$ is a complete basis of H. Define

$$
\begin{aligned}
\boldsymbol{P}_j\boldsymbol{\omega} &:= \left\langle \boldsymbol{\varphi}_j^*, \boldsymbol{\omega} \right\rangle_H \boldsymbol{\varphi}_j, \qquad & Q_j\boldsymbol{\omega} &:= \left\langle \boldsymbol{\varphi}_j^*, \boldsymbol{\omega} \right\rangle_H, \\
\boldsymbol{\omega}_j &:= \boldsymbol{P}_j\boldsymbol{\omega}, & w_j &:= Q_j\boldsymbol{\omega}.
\end{aligned}
$$

Thus $\boldsymbol{\omega}$ can be expanded by

$$
\boldsymbol{\omega} = \sum_{j=1}^{\infty} \boldsymbol{\omega}_j = \sum_{j=1}^{\infty} w_j \boldsymbol{\varphi}_j.
$$

We also define the operator $\hat{}$ from H to H by

$$
\hat{\boldsymbol{\omega}} := (I - \boldsymbol{P}_1)\boldsymbol{\omega} = \sum_{j=2}^{\infty} \boldsymbol{\omega}_j.
$$

We seek a locally invariant manifold with one codimension such that

$$
\boldsymbol{\omega}_1 = \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}).
$$

Thus we split (2.3) into

$$
(2.8) \qquad \begin{cases} \boldsymbol{\omega}_{1t} = -\lambda_1 \boldsymbol{\omega}_1 + \boldsymbol{P}_1 \boldsymbol{F}(\boldsymbol{\omega}_1 + \hat{\boldsymbol{\omega}}), \\ \hat{\boldsymbol{\omega}}_t = -\boldsymbol{A}\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{F}}(\boldsymbol{\omega}_1 + \hat{\boldsymbol{\omega}}). \end{cases}
$$

THEOREM 2.2. *Assume $\lambda_1 < 2\lambda_2$. Then there exists a C^2-function $\boldsymbol{\Phi}$ from $(I - \boldsymbol{P}_1)D(\boldsymbol{A})$ into $\boldsymbol{P}_1 D(\boldsymbol{A})$ such that the graph of $\boldsymbol{\Phi}$ is a locally invariant manifold to* (2.8) *near ${}^t(\bar{u}, \bar{v})$,*

$$
(2.9) \qquad \boldsymbol{\Phi} = \boldsymbol{\Psi} + o(\|\hat{\boldsymbol{\omega}}\|_X^2), \quad \psi(\hat{\boldsymbol{\omega}}) = Q_1 \boldsymbol{\Psi}(\hat{\boldsymbol{\omega}})
$$

and

$$
(2.10)\qquad \begin{aligned}\psi(\hat{\boldsymbol{\omega}}) = \;& \zeta_0 \frac{\alpha(\boldsymbol{e}_{0,+}, \boldsymbol{e}_{0,+}) w_{j+(0)}^2}{2\mu_{0,+} - \mu_{0,-}} \\ & + \zeta_0 \sum_{k=1}^{\infty} \left\{ \frac{\alpha(\boldsymbol{e}_{k,+}, \boldsymbol{e}_{k,+}) w_{j+(k)} w_{j+(k)}}{2\mu_{k,+} - \mu_{0,-}} + \frac{\alpha(\boldsymbol{e}_{k,+}, \boldsymbol{e}_{k,-}) w_{j+(k)} w_{j-(k)}}{\mu_{k,+} + \mu_{k,-} - \mu_{0,-}} \right. \\ & \left. + \frac{\alpha(\boldsymbol{e}_{k,-}, \boldsymbol{e}_{k,+}) w_{j-(k)} w_{j+(k)}}{\mu_{k,+} + \mu_{k,-} - \mu_{0,-}} + \frac{\alpha(\boldsymbol{e}_{k,-}, \boldsymbol{e}_{k,-}) w_{j-(k)} w_{j-(k)}}{2\mu_{k,-} - \mu_{0,-}} \right\},\end{aligned}
$$

where

$$
(2.11)\qquad \alpha\left(\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} \right) = u_{0,-}^* \xi_1(\xi_2 + b\eta_2) + v_{0,-}^* \eta_1(c\xi_2 + \eta_2).
$$

We note that the denominators of the terms in the right hand side of (2.10) are positive, because

$$
\lambda_{j_1} + \lambda_{j_2} - \lambda_1 \geq 2\lambda_2 - \lambda_1 > 0.
$$

Remark. The principal part of the separatrix of (1.5) is given by

$$
\boldsymbol{e}_{0,-}^* \cdot {}^t(u - \bar{u}, v - \bar{v}) = \alpha(\boldsymbol{e}_{0,+}, \boldsymbol{e}_{0,+}) \frac{\{\boldsymbol{e}_{0,+}^* \cdot {}^t(u - \bar{u}, v - \bar{v})\}^2}{2\mu_{0,+} - \mu_{0,-}} + o(|\boldsymbol{e}_{0,+}^* \cdot {}^t(u - \bar{u}, v - \bar{v})|^2)
$$

near ${}^t(u, v) = {}^t(\bar{u}, \bar{v})$.

The locally invariant manifold in Theorem 2.2 is a separatrix in the following sense.

THEOREM 2.3. *For sufficiently small $\epsilon > 0$, there exists a positive constant r_1 such that*

(i) *if*

$$
Q_1 \left\{ \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} - \boldsymbol{\Psi}\left((I - \boldsymbol{P}_1) \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} \right) \right\} \geq \epsilon \left\| \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} \right\|_X^2,
$$

then ${}^t(u(x,t), v(x,t))$ converges to ${}^t(a, 0)$ as $t \to \infty$;

(ii) *if*

$$
Q_1 \left\{ \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} - \boldsymbol{\Psi}\left((I - \boldsymbol{P}_1) \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} \right) \right\} \leq -\epsilon \left\| \begin{pmatrix} u(\cdot, 0) - \bar{u} \\ v(\cdot, 0) - \bar{v} \end{pmatrix} \right\|_X^2,
$$

then ${}^t(u(x,t), v(x,t))$ converges to ${}^t(0, 1)$ as $t \to \infty$, *where* ${}^t(u(x,t), v(x,t))$ *is a solution to* (1.3) *satisfying*

$$\left\| \begin{pmatrix} u(\cdot,0)-\bar u \\ v(\cdot,0)-\bar v \end{pmatrix} \right\|_X \le r_1.$$

Proofs of Theorems 2.2 and 2.3 are stated in §4.

3. Applications

In this section we apply Theorems 2.2 and 2.3 to some special cases and we give the observation of (2.10). Before presenting the applications, we give the following elementary lemma:

LEMMA 3.1. *The following hold:*

$$\zeta_j^2 = \frac{1}{\sqrt{L}}\zeta_0 + \frac{1}{\sqrt{2L}}\zeta_{2j},$$
$$\langle \zeta_j\zeta_k, \zeta_0\rangle = \begin{cases} \zeta_0 & \textit{if } \ j=k, \\ 0 & \textit{otherwise.} \end{cases}$$

This lemma can be easily shown by (2.4) so that the proof is omitted.

3.1. Separatrices for the same diffusion coefficients.

In this subsection we assume $d=1$. Then we have

$$\mu_{k,\pm} = \sigma_k + \mu_{0,\pm}, \quad \boldsymbol{e}_{k,\pm} = \boldsymbol{e}_{0,\pm} \tag{3.1}$$

by the definition of $\mu_{k,\pm}$ and $\boldsymbol{e}_{k,\pm}$. For simplicity, we write $\mu_{0,\pm}$, $\boldsymbol{e}_{0,\pm} = {}^t(u_{0,\pm}, v_{0,\pm})$, and $\boldsymbol{e}^*_{0,\pm} = {}^t(u^*_{0,\pm}, v^*_{0,\pm})$ by $\mu_\pm$, $\boldsymbol{e}_\pm = {}^t(u_\pm, v_\pm)$, and $\boldsymbol{e}^*_\pm = {}^t(u^*_\pm, v^*_\pm)$, respectively. Especially we can take

$$\begin{cases} u_\pm = 1, \quad v_\pm = \dfrac{\mu_\pm - \bar u}{b\bar u} = \dfrac{c\bar v}{\mu_\pm - \bar v}, \\ u^*_\pm = \beta_\pm, \quad v^*_\pm = \beta_\pm \dfrac{\mu_\pm - \bar u}{c\bar v} = \beta_\pm \dfrac{b\bar u}{\mu_\pm - \bar v}, \end{cases} \tag{3.2}$$

where

$$\frac{1}{\beta_\pm} = 1 + \frac{(\mu_\pm - \bar u)^2}{bc\bar u\bar v} > 0,$$

because ${}^t(0,1)$ cannot be eigenvectors (see Fig. 3). We can check

$$e^*_\pm \cdot e_\pm = 1, \quad e^*_\mp \cdot e_\pm = 0. \tag{3.3}$$

Similarly in §2 we set

$$\begin{aligned} &y = e_- \cdot \omega, \quad z = e_+ \cdot \omega, \\ &y_k = \langle y, \zeta_k \rangle, \quad z_k = \langle z, \zeta_k \rangle, \quad \hat{y} = y - y_0. \end{aligned}$$

We prepare the following lemma:

LEMMA 3.2.
(i) $\alpha(e_+, e_+) = \mu_+ \left(\frac{u^*_- u^2_+}{\bar{u}} + \frac{v^*_- v^2_+}{\bar{v}} \right) = \mu_+ \beta_- \frac{\bar{u} + b\bar{v} - \mu_+}{b\bar{u}\bar{v}}.$

(ii) $\alpha(e_+, e_-) + \alpha(e_-, e_+) = (\mu_+ + \mu_-) \left(\frac{u^*_- u_- u_+}{\bar{u}} + \frac{v^*_- v_- v_+}{\bar{v}} \right) = (\mu_+ + \mu_-) \beta_- \frac{\bar{u} + b\bar{v} - \mu_-}{b\bar{u}\bar{v}}.$

(iii) $\alpha(e_-, e_-) = \mu_- \left(\frac{u^*_- u^2_-}{\bar{u}} + \frac{v^*_- v^2_-}{\bar{v}} \right).$

Proof. We prove (i) only. We obtain the first equality of (i), substituting $u_\pm$ and $u^*_\pm$ into (2.11). For the last equality of (i), by definition, we have

$$\begin{aligned} \frac{u^*_- u^2_+}{\bar{u}} + \frac{v^*_- v^2_+}{\bar{v}} &= \frac{\beta_-}{\bar{u}} + \frac{\beta_-(\mu_- - \bar{u})(\mu_+ - \bar{u})^2}{b^2 c \bar{u}^2 \bar{v}^2} \\ &= \frac{\beta_-}{\bar{u}} + \frac{\beta_-(\mu_+\mu_- - (\mu_+ + \mu_-)\bar{u} + \bar{u}^2)(\mu_+ - \bar{u})}{b^2 c \bar{u}^2 \bar{v}^2} \\ &= \beta_- \frac{\bar{u} + b\bar{v} - \mu_+}{b\bar{u}\bar{v}}. \end{aligned}$$

The others can be proved similarly by (2.11) and (3.2). □

This lemma and Theorem 2.2 imply the following.

THEOREM 3.3. *Assume $\mu_- > -2\sigma_1$ and $d = 1$. The separatrix for (1.3) is represented by the graph of the function Φ satisfying $\Phi = \psi e_- \zeta_0 + o(\|{}^t(\hat{y}, z)\|^2_X)$ and*

$$\begin{aligned} \psi(\hat{y}, z) = \; & \left(\frac{u^*_- u^2_-}{\bar{u}} + \frac{v^*_- v^2_-}{\bar{v}}\right) \sum_{k=1}^{\infty} \frac{\zeta_0 \mu_- y^2_k}{2\sigma_k + \mu_-} + \left(\frac{u^*_- u^2_+}{\bar{u}} + \frac{v^*_- v^2_+}{\bar{v}}\right) \sum_{k=0}^{\infty} \frac{\zeta_0 \mu_+ z^2_k}{2\sigma_k + 2\mu_+ - \mu_-} \\ & + \left(\frac{u^*_- u_- u_+}{\bar{u}} + \frac{v^*_- v_- v_+}{\bar{v}}\right) \sum_{k=1}^{\infty} \frac{\zeta_0(\mu_- + \mu_+) y_k z_k}{2\sigma_k + \mu_+}. \end{aligned} \tag{3.4}$$

Remark. The principal part of the separatrix $v = h(u)$ for (1.5) is given by

$$y_0 = \frac{\zeta_0 \mu_+}{2\mu_+ - \mu_-} \left(\frac{u^*_- u^2_+}{\bar{u}} + \frac{v^*_- v^2_+}{\bar{v}}\right) z^2_0 = \frac{\zeta_0 \mu_+}{2\mu_+ - \mu_-} \beta_- \frac{\bar{u} + b\bar{v} - \mu_+}{b\bar{u}\bar{v}} z^2_0 \tag{3.5}$$

by means of Lemma 3.2 (i). In order to know the sign of the last term of (3.5), we substitute $\bar{u} + b\bar{v}$ into the left hand side of (2.6):

$$\begin{aligned}(\bar{u} + b\bar{v})^2 - (\bar{u} + \bar{v})(\bar{u} + b\bar{v}) + \bar{u}\bar{v}(1 - bc) &= b\bar{v}(-(c-1)\bar{u} + (b-1)\bar{v}) \\ &= (a-1)b\bar{v}.\end{aligned}$$

This implies that

$$h''(u) < 0 \quad \text{near } u = \bar{u},$$

if and only if $a > 1$ (see [7]).

In particular, we consider the case $a = d = 1$. In this case we can easily calculate the eigenvalues and the eigenvectors. The eigenvalues of the matrix $\boldsymbol{M}_0$ are

$$\mu_- = -\frac{(b-1)(c-1)}{bc-1}, \quad \mu_+ = 1. \tag{3.6}$$

The corresponding eigenvectors of $\boldsymbol{M}_0$ and ${}^t\boldsymbol{M}_0$ are

$$\begin{cases} \boldsymbol{e}_- = \begin{pmatrix} u_- \\ v_- \end{pmatrix} = \begin{pmatrix} 1 \\ -\dfrac{c}{b} \end{pmatrix}, & \boldsymbol{e}_+ = \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{c-1}{b-1} \end{pmatrix}, \\ \boldsymbol{e}_-^* = \begin{pmatrix} u_-^* \\ v_-^* \end{pmatrix} = \dfrac{b(c-1)}{2bc-b-c} \begin{pmatrix} \dfrac{1}{b-1} \\ -\dfrac{1}{c-1} \end{pmatrix}, & \boldsymbol{e}_+^* = \begin{pmatrix} u_+^* \\ v_+^* \end{pmatrix} = \dfrac{(b-1)c}{2bc-b-c} \begin{pmatrix} \frac{1}{b} \\ \frac{1}{c} \end{pmatrix}. \end{cases} \tag{3.7}$$

Recall

$$y = \frac{b(c-1)}{2bc-b-c}u - \frac{b(b-1)}{2bc-b-c}v, \quad z = \frac{(b-1)c}{2bc-b-c}u + \frac{b(b-1)}{2bc-b-c}v.$$

Substituting the above into Theorem 3.3, we get the following corollary:

COROLLARY 3.4. *If*

$$\frac{(b-1)(c-1)}{bc-1} < 2\left(\frac{\pi}{L}\right)^2 \quad \textit{and} \quad a = d = 1,$$

then ψ *is given by*

$$\psi(\hat{y}, z) = \frac{b-c}{b}\zeta_0 \sum_{k=1}^{\infty} \frac{y_k^2}{2\sigma_k + \mu_-} + \frac{b+c-2}{b-1}\zeta_0 \sum_{k=1}^{\infty} \frac{y_k z_k}{2\sigma_k + \mu_+}. \tag{3.8}$$

We address the question: Which species of the two becomes extinct when the initial distributions for them are given in Fig. 1 ? Let us consider the case $a = 1, b = 2$, and $c = 2$ to pay attention only to the influence of the initial states on the asymptotic states. In this case we note that

$$\bar{u} = \bar{v} = \frac{1}{3}, \quad y = \frac{u - v}{2}, \quad z = \frac{u + v}{2} - \frac{1}{3}.$$

Then we have

$$\psi(\hat{\omega}) = \frac{\zeta_0}{2} \sum_{k=1}^{\infty} \frac{u_k^2 - v_k^2}{2\sigma_k + 1} \tag{3.9}$$

by this corollary where $u_k = \langle u, \zeta_k \rangle$ and $v_k = \langle v, \zeta_k \rangle$. Since

$$u_0 = v_0, \quad u_k = 0 \quad (k \geq 1), \quad v_1 \neq 0 \quad \text{and } v_k = 0 \quad (k \geq 2)$$

at the initial data, we have

$$Q_1\{\omega_1 - \Psi(\hat{\omega})\} = \frac{\zeta_0}{2} \frac{v_1^2}{2\sigma_1 + 1} > \epsilon \|\hat{\omega}\|_X^2 .$$

Theorem 2.3 implies that the species u wins out v, namely, that the species which distributes uniformly near the equilibrium point ${}^t(\bar{u}, \bar{v})$ survives and the other becomes extinct (see Fig. 2). Consider the initial distributions in Fig. 4. By the effect of the diffusion, u easily become spatial homogeneous. So u dominates (see Fig. 5).

Next we present two different initial data where the images of them in $\mathbb{R}^2$ coincide together and each solution converges to the different equilibrium. In other word, it is impossible to select equilibria to which solutions converge, by means of the information of the (u, v) plane of initial data. If we specify the initial data such that

$$\begin{pmatrix} u^1(x,0) \\ v^1(x,0) \end{pmatrix} = \omega^1(x,0) + \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \bar{u} - s^2 \rho \\ \bar{v} + s \cos \dfrac{\pi x}{L} \end{pmatrix},$$

and

$$\begin{pmatrix} u^2(x,0) \\ v^2(x,0) \end{pmatrix} = \omega^2(x,0) + \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \bar{u} - s^2 \rho \\ \bar{v} + s \cos \dfrac{2\pi x}{L} \end{pmatrix},$$

where

$$\frac{1}{2\sigma_2 + 1} < 2\rho < \frac{1}{2\sigma_1 + 1}$$

for sufficiently small $s > 0$, then the image of the initial data ${}^t(u^1(x,0), v^1(x,0))$ coincides with that of ${}^t(u^2(x,0), v^2(x,0))$. However, since

$$
\begin{aligned}
Q_1\{\omega_1^1 - \Psi(\hat{\omega}^1)\} &= \frac{\sqrt{L}}{4}\left(-2\rho + \frac{1}{2\sigma_1 + 1}\right)s^2,\\
Q_1\{\omega_1^2 - \Psi(\hat{\omega}^2)\} &= \frac{\sqrt{L}}{4}\left(-2\rho + \frac{1}{2\sigma_2 + 1}\right)s^2,
\end{aligned}
$$

the former solution ${}^t(u^1, v^1)$ converges to ${}^t(a, 0)$, while the latter ${}^t(u^2, v^2)$ converges to ${}^t(0,1)$ by Corollary 3.4 and Theorem 2.3.

3.2. Dependency on diffusion coefficients. In this subsection, we focus ourselves to the phenomena which are exhibited by the difference between two diffusion coefficients. Hence we denote $\Phi, \Psi, \psi, \varphi_j$ and Q_1 in Theorem 2.2 by $\Phi^{a,d}, \Psi^{a,d}, \psi^{a,d}, \varphi_j^{a,d}$ and $Q_1^{a,d}$ respectively.

First we consider the case where a and d are close to 1. Put $a = 1 + \tilde{a}, d = 1 + \tilde{d}$. Note that the function Ψ given by (3.4) converges to (3.8) as a tends to 1. Since $e_{0,-}$ is independent of d, we also note that

$$Q_1^{a,1-\tilde{d}} = Q_1^{a,1}.$$

Then we have the following proposition.

PROPOSITION 3.5. *Set*

$$\Xi^a = -\left.\frac{\partial \Psi^{a,d}}{\partial d}\right|_{d=1}.$$

Then

$$
\begin{aligned}
&\Xi^1(\hat{y}, z)\\
&= \frac{\zeta_0}{2bc - b - c}\varphi_1^{1,1}\sum_{k=1}^{\infty}\sigma_k\left(\frac{2(b-c)(c-1)}{2\sigma_k + \mu_-} + \frac{c(b+c-2)}{2\sigma_k + \mu_+}\right)\left(\frac{y_k^2}{2\sigma_k + \mu_-} + \frac{y_k z_k}{2\sigma_k + \mu_+}\right)\\
&+ \frac{\zeta_0 b(b+c-2)(c-1)}{(2bc - b - c)(b-1)}\varphi_1^{1,1}\sum_{k=1}^{\infty}\frac{\sigma_k}{2\sigma_k + \mu_+}\left(\frac{y_k z_k}{2\sigma_k + \mu_+} + \frac{z_k^2}{2\sigma_k + 2\mu_+ - \mu_-}\right)
\end{aligned}
\tag{3.10}
$$

The proof is stated in the successive section.

In particular, if we put $a = 1,\ b = c = 2$ and $u_0 = v_0 = 0$, then we obtain

$$
\begin{aligned}
\psi^{1,1-\tilde{d}}(\hat{\omega}) = \ & \frac{\zeta_0}{2}\sum_{k=1}^{\infty}\frac{u_k^2 - v_k^2}{2\sigma_k+1} \\
& + \tilde{d}\frac{\zeta_0}{4}\sum_{k=1}^{\infty}\frac{\sigma_k}{2\sigma_k+1}\left\{\frac{(u_k-v_k)^2}{2\sigma_k-\frac{1}{3}} + \frac{2(u_k^2-v_k^2)}{2\sigma_k+1} + \frac{(u_k+v_k)^2}{2\sigma_k+\frac{7}{3}}\right\} + o(\tilde{d}\,\|\hat{\omega}\|_X^2).
\end{aligned}
\tag{3.11}
$$

Since

$$
\frac{X^2}{2\sigma_k-\frac{1}{3}} + \frac{2XY}{2\sigma_k+1} + \frac{Y^2}{2\sigma_k+\frac{7}{3}} \geq \left(\frac{|X|}{\sqrt{2\sigma_k-\frac{1}{3}}} - \frac{|Y|}{\sqrt{2\sigma_k+\frac{7}{3}}}\right)^2 \geq 0,
$$

it turns out that

$$
\frac{\zeta_0}{4}\sum_{k=1}^{\infty}\frac{\sigma_k}{2\sigma_k+1}\left\{\frac{(u_k-v_k)^2}{2\sigma_k-\frac{1}{3}} + \frac{2(u_k^2-v_k^2)}{2\sigma_k+1} + \frac{(u_k+v_k)^2}{2\sigma_k+\frac{7}{3}}\right\} > 0.
$$

This implies that if the diffusion coefficient of one species decreases in the case where $a = 1$ and d is close to 1, then the species tends to become extinct.

Iida et at [7] shows the diffusion-induced extinction in the case with the same diffusion coefficients, namely, the species v can become extinct even if the species v is superior to u everywhere at $t = 0$, i.e., $v(x,0) > h(u(x,0))$. This phenomenon occurs by the effect of diffusion and the concavity of the separatrix. If the diffusion coefficients are different, it may occurs without the concavity. Actually, pick the initial data

$$
\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \omega(x,0) + \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \bar{u} + \rho\zeta_0 + s\zeta_1 \\ \bar{v} + s\zeta_1 \end{pmatrix}
\tag{3.12}
$$

for sufficiently small s where $a = 1,\ b = c = 2,\ d = 1 + \tilde{d}$ and

$$
0 < \rho < \frac{2\tilde{d}s^2\sigma_1\zeta_0}{(2\sigma_1+1)(2\sigma_1+\frac{7}{3})}.
$$

Since

$$
v = h(u) \equiv u
$$

in this case, we have

$$
v(x,0) - h(u(x,0)) = -\rho\zeta_0 < 0 \quad \text{for each } x.
$$

By (3.11), however, we have

$$Q_1^{1,1}\{\omega_1(0)-\Psi^{1,1-\tilde{d}}(\hat{\omega}(\cdot,0))\}=\frac{\rho}{2}-\frac{\tilde{d}\sigma_1 s^2\zeta_0}{(2\sigma_1+1)(2\sigma_1+\frac{7}{3})}+o(\tilde{d}s^2)<0,$$

from which it follows that the solution with the initial data (3.12) converges to ${}^t(0,1)$. Ecologically speaking, the species u becomes extinct, nevertheless u is superior to v everywhere at the initial state. Thus the diffusion-induced extinction can occur even in the case with the same growth rates (see Figs. 6, 7).

It is natural that the species of which the growth rate decreases becomes extinct. As seen in (3.11), the species tends to win out, if its diffusion rate increases.

Let us consider the relationship locally near $(a,d)=(1,1)$ when the two species are equally balanced. As neutral initial data, we pick the initial data below for (1.3) on the separatrix of (1.5):

LEMMA 3.6. *There exists initial data* ${}^t(u^a(x,0),v^a(x,0))=\boldsymbol{\omega}^a(x,0)+{}^t(\bar{u},\bar{v})$ *placed on the separatrix for* (1.5) *such that*

$$\begin{pmatrix}u^a(x,0)\\v^a(x,0)\end{pmatrix}=\begin{pmatrix}\bar{u}\\\bar{v}\end{pmatrix}+s\zeta_l\boldsymbol{e}_{0,+}+\gamma^a s^2\zeta_0\boldsymbol{e}_{0,-}+\frac{\gamma^a s^2}{\sqrt{2}}\zeta_{2l}\boldsymbol{e}_{0,-}+o(s^2) \tag{3.13}$$

where

$$\gamma^a=\frac{\alpha(\boldsymbol{e}_{0,+},\boldsymbol{e}_{0,+})\zeta_0}{2\mu_{0,+}-\mu_{0,-}}.$$

Proof. Recall that the separatrix for (1.5) is given near ${}^t(\bar{u},\bar{v})$ by the graph:

$$\sqrt{L}\boldsymbol{e}^*_{0,-}\cdot\boldsymbol{\omega}=\gamma^a\left(\sqrt{L}\boldsymbol{e}^*_{0,+}\cdot\boldsymbol{\omega}\right)^2+o(|\boldsymbol{e}^*_{0,+}\cdot\boldsymbol{\omega}|^2). \tag{3.14}$$

Substitution of

$$\boldsymbol{\omega}=s\zeta_l\boldsymbol{e}_{0,+}+O(s^2)$$

into (3.14) yields

$$\boldsymbol{e}^*_{0,-}\cdot\boldsymbol{\omega}=\sqrt{L}\gamma^a\zeta_l^2 s^2+o(s^2).$$

Lemma 3.1 immediately implies (3.13). □

Substitute the initial data (3.13) into (2.10), i.e.,

$$\omega_1^a - \Phi^{a,d}(\hat{\omega}^a) = 0.$$

From the condition that the leading term of $\omega_1^a - \Phi^{a,d}(\hat{\omega}^a)$ vanishes, we obtain the relationship between a and d, which is given by the following implicit form:

$$\lim_{s\to 0} \frac{Q_1^{a,d}\{\omega_1^a - \Phi^{a,d}(\hat{\omega}^a)\}}{s^2} = \gamma^a - \psi^{a,d}(\zeta_l e_{0,+}) = 0.$$

More precisely,

$$\begin{aligned} \frac{\alpha(e_{0,+}, e_{0,+})}{2\mu_{0,+} - \mu_{0,-}} = & \frac{\alpha(e_{l,+}, e_{l,+})(e_{l,+}^* \cdot e_{0,+})^2}{2\mu_{l,+} - \mu_{0,-}} + \frac{\alpha(e_{l,-}, e_{l,-})(e_{l,-}^* \cdot e_{0,+})^2}{2\mu_{l,-} - \mu_{0,-}} \\ & + \frac{(\alpha(e_{l,-}, e_{l,+}) + \alpha(e_{l,+}, e_{l,-}))(e_{l,-}^* \cdot e_{0,+})(e_{l,+}^* \cdot e_{0,+})}{\mu_{l,+} + \mu_{l,-} - \mu_{0,-}}. \end{aligned} \tag{3.15}$$

If we take $a = 1$ and $d = 1$, then this equation holds. Actually, it is shown by (3.1), (3.3) and $\alpha(e_+, e_+) = 0$, if $a = d = 1$. We want to seek the function $a(d)$ satisfying (3.15) and $a(1) = 1$, if it exists.

It seems that (3.15) is complicated. First we deal with the case with $a = 1 + \tilde{a}$ and $d = 1 + \tilde{d}$ where $\tilde{a}$ and $\tilde{d}$ are sufficiently small. The implicit function theorem implies that

$$\frac{\partial a}{\partial d}(1) = \left. \frac{\dfrac{\partial \psi^{a,d}(\zeta_l e_{0,+})}{\partial d}}{\dfrac{\partial(\gamma^a - \psi^{a,d}(\zeta_l e_{0,+}))}{\partial a}} \right|_{(a,d)=(1,1)}. \tag{3.16}$$

Thus

$$\gamma^a - \psi^{a,1}(\zeta_l e_{0,+}) = \gamma^a - \frac{2\mu_+ - \mu_-}{2\sigma_l + 2\mu_+ - \mu_-}\gamma^a = \frac{2\sigma_l \zeta_0 \mu_+ \beta_-(\bar{u} + b\bar{v} - \mu_+)}{(2\sigma_l + 2\mu_+ - \mu_-)(2\mu_+ - \mu_-)b\bar{u}\bar{v}}$$

by (3.1) and (3.3). Since

$$\left.\frac{\partial \mu_+}{\partial a}\right|_{a=1} = \frac{(b-1)c}{2bc - b - c}$$

by (2.6), we obtain

$$(\bar{u} + b\bar{v} - \mu_+)|_{a=1} = 0, \quad \left.\frac{\partial(\bar{u} + b\bar{v} - \mu_+)}{\partial a}\right|_{a=1} = \frac{b(c-1)}{2bc - b - c}.$$

Substituting (3.10) into (3.16), we get

$$a(1+\tilde{d}) = 1 - \frac{(3bc - b - c - 1)(2bc - b - c)(b + c - 2)}{2(bc-1)^3(2\sigma_l + 1)}\tilde{d} + o(|\tilde{d}|). \tag{3.17}$$

We present the relationship computed numerically. In Fig. 8, the nullcline of (3.15) has been plotted in the cases with $b = c = 2$. The relation (3.17) indicates the graph near $a = 1$.

4. Proofs

We assume $\lambda_1 < 2\lambda_2 < 0$ in this section, because we can prove the case $\lambda_2 \geq 0$ more easily.

First we give the proof of Theorem 2.2.

Proof of Theorem 2.2.

We make a modification of the system (2.3) outside certain neighborhood near $\boldsymbol{\omega} = 0$. Consider the following modified system instead of (2.3):

$$\boldsymbol{\omega}_t = -\boldsymbol{A}\boldsymbol{\omega} + \boldsymbol{f}(\boldsymbol{\omega}) \tag{4.1}$$

where χ is a smooth function satisfying

$$\chi(x) = \begin{cases} 1 & (x < 1), \\ 0 & (x > 2) \end{cases}$$

and

$$\boldsymbol{f}(\boldsymbol{\omega}) = \chi\left(\frac{|\boldsymbol{\omega}_1|}{r}\right)\chi\left(\frac{\|\hat{\boldsymbol{\omega}}\|_X}{r}\right)\boldsymbol{F}(\boldsymbol{\omega}).$$

Recall that

$$\boldsymbol{\omega}_1 = w_1\boldsymbol{\varphi}_1 = w_1\begin{pmatrix} u_{0,-} \\ v_{0,-} \end{pmatrix}\zeta_0.$$

and hence that

$$|\omega_1| = \zeta_0 |e_{0,-}|\ |w_1| = \zeta_0 \sqrt{u_{0,-}^2 + v_{0,-}^2} |w_1|.$$

Note that there exists a positive constant $K_1 > 1$ satisfying

$$(4.2)\quad \begin{cases} \left\| (I - \boldsymbol{P}_1) e^{-(I-P_1)At} \right\|_X \le K_1 e^{-(\lambda_2 - K_1 r)t} \quad \text{for } t \ge 0 \\ |\boldsymbol{P}_1 \boldsymbol{f}(\boldsymbol{\omega}^1) - \boldsymbol{P}_1 \boldsymbol{f}(\boldsymbol{\omega}^2)| \le K_1 r (|\boldsymbol{\omega}_1^1 - \boldsymbol{\omega}_1^2| + \left\| \hat{\boldsymbol{\omega}}^1 - \hat{\boldsymbol{\omega}}^2 \right\|_X), \\ \|(I - \boldsymbol{P}_1)\boldsymbol{f}(\boldsymbol{\omega}^1) - (I - \boldsymbol{P}_1)\boldsymbol{f}(\boldsymbol{\omega}^2)\|_X \le K_1 r (|\boldsymbol{\omega}_1^1 - \boldsymbol{\omega}_1^2| + \left\| \hat{\boldsymbol{\omega}}^1 - \hat{\boldsymbol{\omega}}^2 \right\|_X), \end{cases}$$

for any $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2 \in X$. Especially,

$$(4.3)\quad \begin{cases} |\boldsymbol{P}_1 \boldsymbol{f}(\boldsymbol{\omega})| \le K_1 r (|\boldsymbol{\omega}_1| + \|\hat{\boldsymbol{\omega}}\|_X), \\ \|(I - \boldsymbol{P}_1)\boldsymbol{f}(\boldsymbol{\omega})\|_X \le K_1 r (|\boldsymbol{\omega}_1| + \|\hat{\boldsymbol{\omega}}\|_X). \end{cases}$$

The asymptotic behaviors of solutions to this system coincides with those of solutions to (2.3) in the neighborhood D_r of origin given by

$$D_r = \{\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \hat{\boldsymbol{\omega}} \in X |\ |\boldsymbol{\omega}_1| \le r,\ \|\hat{\boldsymbol{\omega}}\|_X \le r\}.$$

The existence of such a local invariant manifold to (4.1) follows from standard methods of the construction of invariant manifolds, the Lyapunov-Perron method (see [5], or [2]). That is, there is a C^2-function $\boldsymbol{\Phi}$ from $(I - \boldsymbol{P}_1)D(\boldsymbol{A}) \cap B_r$ to $\boldsymbol{P}_1 D(\boldsymbol{A})$ whose graph is locally invariant under the semiflow defined by (4.1) where

$$B_r = \{\hat{\boldsymbol{\omega}} \in (I - \boldsymbol{P}_1)X |\ \|\hat{\boldsymbol{\omega}}\|_X < r\}$$

and $r (> 0)$ is sufficiently small. If suffices to show the properties of $\boldsymbol{\Phi}$. Review a cone property, which will be useful in several contexts as well as the construction of the manifold.

LEMMA 4.1. *If X, Y are positive continuous functions satisfying*

$$(4.4)\quad \begin{cases} X(t+\tau) e^{\bar{\lambda}_1 (t+\tau)} \le X(t) e^{\bar{\lambda}_1 t} + \kappa_1 r \int_\tau^0 \{X(t+s) + Y(t+s)\} e^{\bar{\lambda}_1 (t+s)} ds, \\ Y(t) e^{\bar{\lambda}_2 t} \le Y(0) + \kappa_1 r \int_0^t \{X(s) + Y(s)\} e^{\bar{\lambda}_2 s} ds, \end{cases}$$

for $0 \le t + \tau \le t$, then

(i) $Y(t) \le Y(0)e^{-(\tilde\lambda_2-\kappa_1(1+\kappa_2)r)t}$ *provided* $X(s) \le \kappa_2 Y(s)$ *for* $0 \le s \le t$;

(ii) $X(t) \ge X(0)e^{-(\tilde\lambda_1+\kappa_1(1+\kappa_3)r)t}$ *provided* $Y(s) \le \kappa_3 X(s)$ *for* $0 \le s \le t$,

where κ_i $(i = 1,2,3)$ *are positive constants. Moreover, if*

$$\tilde\lambda_2 - \tilde\lambda_1 - \kappa_1\left(2 + \kappa_2 + \frac{1}{\kappa_2}\right) r > 0,$$

then the region $\Gamma_{\kappa_2} = \{(X,Y) \in \mathbb{R}^2 |\ 0 \le \kappa_2 Y < X\}$ *is positively invariant.*

This lemma follows from Gronwall's inequality. See [2, Lemmas 2.3–2.5]. This property is called a *cone property.*

By the variation-of-constants formula, we have

$$(4.5)\quad \begin{cases} |\omega_1(t+\tau)|e^{\lambda_1(t+\tau)} \\ \qquad\qquad \le |\omega_1(t)|e^{\lambda_1 t} + K_1 r \displaystyle\int_\tau^0 (|\omega_1(t+s)| + \|\hat\omega(\cdot,t+s)\|_X)e^{\lambda_1(t+s)}ds, \\ \hat\omega(\cdot,t) = e^{-At}\hat\omega(\cdot,0) + \displaystyle\int_0^t e^{-A(t-s)}\hat{f}(\omega_1,\hat\omega)ds, \end{cases}$$

for a solution $\omega_1 + \hat\omega$ to (4.1). Set

$$\tilde\lambda_1 = \lambda_1, \quad \tilde\lambda_2 = \lambda_2 - K_1 r, \quad X(t) = |\omega_1(t)|, \quad Y(t) = \sup_{s\ge 0} e^{\tilde\lambda_2 s}\left\|e^{-As}\hat\omega(\cdot,t)\right\|_X.$$

Note that

$$(4.6)\qquad \|\hat\omega(\cdot,t)\|_X \le Y(t) \le K_1\|\hat\omega(\cdot,t)\|_X.$$

It is easily seen from (4.5) that $(X(t),Y(t))$ satisfies (4.4) with $\kappa_1 = K_1^2$, and $\kappa_2 = \kappa_3 = 1$. Thus we get

$$|\Phi(\hat\omega)| \le \|\hat\omega\|_X$$

such that

$$(4.7)\qquad \lambda_2 - \lambda_1 - K_1 r - 4K_1^2 r > 0.$$

Hence, Lemma 4.1 (i) implies

$$(4.8)\qquad \|\hat\omega(\cdot,t)\|_X \le K_1\|\hat\omega(\cdot,0)\|_X e^{-\nu_1 t}$$

where

$$\nu_1 = \tilde{\lambda}_2 - 2K_1^2 r,$$

if $\boldsymbol{\omega}_1(t) + \hat{\boldsymbol{\omega}}(\cdot, t)$ is a solution on the manifold, i.e., $\boldsymbol{\omega}_1(t) = \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot, t))$.

PROPOSITION 4.2. *Assume that $\tilde{\boldsymbol{\Phi}}$ is a C^2-function from $(I - \boldsymbol{P}_1)D(\boldsymbol{A}) \cap B_r$ to $\boldsymbol{P}_1 D(\boldsymbol{A})$ such that*

$$\boldsymbol{G}(\hat{\boldsymbol{\omega}}; \tilde{\boldsymbol{\Phi}}) := \frac{\partial \tilde{\boldsymbol{\Phi}}}{\partial \hat{\boldsymbol{\omega}}}\{-\boldsymbol{A}\hat{\boldsymbol{\omega}} + (I - \boldsymbol{P}_1)\boldsymbol{f}(\tilde{\boldsymbol{\Phi}} + \hat{\boldsymbol{\omega}})\} + \lambda_1 \tilde{\boldsymbol{\Phi}} - \boldsymbol{P}_1 \boldsymbol{f}(\tilde{\boldsymbol{\Phi}} + \hat{\boldsymbol{\omega}})$$

with

$$|\boldsymbol{G}(\hat{\boldsymbol{\omega}}; \tilde{\boldsymbol{\Phi}})| \leq C \, \|\hat{\boldsymbol{\omega}}\|_X^p$$

in B_r for some p satisfying $p > 1$ and $\lambda_1 < p\lambda_2$. Then there exits a positive constant C'

$$|\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}) - \tilde{\boldsymbol{\Phi}}(\hat{\boldsymbol{\omega}})| \leq C' \, \|\hat{\boldsymbol{\omega}}\|_X^p$$

in $\hat{\boldsymbol{\omega}} \in B_r$ with sufficiently small $r > 0$. If

$$|\boldsymbol{G}(\hat{\boldsymbol{\omega}}; \tilde{\boldsymbol{\Phi}})| \leq C(\|\hat{\boldsymbol{\omega}}\|_X^p + (d-1)^{p_1} \, \|\hat{\boldsymbol{\omega}}\|_X^{p_2})$$

as d is close to 1 where $p_1 > 0, p_2 \leq p$, then

$$|\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}) - \tilde{\boldsymbol{\Phi}}(\hat{\boldsymbol{\omega}})| \leq C'(\|\hat{\boldsymbol{\omega}}\|_X^p + (d-1)^{p_1} \, \|\hat{\boldsymbol{\omega}}\|_X^{p_2})$$

Proof. This proposition can be proved by the argument similar to the center manifold theory in [3], [5] and [13] except for the infinite-dimensional invariant manifold. So, we give the sketch of the proof of the latter part only. Let $\hat{\boldsymbol{\omega}}(\cdot, t)$ be the solution of

$$\hat{\boldsymbol{\omega}}_t = -\boldsymbol{A}\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{f}}(\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}) + \hat{\boldsymbol{\omega}}).$$

Suppose that $\tilde{\boldsymbol{\Phi}}$ is as in the lemma, and is extended to $(I - \boldsymbol{P}_1)X$ subject to the same condition in $(I - \boldsymbol{P}_1)X$, if necessary, by multiplying the cut-off function. Set $\boldsymbol{\omega}_1(t) = \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot, t)) - \tilde{\boldsymbol{\Phi}}(\hat{\boldsymbol{\omega}}(\cdot, t))$, which satisfies

$$\boldsymbol{\omega}_{1t} = -\lambda_1 \boldsymbol{\omega}_1 + \boldsymbol{P}_1(\boldsymbol{f}(\boldsymbol{\Phi} + \hat{\boldsymbol{\omega}}) - \boldsymbol{f}(\tilde{\boldsymbol{\Phi}} + \hat{\boldsymbol{\omega}})) + \frac{\partial \tilde{\boldsymbol{\Phi}}}{\partial \hat{\boldsymbol{\omega}}}(\hat{\boldsymbol{f}}(\boldsymbol{\Phi} + \hat{\boldsymbol{\omega}}) - \hat{\boldsymbol{f}}(\tilde{\boldsymbol{\Phi}} + \hat{\boldsymbol{\omega}})) - \boldsymbol{G}(\hat{\boldsymbol{\omega}}; \tilde{\boldsymbol{\Phi}}).$$

Thus, we obtain

$$
\begin{aligned}
&|\omega_1(t+\tau)|e^{\lambda_1(t+\tau)} \\
&\quad \leq |\omega_1(t)|e^{\lambda_1 t} + \int_\tau^0 \{K_1 r(1+K_2)|\omega_1(t+s)| + C\,\|\hat{\omega}\|_X^p + C(d-1)^{p_1}\,\|\hat{\omega}\|_X^{p_2}\}\, e^{\lambda_1(t+s)} ds
\end{aligned}
$$

where

$$K_2 = \sup_{\hat{\omega}^1, \|\hat{\omega}^2\|_X = 1} \left| \frac{\partial \tilde{\Phi}(\hat{\omega}^1)}{\partial \hat{\omega}} \hat{\omega}^2 \right|.$$

Substitution of (4.8) into the above inequality yields

$$
\begin{aligned}
X(t+\tau) \leq \;& X(t) + K_1 r \int_\tau^0 (1+K_2) X(t+s) ds \\
&+ \frac{CK_1^p\,\|\hat{\omega}(0)\|_X^p + CK_1^{p_2}(d-1)^{p_1}\,\|\hat{\omega}(0)\|_X^{p_2}}{\nu_2} e^{-\nu_2(t+\tau)},
\end{aligned}
$$

where

$$X(t) = |\omega_1(t)|e^{\lambda_1 t}, \nu_2 = \min(p(\tilde{\lambda}_2 - 2K_1^2 r) - \lambda_1, p_2(\tilde{\lambda}_2 - 2K_1^2 r) - \lambda_1).$$

Gronwall's inequality implies that

$$X(t+\tau) \leq X(t)e^{-K_1 r(1+K_2)\tau} + K_3(\|\hat{\omega}(0)\|_X^p + (d-1)^{p_1}\,\|\hat{\omega}(0)\|_X^{p_2}) e^{-\nu_2(t+\tau)},$$

where K_3 is a positive constant independent of r. Since

$$X(t)e^{K_1 r(1+K_2)t} \longrightarrow 0 \quad \text{as } t \to \infty,$$

we have

$$|\Phi(\hat{\omega}(0)) - \tilde{\Phi}(\hat{\omega}(0))| = X(0) \leq K_3\{\|\hat{\omega}(0)\|_X^p + (d-1)^{p_1}\,\|\hat{\omega}(0)\|_X^{p_2}\},$$

taking $\tau = -t$ and letting $t \to \infty$, where r is chosen such that

$$\lambda_1 - K_1(1+K_2)r < 0, \quad \nu_2 - K_1(1+K_2)r > 0,$$

and (4.7) hold. □

Since

$$G(\hat{\omega}; 0) = -P_1 f(\hat{\omega}) = O(\|\hat{\omega}\|_X^2),$$

we have

$$
|\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})| \le K_4 \|\hat{\boldsymbol{\omega}}\|_X^2 , \tag{4.9}
$$

using Proposition 4.2. Next we construct the approximate function $\boldsymbol{\Psi}$ of $\boldsymbol{\Phi}$. The principal part of $\boldsymbol{G}(\hat{\boldsymbol{\omega}};\boldsymbol{\Psi}) = 0$ is

$$
\frac{\partial \boldsymbol{\Psi}}{\partial \hat{\boldsymbol{\omega}}}\{-\boldsymbol{A}\hat{\boldsymbol{\omega}} + (I - \boldsymbol{P}_1)\boldsymbol{F}(\hat{\boldsymbol{\omega}})\} + \lambda_1 \boldsymbol{\Psi} - \boldsymbol{P}_1 \boldsymbol{F}(\hat{\boldsymbol{\omega}}) = 0. \tag{4.10}
$$

This argument is slightly different from the theory of center manifolds. Actually, in constructing (finite-dimensional) center manifolds, we can easily seek $\boldsymbol{\Psi}$ as the solution of (4.10), substituting the formal expansion for $\boldsymbol{\Psi}$. In our case, however, it seems to be difficult to seek it because $\boldsymbol{\Psi}$ maps from the infinite-dimensional space. We do it as follows. We can construct the solution of (4.10) by a method of characteristics. Namely, we solve the invariant manifold to the system

$$
\begin{cases} w_{1t} = -\lambda_1 w_1 + Q_1 \boldsymbol{F}(\hat{\boldsymbol{\omega}}), \\ w_{jt} = -\lambda_j w_j \qquad (j \ge 2). \end{cases} \tag{4.11}
$$

We substitute solutions of the latter equations into the former equation and we get

$$
w_{1t} = -\lambda_1 w_1 + Q_1 \boldsymbol{F}\left(\sum_{j=2}^{\infty} w_j(0) e^{-\lambda_j t} \boldsymbol{\varphi}_j\right).
$$

Using the variation-of-constants formula and letting $t \to \infty$ yield

$$
w_1(0) = -\int_0^{\infty} e^{\lambda_1 s} Q_1 \boldsymbol{F}\left(\sum_{j=2}^{\infty} w_j(0) e^{-\lambda_j s} \boldsymbol{\varphi}_j\right) ds. \tag{4.12}
$$

Recall that

$$
Q_1 \boldsymbol{F}(\boldsymbol{\omega}) = -u_{0,-}^{*} \langle u(u+bv), \zeta_0 \rangle - v_{0,-}^{*} \langle v(cu+v), \zeta_0 \rangle
$$

where

$$
\boldsymbol{\omega} = \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k=0}^{\infty} \left\{ w_{j_+(k)} \begin{pmatrix} u_{k,+} \\ v_{k,+} \end{pmatrix} \zeta_k + w_{j_-(k)} \begin{pmatrix} u_{k,-} \\ v_{k,-} \end{pmatrix} \zeta_k \right\} \in X.
$$

We have

$$
\begin{aligned}
&\langle u(u+bv),\zeta_0\rangle \\
&= \left\langle \sum_{k=0}^{\infty}(w_{j+(k)}u_{k,+}+w_{j-(k)}v_{k,-})\zeta_k\sum_{l=0}^{\infty}\{w_{j+(l)}(u_{l,+}+bv_{l,+})+w_{j-(l)}(u_{l,-}+bv_{l,-})\}\zeta_l,\zeta_0\right\rangle \\
&= \sum_{k=0}^{\infty}(w_{j+(k)}u_{k,+}+w_{j-(k)}v_{k,-})\{w_{j+(k)}(u_{k,+}+bv_{k,+})+w_{j-(k)}(u_{k,-}+bv_{k,-})\}\zeta_0,
\end{aligned}
$$

using Lemma 3.1. By (2.11),

$$
\begin{aligned}
Q_1\boldsymbol{F}(\hat{\omega}) = \; & -\zeta_0\alpha(\boldsymbol{e}_{0,+},\boldsymbol{e}_{0,+})w_{j+(0)}w_{j+(0)} \\
& -\zeta_0\sum_{k=1}^{\infty}\{\alpha(\boldsymbol{e}_{k,+},\boldsymbol{e}_{k,+})w_{j+(k)}w_{j+(k)}+\alpha(\boldsymbol{e}_{k,+},\boldsymbol{e}_{k,-})w_{j+(k)}w_{j-(k)} \\
& +\alpha(\boldsymbol{e}_{k,-},\boldsymbol{e}_{k,+})w_{j-(k)}w_{j+(k)}+\alpha(\boldsymbol{e}_{k,-},\boldsymbol{e}_{k,-})w_{j-(k)}w_{j-(k)}\}.
\end{aligned}
$$

Substitution of this into (4.12) yields (2.10). Let $\delta>0$ be so small that

$$
(2+\delta)\lambda_2-\lambda_1>0. \tag{4.13}
$$

Since

$$
\boldsymbol{G}(\hat{\omega};\boldsymbol{\Psi})=O(\|\hat{\omega}\|_X^{2+\delta})
$$

where $\boldsymbol{\Psi}$ is given by (2.10), Proposition 4.2 implies

$$
\boldsymbol{\Phi}=\boldsymbol{\Psi}+O(\|\hat{\omega}\|_X^{2+\delta}). \;\square
$$

Next we prepare two lemmas to prove Theorem 2.3.

LEMMA 4.3. *There exists a positive constant $R>1$ independent of r such that the solution $\boldsymbol{\omega}(t)+{}^t(\bar{u},\bar{v})$ to* (1.3) *satisfying*

$$
Q_1\omega_1(0)>R\,\|\hat{\omega}(\cdot,0)\|_X
$$

converges to ${}^t(a,0)$.

Proof. There exists a positive constant K_5 such that

$$
\sup_x(|u(x)|+|v(x)|)\le K_5\left\|\begin{pmatrix}u\\v\end{pmatrix}\right\|_X \quad \text{for } \begin{pmatrix}u\\v\end{pmatrix}\in X.
$$

Recall that

$$
u_{0,-}>0 \text{ and } v_{0,-}<0.
$$

If we take

$$R = \frac{2K_5}{\zeta_0} \max\left\{\frac{1}{u_{0,-}}, -\frac{1}{v_{0,-}}\right\},$$

we have

$$\begin{aligned} w_1(0)\zeta_0 u_{0,-} &> R\zeta_0 u_{0,-} \,\|\hat{\boldsymbol{\omega}}(\cdot,0)\|_X \geq 2\sup_x |\tilde{u}(x,0)|, \\ -w_1(0)\zeta_0 v_{0,-} &> -R\zeta_0 v_{0,-} \,\|\hat{\boldsymbol{\omega}}(\cdot,0)\|_X \geq 2\sup_x |\tilde{v}(x,0)| \end{aligned}$$

where

$$\boldsymbol{\omega}(x,t) = w_1(t)\zeta_0 \boldsymbol{e}_{0,-} + \hat{\boldsymbol{\omega}}(x,t) = w_1(t)\zeta_0 \boldsymbol{e}_{0,-} + \begin{pmatrix} \tilde{u}(x,t) \\ \tilde{v}(x,t) \end{pmatrix}.$$

Namely,

$$\begin{aligned} &\frac{3}{2}w_1(0)\zeta_0 u_{0,-} \geq w_1(0)\zeta_0 u_{0,-} + \tilde{u}(x,0) \geq \frac{1}{2}w_1(0)\zeta_0 u_{0,-}, \\ &\frac{3}{2}w_1(0)\zeta_0 v_{0,-} \leq w_1(0)\zeta_0 v_{0,-} + \tilde{v}(x,0) \leq \frac{1}{2}w_1(0)\zeta_0 v_{0,-}. \end{aligned}$$

Let $\bar{\boldsymbol{\omega}}^i(t) + {}^t(\bar{u},\bar{v})$ $(i = 1,2)$ be a spatial homogeneous solution to (1.5) with

$$\bar{\boldsymbol{\omega}}^1(0) = \frac{1}{2}w_1(0)\zeta_0 \boldsymbol{e}_{0,-}, \quad \bar{\boldsymbol{\omega}}^2(0) = \frac{3}{2}w_1(0)\zeta_0 \boldsymbol{e}_{0,-}.$$

Comparing $\boldsymbol{\omega}(x,t) + {}^t(\bar{u},\bar{v})$ with $\bar{\boldsymbol{\omega}}^i(t) + {}^t(\bar{u},\bar{v})$ $(i = 1,2)$, we can show that $\boldsymbol{\omega}(x,t) + {}^t(\bar{u},\bar{v})$ converges to ${}^t(a,0)$. □

LEMMA 4.4. *Set*

$$X(t) = Q_1\{\boldsymbol{\omega}_1(t) - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot,t))\}, \quad Y(t) = \sup_{s\geq 0} e^{\tilde{\lambda}_2 s} \left\| e^{-As}\hat{\boldsymbol{\omega}}(\cdot,t)\right\|_X$$

for a solution $\boldsymbol{\omega}_1 + \hat{\boldsymbol{\omega}}$ to (4.1). *If*

$$\tilde{\lambda}_2 - \tilde{\lambda}_1 - K_6\left(2 + 2R + \frac{1}{2R}\right) r > 0,$$

then, the region

$$\Gamma_R = \{(X,Y) \in \mathbb{R}^2 |\ 0 \leq 2RY < X\}$$

is positively invariant where

$$K_6 = 2K_1^2 \max\left\{\frac{1}{|\boldsymbol{e}_{0,-}|\zeta_0}, 1, |\boldsymbol{e}_{0,-}|\zeta_0\right\}.$$

Moreover, if (4.13) *holds and* $Q_1\{\omega_1(t_1)-\Phi(\hat{\omega}(\cdot,t_1))\} \le 2R\,\|\hat{\omega}(\cdot,t_1)\|_X$ *for some* $t_1>0$, *then*

$$\frac{Q_1\{\omega_1(t)-\Phi(\hat{\omega}(\cdot,t))\}}{\|\hat{\omega}(\cdot,t)\|_X^{2+\delta}} \ge K_1^{-(2+\delta)}\frac{Q_1\{\omega_1(0)-\Phi(\hat{\omega}(\cdot,0))\}}{\|\hat{\omega}(\cdot,0)\|_X^{2+\delta}}$$

for $0\le t\le t_1$ *and sufficiently small* $r>0$.

Proof. We can assume that $X(t)$ is positive. The following inequalities are easily obtained:

$$(4.14)\qquad \begin{cases} X(t+\tau)e^{\tilde{\lambda}_1(t+\tau)} \le X(t)e^{\tilde{\lambda}_1 t} + K_6 r\displaystyle\int_\tau^0 X(t+s)e^{\tilde{\lambda}_1(t+s)}ds, \\ Y(t)e^{\tilde{\lambda}_2 t} \le Y(0) + K_6 r\displaystyle\int_0^t \{X(s)+Y(s)\}e^{\tilde{\lambda}_2 s}ds. \end{cases}$$

The first part of this lemma follows from the above inequalities and Gronwall's inequality (see Lemma 4.1). We prove the latter part. By assumption, $X(t)\le 2RY(t)$ holds for $0\le t\le t_1$. Hence, we have

$$|\omega_1(t)-\Phi(\hat{\omega}(\cdot,t))| \ge |\omega_1(0)-\Phi(\hat{\omega}(\cdot,0))|e^{-(\tilde{\lambda}_1+K_6 r)t},\ \|\hat{\omega}(\cdot,t)\|_X \le K_1\,\|\hat{\omega}(\cdot,0)\|_X\, e^{-\nu_3 t}$$

for $0\le t\le t_1$, where

$$\nu_3 = \{\tilde{\lambda}_2 - K_6(1+2R)r\}.$$

Since

$$(2+\delta)\nu_3 - \tilde{\lambda}_1 - K_6 r > 0$$

for sufficiently small $r>0$, we obtain

$$\frac{Q_1\{\omega_1(t)-\Phi(\hat{\omega}(\cdot,t))\}}{\|\hat{\omega}(\cdot,t)\|_X^{2+\delta}} \ge \frac{Q_1\{\omega_1(0)-\Phi(\hat{\omega}(\cdot,0))\}}{K_1^{2+\delta}\,\|\hat{\omega}(\cdot,0)\|_X^{2+\delta}},$$

if $Q_1\{\omega_1(0)-\Phi(\hat{\omega}(\cdot,0))\}\ge 0$. □

Proof of Theorem 2.3.

Let $\omega_1(t)+\hat{\omega}(x,t)$ be a solution to (4.1) satisfying the assumptions of Theorem 2.3 (i). If there exists a positive time $t_2>0$ such that

$$Q_1\{\omega_1(t_2)-\Phi(\hat{\omega}(\cdot,t_2))\} \ge 2R\,\|\hat{\omega}(\cdot,t_2)\|_X\,,\quad \omega_1(t)+\hat{\omega}(\cdot,t)\in D_r \quad \text{for } 0\le t\le t_1,$$

then the solution ${}^t(\bar{u},\bar{v}) + \boldsymbol{\omega}_1(t) + \hat{\boldsymbol{\omega}}(x,t)$ converges to ${}^t(a,0)$ by Lemma 4.3. So, we suppose that

$$Q_1\{\boldsymbol{\omega}_1(t) - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot,t))\} \le 2R\,\|\hat{\boldsymbol{\omega}}(\cdot,t)\|_X$$

as long as $\boldsymbol{\omega}_1(t) + \hat{\boldsymbol{\omega}}(\cdot,t) \in D_r$. By Lemma 4.4, we have

$$\frac{Q_1\{\boldsymbol{\omega}_1(t) - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot,t))\}}{\|\hat{\boldsymbol{\omega}}(\cdot,t)\|_X^{2+\delta}} \ge \frac{Q_1\{\boldsymbol{\omega}_1(0) - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}}(\cdot,0))\}}{K_1^{2+\delta}\,\|\hat{\boldsymbol{\omega}}(\cdot,0)\|_X^{2+\delta}} > \frac{\epsilon}{K_1^{2+\delta}\,\|\hat{\boldsymbol{\omega}}(\cdot,0)\|_X^{\delta}}$$

We pick r_1 satisfying

$$K_7 r_1 < \frac{r}{|\boldsymbol{e}_{0,-}|\zeta_0 R}, \quad \frac{\epsilon|\boldsymbol{e}_{0,-}|\zeta_0}{(K_7 r_1)^\delta}\left(\frac{r}{K_1|\boldsymbol{e}_{0,-}|\zeta_0 R}\right)^{2+\delta} > 2r,$$

where $K_7(>1)$ is a positive constant such that

$$\|\hat{\boldsymbol{\omega}}\|_X \le K_7\,\|\boldsymbol{\omega}\|_X \quad \text{for} \ \ \boldsymbol{\omega} \in X.$$

If $\|\boldsymbol{\omega}(\cdot,0)\|_X \le r_1$ and $\|\hat{\boldsymbol{\omega}}(\cdot,t)\|_X = r/(|\boldsymbol{e}_{0,-}|\zeta_0 R)$ at some positive time t, then

$$|\boldsymbol{\omega}_1 - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})| = |\boldsymbol{e}_{0,-}|\zeta_0\; Q_1\{\boldsymbol{\omega}_1 - \boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})\} \ge \frac{\epsilon|\boldsymbol{e}_{0,-}|\zeta_0}{(K_7 r_1)^\delta}\left(\frac{r}{K_1|\boldsymbol{e}_{0,-}|\zeta_0 R}\right)^{2+\delta} > 2r.$$

This implies that

$$|\boldsymbol{\omega}_1| > r \ \text{ or } \ |\boldsymbol{\Phi}(\hat{\boldsymbol{\omega}})| > r.$$

The latter inequality contradicts (4.9) if $r < 1/K_4$. The former inequality also contradicts $\boldsymbol{\omega} \in D_r$. Since $\|\hat{\boldsymbol{\omega}}(\cdot,0)\|_X \le r/(|\boldsymbol{e}_{0,-}|\zeta_0 R)$, $\|\hat{\boldsymbol{\omega}}(\cdot,t)\|_X \le r/(|\boldsymbol{e}_{0,-}|\zeta_0 R)$ as long as $\boldsymbol{\omega}_1(t) + \hat{\boldsymbol{\omega}}(\cdot,t) \in D_r$. The first equation of (4.14) and (4.9) imply that there exists $t_3 > 0$ satisfying $|\boldsymbol{\omega}_1(t_3)| = r$. Thus

$$R\,\|\hat{\boldsymbol{\omega}}(\cdot,t_3)\|_X \le \frac{r}{|\boldsymbol{e}_{0,-}|\zeta_0} = \frac{|\boldsymbol{\omega}_1(t_3)|}{|\boldsymbol{e}_{0,-}|\zeta_0} = Q_1\boldsymbol{\omega}_1(t_3).$$

Using Lemma 4.3, we complete the proof of Theorem 2.3 (i). Theorem 2.3 (ii) can be shown similarly. □

Hereafter we consider the case with d close to 1. Put

$$d = 1 + \tilde{d}.$$

The function $\psi^{a,d}$ given by (2.10) converges to (3.4) as d tends to 1. Then we have

PROPOSITION 4.5. *The following holds:*

$$\boldsymbol{\Phi}^{a,1+\tilde{d}} = \boldsymbol{\Psi}^{a,1} - \tilde{d}\boldsymbol{\Xi}^a + O(\|\hat{\boldsymbol{\omega}}\|_X^{2+\delta} + \tilde{d}^{1+\delta}\|\hat{\boldsymbol{\omega}}\|_X^2) \tag{4.15}$$

where $\boldsymbol{\Psi}^{a,1} = \boldsymbol{\Psi}^{a,d}|_{d=1}$ *and*

$$\begin{aligned}
\boldsymbol{\Xi}^a = \;& \varphi_1^{a,1}\sum_{k=1}^{\infty}\{(2u_-^*\gamma_1(k) + u_+^*\gamma_2(k))\frac{\sigma_k u_- y_k^2}{2\sigma_k + \mu_-} \\
&+(2u_-^*u_+\gamma_1(k) + (u_+^*u_+ + u_-^*u_-)\gamma_2(k) + 2u_+^*u_-\gamma_3(k))\frac{\sigma_k y_k z_k}{2\sigma_k + \mu_+} \\
&+(u_-^*\gamma_2(k) + 2u_+^*\gamma_3(k))\frac{\sigma_k u_+ z_k^2}{2\sigma_k + 2\mu_+ - \mu_-}\}, \\
\gamma_1(k) = \;& (\frac{u_-^*u_-^2}{\bar{u}} + \frac{v_-^*v_-^2}{\bar{v}})\frac{\zeta_0\mu_-}{2\sigma_k + \mu_-}, \\
\gamma_2(k) = \;& (\frac{u_-^*u_-u_+}{\bar{u}} + \frac{v_-^*v_-v_+}{\bar{v}})\frac{\zeta_0(\mu_- + \mu_+)}{2\sigma_k + \mu_+}, \\
\gamma_3(k) = \;& (\frac{u_-^*u_+^2}{\bar{u}} + \frac{v_-^*v_+^2}{\bar{v}})\frac{\zeta_0\mu_+}{2\sigma_k + 2\mu_+ - \mu_-}.
\end{aligned}$$

Proof. The system (2.3) is rewritten as

$$\begin{cases}
y_{1t} = -\mu_- y_1 + \left\langle \boldsymbol{e}_-^*\boldsymbol{F}, \zeta_0\right\rangle, \\
\hat{y}_t = \hat{y}_{xx} - \mu_-\hat{y} + \boldsymbol{e}_-^*\boldsymbol{F} - \left\langle \boldsymbol{e}_-^*\boldsymbol{F}, \zeta_0\right\rangle + \tilde{d}u_-^*u_{xx}, \\
z_t = z_{xx} - \mu_+ z + \boldsymbol{e}_+^*\boldsymbol{F} + \tilde{d}u_+^*u_{xx}.
\end{cases}$$

Thus $\boldsymbol{\Psi}^{a,1+\tilde{d}}$ satisfies

$$\begin{aligned}
&\frac{\partial\boldsymbol{\Psi}^{a,1+\tilde{d}}}{\partial\hat{y}}(\hat{y}_{xx} - \mu_-\hat{y} + \tilde{d}u_-^*u_{xx}) + \frac{\partial\boldsymbol{\Psi}^{a,1+\tilde{d}}}{\partial z}(z_{xx} - \mu_+ z + \tilde{d}u_+^*u_{xx}) \\
&\quad = -\mu_-\boldsymbol{\Psi}^{a,1+\tilde{d}} + \left\langle \boldsymbol{e}_-^*\boldsymbol{F}(\hat{\boldsymbol{\omega}}), \zeta_0\right\rangle.
\end{aligned}$$

Substituting (4.15) into the above equation and taking the principal part yield

$$\frac{\partial\boldsymbol{\Xi}^a}{\partial\hat{y}}(\hat{y}_{xx} - \mu_-\hat{y}) + \frac{\partial\boldsymbol{\Xi}^a}{\partial z}(z_{xx} - \mu_+ z) - \frac{\partial\boldsymbol{\Psi}^{a,1}}{\partial\hat{y}}u_-^*u_{xx} - \frac{\partial\boldsymbol{\Psi}^{a,1}}{\partial z}u_+^*u_{xx} = -\mu_-\boldsymbol{\Xi}^a.$$

We use the same argument as in the proof of Theorem 2.2 in order to solve the above equation. That is, one can find Ξ^a as an invariant manifold of the system

$$
\left\{
\begin{array}{l}
y_{1t} = -\mu_- y_1 + \dfrac{\partial \Psi^{a,1}}{\partial \hat{y}} u_-^* u_{xx} + \dfrac{\partial \Psi^{a,1}}{\partial z} u_+^* u_{xx}, \\
\hat{y}_t = \hat{y}_{xx} - \mu_- \hat{y}, \\
z_t = z_{xx} - \mu_+ z.
\end{array}
\right.
\tag{4.16}
$$

Recall that

$$\Psi^{a,1} = \sum_{k=1}^{\infty} (\gamma_1(k) y_k^2 + \gamma_2(k) y_k z_k) + \sum_{k=0}^{\infty} \gamma_3(k) z_k^2$$

by (3.4). Substitute

$$y_k(t) = e^{-(\sigma_k + \mu_-)t} y_k(0), \quad z_k(t) = e^{-(\sigma_k + \mu_+)t} z_k(0)$$

into the first equation of (4.16) and integrate over $[0, \infty)$. Then we obtain Ξ^a. Proposition 4.2 implies

$$|\Phi^{a,1+\tilde{d}} - (\Psi^{a,1} - \tilde{d}\Xi^a)| \le C(\|\hat{\omega}\|_X^{2+\delta} + \tilde{d}^{1+\delta} \|\hat{\omega}\|_X^2).$$

Since $\Psi^{a,d}$ is bilinear in $\hat{\omega}$, we can show

$$\Xi^a = -\left.\frac{\partial \Psi^{a,d}}{\partial d}\right|_{d=1}. \quad \Box$$

Proof of Proposition 3.5. Since $a = d = 1$, we have

$$
\begin{aligned}
\gamma_1(k) &= \frac{(b-c)\zeta_0}{b(2\sigma_k + \mu_-)}, \\
\gamma_2(k) &= \frac{(b+c-2)\zeta_0}{(b-1)(2\sigma_k + \mu_+)}, \\
\gamma_3(k) &= 0,
\end{aligned}
$$

by (3.8). Substituting $\gamma_i(k)$ $(i = 1, 2, 3)$ into Ξ^a, we obtain (3.10). □

Acknowledgements. The author would like to thank Professor Takaaki Nishida (Kyoto University) and Professor Masayasu Mimura (University of Tokyo) for their useful comments.

DEPARTMENT OF APPLIED PHYSICS

TOKYO INSTITUTE OF TECHNOLOGY

References

[1] S. AHMAD AND A. C. LAZER, *Asymptotic behavior of solutions of periodic competition diffusion system*, Nonlinear Anal., 13 (1989), pp. 263–284 .

[2] P. W. BATES AND C. K. R. T. JONES, *Invariant manifolds for semilinear partial differential equations*, U. Kirchgraber and H. O. Walther, ed., Dynamics Reported Vol. 2, (1989), pp. 1–38.

[3] J. CARR, *Centre manifold theory*, Springer-Verlag, 1981.

[4] S. EI AND E. YANAGIDA, *Dynamics of interfaces in competition-diffusion systems*, SIAM J. Appl. Math., (to appear).

[5] D. HENRY, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math. No. 840 Springer-Verlag, 1981.

[6] M. W. HIRSH, *Differential equations and convergence almost everywhere of strongly monotone semiflows*, Contemp. Math., Amer. Math. Soc., Providence, R. I., 17 (1983), pp. 267–285.

[7] M. IIDA, T. MURAMATSU, H. NINOMIYA, AND E. YANAGIDA, *Diffusion induced extinction of a superior species in a competition model*, (submitted to J. Math. Biol.).

[8] C. S. KAHANE, *On the competition-diffusion equations for closely competing species*, Funkcialaj Ekvacioj, 35 (1992), pp. 51–64.

[9] K. KISHIMOTO AND H. F. WEINBERGER, *The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains*, J. Differential Equations, 58 (1985), pp. 15–21.

[10] H. MATANO AND M. MIMURA, *Pattern formation in competition-diffusion systems in nonconvex domains*, Publ. Res. Inst. Math. Sci., 19 (1983), pp. 1049–1079.

[11] P. de MOTTONI, *Qualitative analysis for some quasilinear parabolic systems*, Institute of Math., Polish Academy Sci., zam. 11/79, 190 (1979).

[12] M. H. PROTTER AND H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall Inc., 1967.

[13] A. VANDERBAUWHEDE AND S. A. VAN GILS, *Center manifolds and contractions on a scale of Banach spaces*, J. Funct. Anal., 72 (1987), pp. 209–224.

Captions

Fig. 1 Example of initial data.

Fig. 2 The solution ${}^t(u(x,t), v(x,t))$ with the initial data as in Fig. 1 in the case $a = 1,\ b = c = 2,\ d = 1$ and $L = 1$.

Fig. 3 Separatrix and vectors $\boldsymbol{e}_{\pm}$.

Fig. 4 Example of initial data.

Fig. 5 The solution ${}^t(u(x,t), v(x,t))$ with the initial data as in Fig. 4 in the case $a = 1,\ b = c = 2,\ d = 1$ and $L = 1$.

Fig. 6 Example of solution with initial data

$$ {}^t(u(x,0), v(x,0)) = {}^t(\bar{u} + 0.0035 + 0.1 \cos \frac{\pi x}{L}, \bar{v} + 0.1 \cos \frac{\pi x}{L}) $$

in the case $a = 1,\ b = c = 2,\ d = 0.01$, and $L = 1$.

Fig. 7 Example of solution with initial data

$$ {}^t(u(x,0), v(x,0)) = {}^t(\bar{u} + 0.0035 + 0.1 \cos \frac{2\pi x}{L}, \bar{v} + 0.1 \cos \frac{2\pi x}{L}) $$

in the case $a = 1,\ b = c = 2,\ d = 0.01$, and $L = 1$. The image of initial distribution in $\mathbb{R}^2$ coincides with that of Fig. 6. However, each solution converges to the different equilibrium.

Fig. 8 Plot of the solution $a(d)$ of (3.15) in the case $b = c = 2$ and $L = 1$.

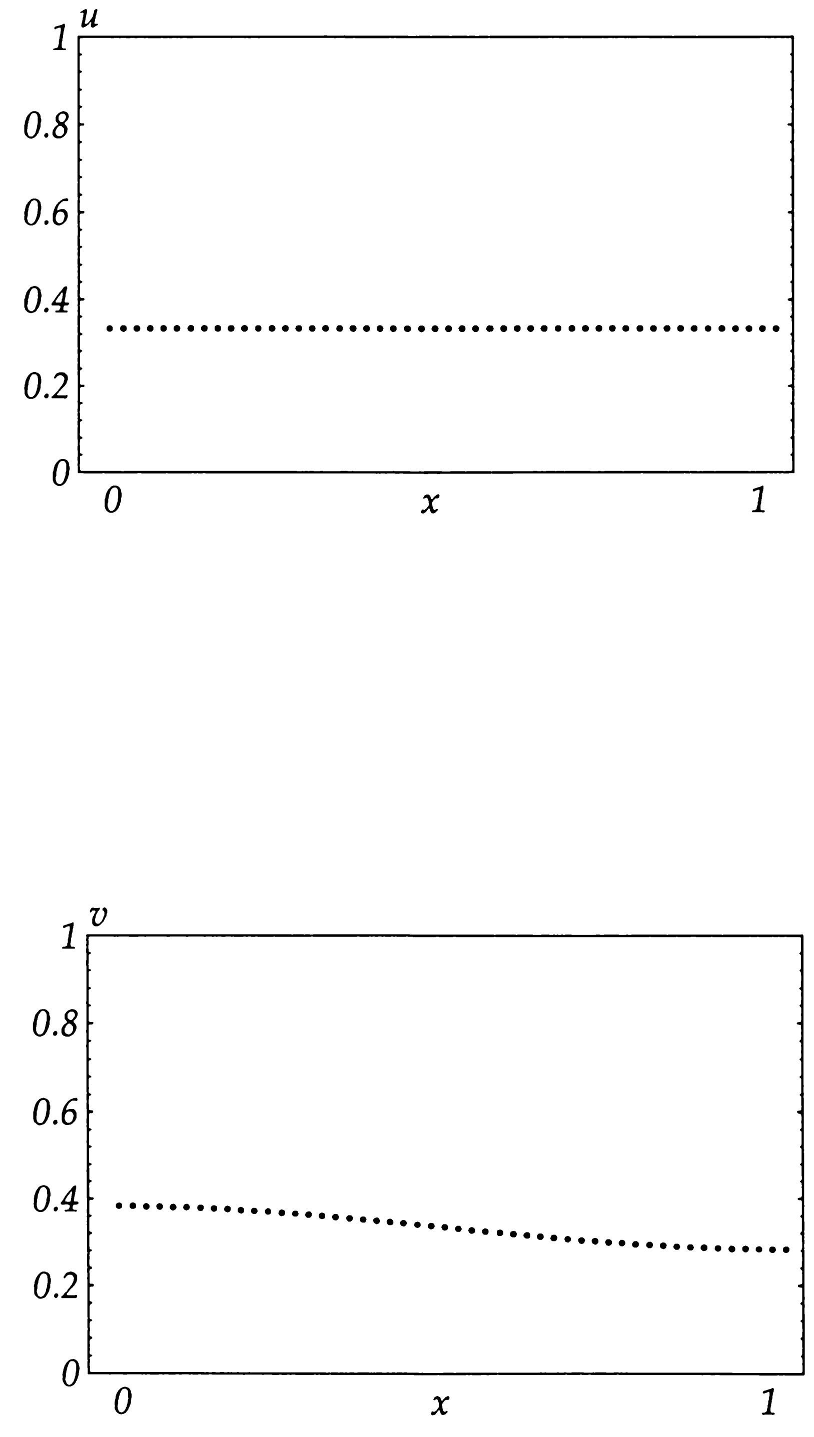


Fig. 1

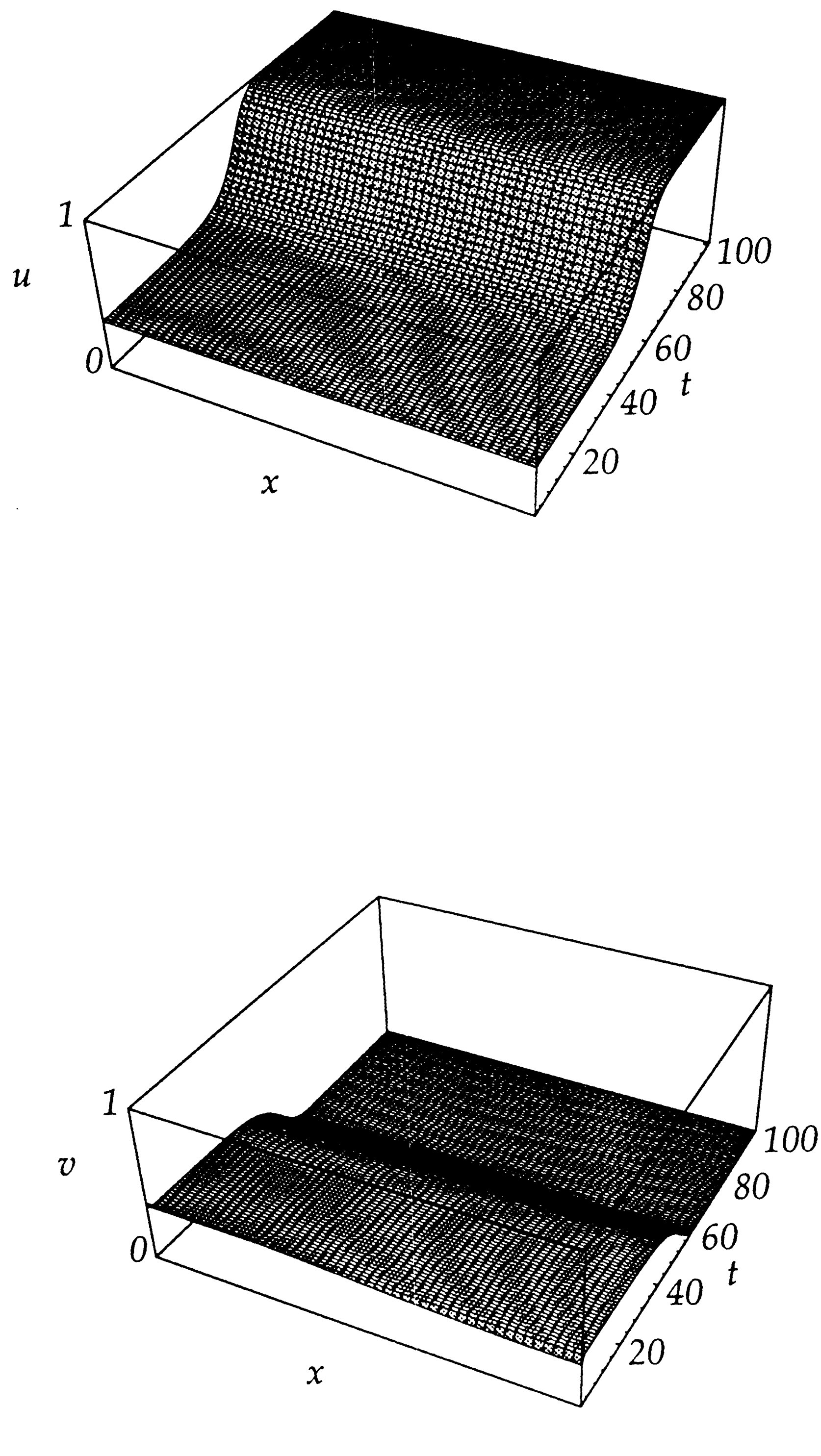


Fig. 2

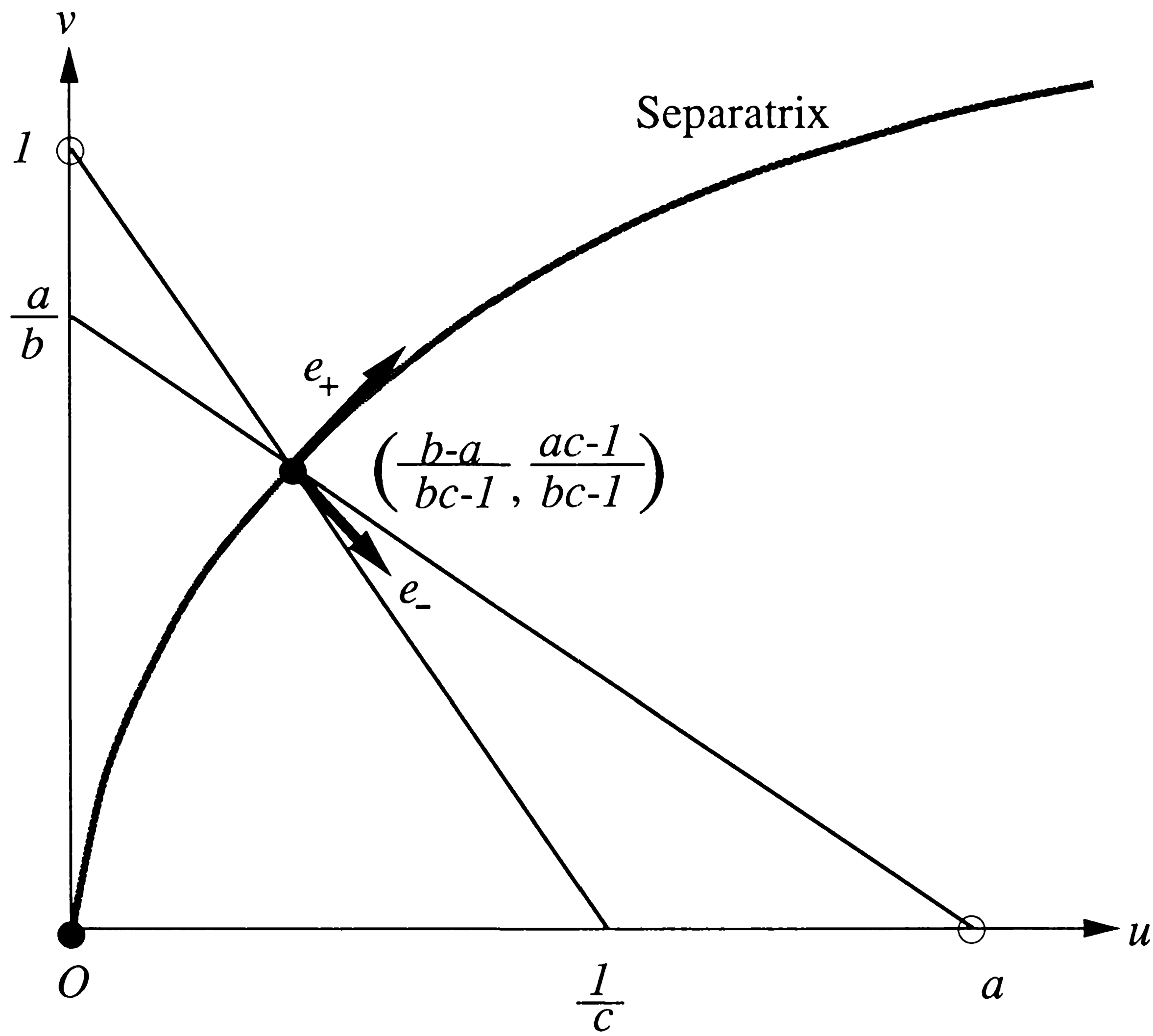
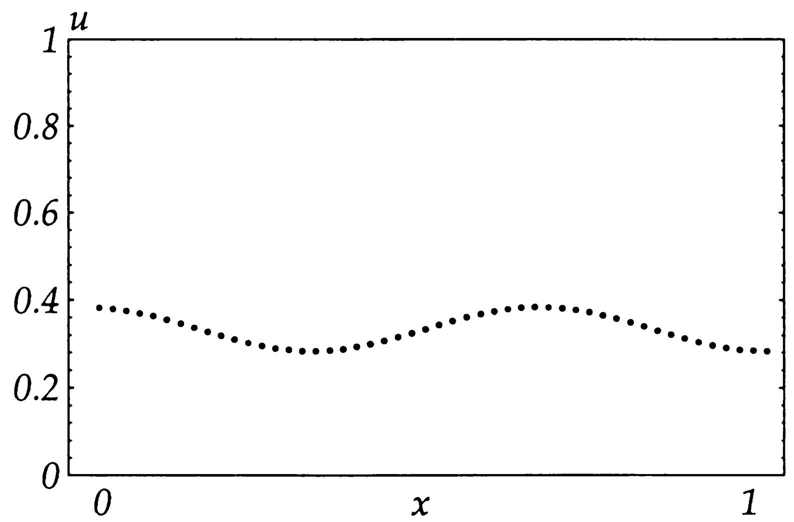

v
u
Separatrix
1
a
b
e_+
e_-
$\left(\frac{b-a}{bc-1}, \frac{ac-1}{bc-1}\right)$
O
$\frac{1}{c}$
a

Fig. 3



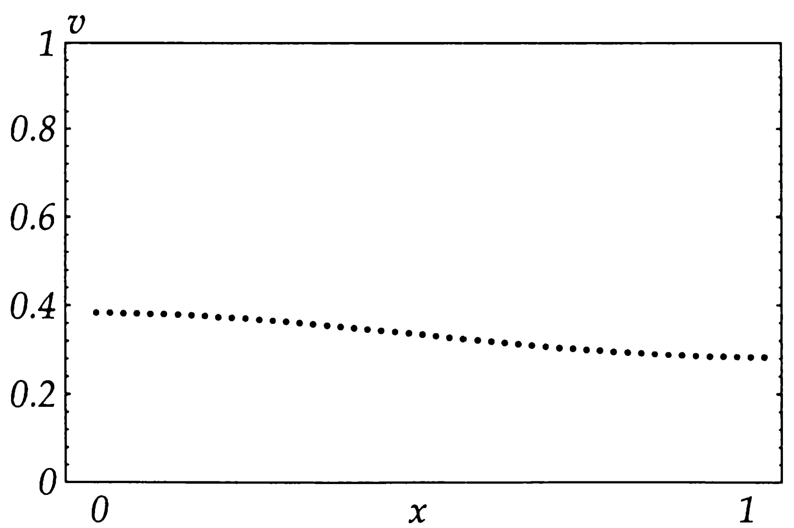
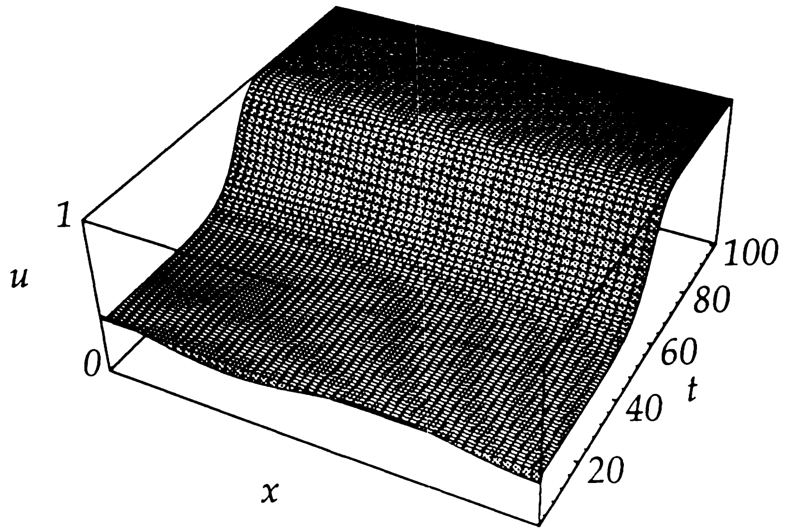


Fig. 4



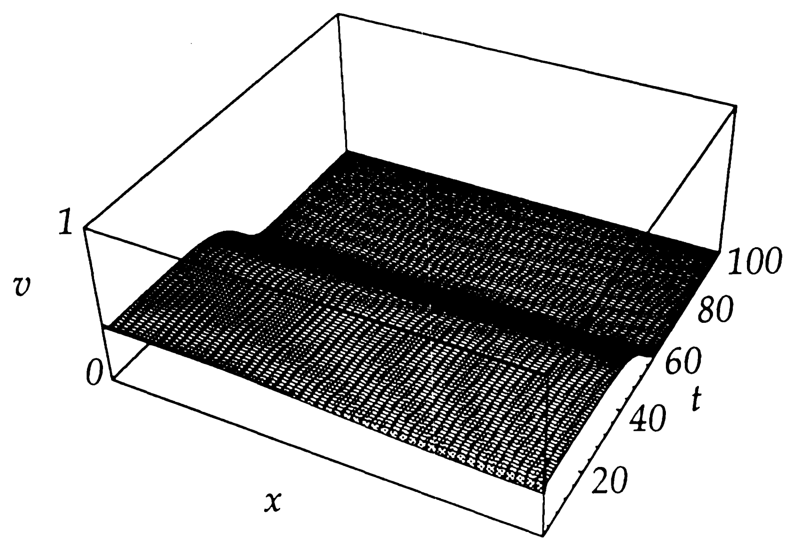


Fig. 5

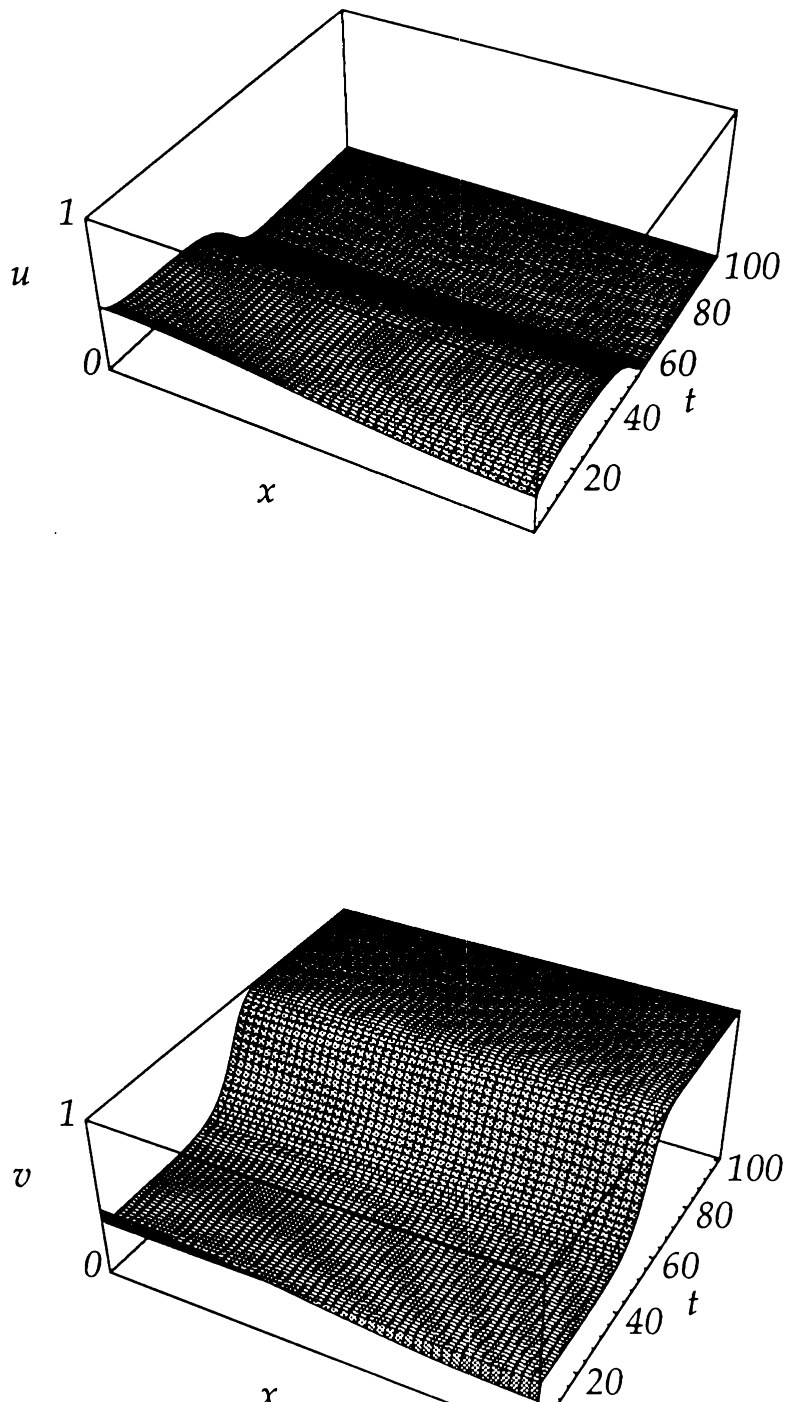
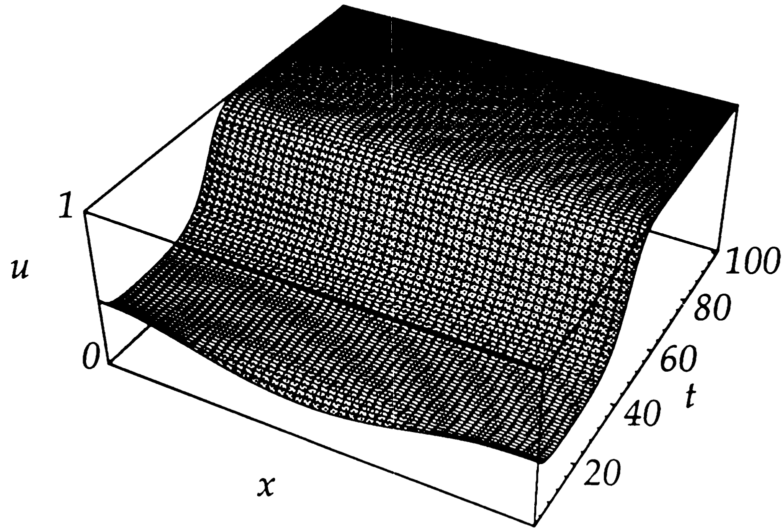


Fig. 6



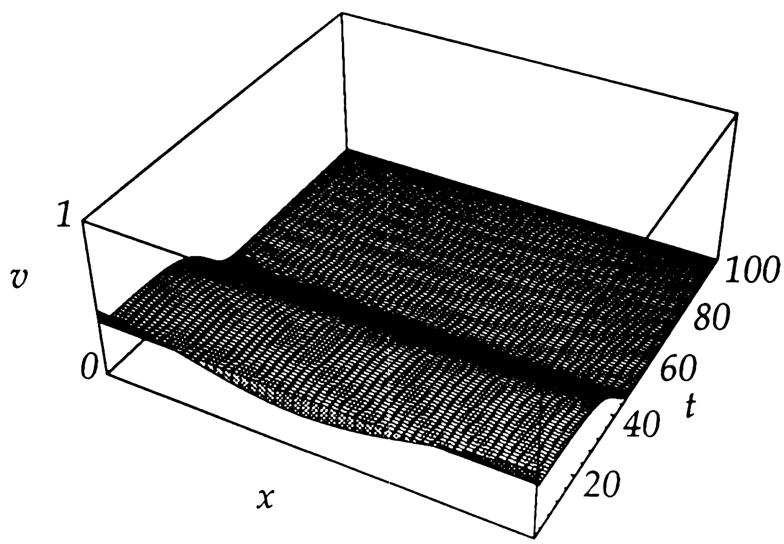


Fig. 7

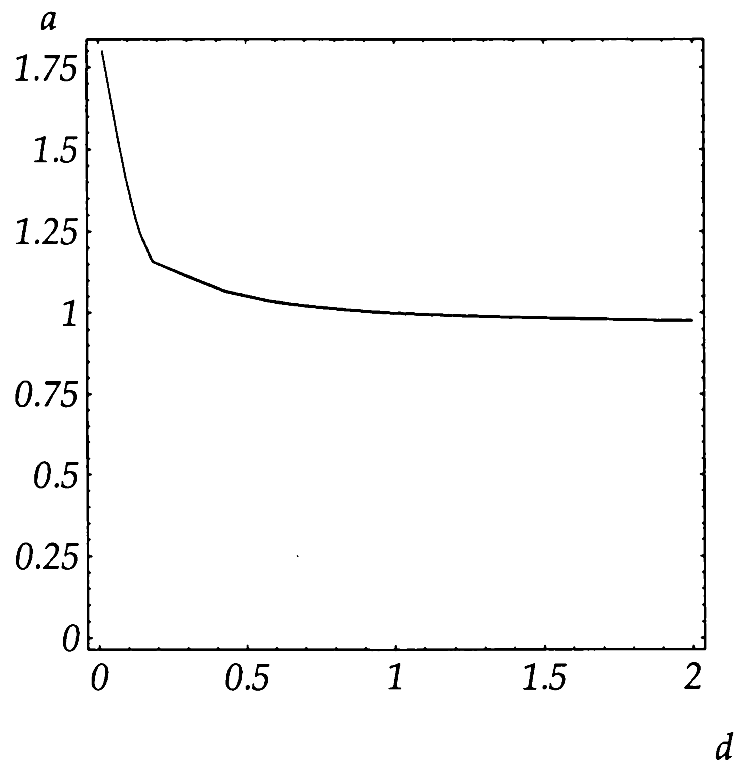


Fig. 8