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A survey on the theory of multiple Bernoulli polynomials and multiple L-functions of root systems

By

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§ 1. Introduction

In [36], Witten found that a certain series of Dirichlet type appear in two dimensional quantum gauge theories with connected compact semisimple Lie groups. Motivated by this observation, Zagier [37] defined the Witten zeta-functions as

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi} \frac{1}{(\dim \varphi)^s} \tag{1.1}$$

for $s \in \mathbb{C}$, where the summation runs over all finite dimensional irreducible representations φ of a given semisimple Lie algebra $\mathfrak{g}$. It is known that a semisimple Lie algebra is a direct sum of simple Lie algebras and simple Lie algebras of rank r are associated to an irreducible root system of type X_r where $X = A, B, \ldots, G$ (see Section 4 for the details). In the case where $\mathfrak{g}$ is of type A_1, the series reduces to the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.2}$$

It is well known that the notion of "zeta-functions" plays an important tool in various areas of modern mathematics.

When s is an even positive integer, then their values are crucial (if s is an odd integer, those with appropriate characters play the same role); mathematically, they

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give the volumes of certain moduli spaces of flat connections, and physically, the 0-th orders of the partition functions of two dimensional quantum gauge theories. Assume that s is an even positive integer $2k$. Witten and Zagier showed that their values are in $\mathbb{Q}\pi^{|\Delta_+|2k}$, where Δ_+ denotes the set of all positive roots. Euler already evaluated them in the A_1 case. The A_2 case was first studied by Tornheim [33] and Mordell [29] independently, and further considered by several authors [7, 31, 34]. In [32], Szenes gave a certain algorithm for the computation in general cases, from the viewpoint of hyperplane arrangements. Gunnells and Sczech also gave another general algorithm and the explicit forms in the A_3 case as an application [6].

This article is a survey on a new approach to this problem proposed in [11,12,15–18, 22, 28] and is an extended and updated version of the informal articles [13, 14]. We will introduce generalizations of Bernoulli polynomials and zeta-functions associated with root systems, which include the Riemann zeta-function, the Euler-Zagier zeta-functions and the Witten zeta-functions. Furthermore we will develop a theory similar to that of the classical Riemann zeta-function.

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§ 2. A Review of Classical Theory

Before stating our results, first we recall the classical theory of the Riemann zeta-function and Bernoulli numbers.

The following is a well-known formula for the Riemann zeta-function and Bernoulli numbers: For $k \in \mathbb{Z}_{\geq 1}$,

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!}, \tag{2.1}$$

where the definition of B_k is given by, for $t \in \mathbb{C}$ with $|t| < 2\pi$,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2.2}$$

Using this formula, we obtain for $k \in \mathbb{Z}_{\geq 1}$,

$$\zeta(2k) + (-1)^{2k}\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!}, \tag{2.3}$$

$$\zeta(2k+1) + (-1)^{2k+1}\zeta(2k+1) = -B_{2k+1}\frac{(2\pi i)^{2k+1}}{(2k+1)!} = 0. \tag{2.4}$$

Hence we have the following important relations: For $k \in \mathbb{Z}_{\geq 2}$,

$$\zeta(k) + (-1)^k\zeta(k) = -B_k\frac{(2\pi i)^k}{k!}, \tag{2.5}$$

that is, value-relations can be written in terms of Bernoulli numbers.

These relations are generalized to the case of Lerch zeta-functions and periodic Bernoulli functions. Let $\varphi(s,y)$ be the Lerch zeta-function defined by

$$\varphi(s,y)=\sum_{n=1}^{\infty}\frac{e^{2\pi iny}}{n^s}. \tag{2.6}$$

Then a formula for Lerch zeta-functions implies that for $k\in\mathbb{Z}_{\geq 2}$ and $y\in\mathbb{R}$,

$$\varphi(k,y)+(-1)^k\varphi(k,-y)=-B_k(\{y\})\frac{(2\pi i)^k}{k!}, \tag{2.7}$$

that is, functional relations as functions in y can be written in terms of periodic Bernoulli functions, which are defined by

$$\frac{te^{t\{y\}}}{e^t-1}=\sum_{k=0}^{\infty}B_k(\{y\})\frac{t^k}{k!}, \tag{2.8}$$

and $\{y\}=y-[y]$ is the fractional part of y.

Once we obtain the notion of periodic Bernoulli functions, we can calculate special values of Dirichlet L-functions $L(s,\chi)$ in terms of them. For a primitive character χ of conductor f and $k\in\mathbb{Z}_{\geq 2}$ satisfying $(-1)^k\chi(-1)=1$, we have

$$L(k,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^k}=\frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k}g(\chi)B_{k,\overline{\chi}}, \tag{2.9}$$

where $g(\chi)$ is the Gauss sum, $\overline{\chi}$ is the complex conjugate of χ, and

$$B_{k,\chi}=f^{k-1}\sum_{a=1}^{f}\chi(a)B_k(a/f). \tag{2.10}$$

Our aim is to find a good class of multiple zeta-functions which generalizes the theory above. First we will introduce zeta- and L-functions associated with semisimple Lie algebras, which are corresponding to simply-connected Lie groups. Moreover besides those, we will study zeta-functions associated with Lie groups that may not be simply-connected.

§ 3. An Overview of Our Results

Based on the observation given in the previous section, we will construct multiple generalizations of Bernoulli polynomials and multiple zeta- and L-functions associated with arbitrary root systems. Before introducing the general theory, we give two simple theorems without using the terminology of root systems. For $s_1, s_2, s_3 \in \mathbb{C}$ with $\Re s_1, \Re s_2, \Re s_3 \geq 2$ and $y_1, y_2 \in \mathbb{R}$, we consider the convergent series

$$\zeta_2(s_1, s_2, s_3, y_1, y_2; A_2) = \sum_{m,n=1}^{\infty} \frac{e^{2\pi i(my_1+ny_2)}}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{3.1}$$

Theorem A. *For $k_1, k_2, k_3 \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned}(3.2)\quad &\zeta_2(k_1, k_2, k_3, y_1, y_2; A_2) + (-1)^{k_1}\zeta_2(k_1, k_3, k_2, -y_1+y_2, y_2; A_2)\\ &+ (-1)^{k_2}\zeta_2(k_3, k_2, k_1, y_1, y_1-y_2; A_2) + (-1)^{k_2+k_3}\zeta_2(k_3, k_1, k_2, -y_1+y_2, -y_1; A_2)\\ &+ (-1)^{k_1+k_3}\zeta_2(k_2, k_3, k_1, -y_2, y_1-y_2; A_2) + (-1)^{k_1+k_2+k_3}\zeta_2(k_2, k_1, k_3, -y_2, -y_1; A_2)\\ &= (-1)^3 \mathcal{P}(k_1, k_2, k_3, y_1, y_2; A_2)\frac{(2\pi i)^{k_1+k_2+k_3}}{k_1! k_2! k_3!},\end{aligned}$$

where $\mathcal{P}(k_1, k_2, k_3, y_1, y_2; A_2)$ is a multiple periodic Bernoulli function (defined later). In particular, we have

$$\zeta_2(2, 2, 2, 0, 0; A_2) = \frac{1}{6}(-1)^3 \frac{1}{3780} \frac{(2\pi i)^{2+2+2}}{2!2!2!} = \frac{\pi^6}{2835}. \tag{3.3}$$

This should be compared with (2.7) and

$$\zeta(2) = \frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}. \tag{3.4}$$

For $s_1, s_2, s_3 \in \mathbb{C}$ with $\Re s_1, \Re s_2, \Re s_3 \geq 2$ and primitive Dirichlet characters χ_1, χ_2, χ_3, consider the convergent series

$$L_2(s_1, s_2, s_3, \chi_1, \chi_2, \chi_3; A_2) = \sum_{m,n=1}^{\infty} \frac{\chi_1(m)\chi_2(n)\chi_3(m+n)}{m^{s_1} n^{s_2} (m+n)^{s_3}}. \tag{3.5}$$

Theorem B. *For $k \in \mathbb{Z}_{\geq 2}$ and a primitive Dirichlet character χ of conductor f such that $(-1)^k \chi(-1) = 1$,*

$$L_2(k, k, k, \chi, \chi, \chi; A_2) = \frac{(-1)^{3k+3}}{6}\left(\frac{(2\pi i)^k}{k! f^k} g(\chi)\right)^3 B_{k,k,k,\overline{\chi},\overline{\chi},\overline{\chi}}(A_2), \tag{3.6}$$

where $B_{k_1,k_2,k_3,\chi_1,\chi_2,\chi_3}(A_2)$ is a multiple generalized Bernoulli number (defined later). In particular, for the quadratic character of conductor 5, namely $\rho_5(1) = \rho_5(4) = 1$ and $\rho_5(2) = \rho_5(3) = -1$, we have

$$L_2(2, 2, 2, \rho_5, \rho_5, \rho_5; A_2) = \frac{(-1)^{6+3}}{6}\left(\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\right)^3 \left(-\frac{28}{125}\right) = -\frac{112\sqrt{5}}{1171875}\pi^6. \tag{3.7}$$

This should be compared with

$$
\begin{aligned}
L(k,\chi) &= \frac{(-1)^{k+1}}{2}\frac{(2\pi i)^k}{k!f^k}g(\chi)B_{k,\overline{\chi}},\\
L(2,\rho_5) &= \frac{(-1)^{2+1}}{2}\frac{(2\pi i)^2}{2!5^2}\sqrt{5}\,\frac{4}{5} = \frac{4\sqrt{5}}{125}\pi^2.
\end{aligned}
\tag{3.8}
$$

Theorems A and B are special cases of our main theorems. In the following sections, we will formulate those theorems.

Remark. Tornheim [33] already showed that for $a,b,c\in\mathbb{N}$, $\zeta_2(a,b,c,0,0;A_2)$ can be expressed as a polynomial in Riemann zeta values with $\mathbb{Q}$-coefficients if $a+b+c$ is odd. On the other hand, it seems difficult to treat the case when $a+b+c$ is even. Actually, in [7], it is stated that only the following four cases can be evaluated in terms of Riemann zeta values: $(a,b,c)=(1,1,N-2)$, $(j,N-j-1,1)$, $(N/3,N/3,N/3)$ and $(N/3,N/3-1,N/3+1)$, where $N\in\mathbb{Z}_{\geq 3}$ is even and $j\in\mathbb{Z}_{\geq 1}$. Hence, for example, it is unknown whether the case $(2p,2q,2r)$ can be evaluated, except for the case $p=q=r$. Especially, as for the double zeta value $\zeta_2(a,0,b,0,0;A_2)$ where $a+b$ is even, it is unknown whether it can be evaluated in terms of Riemann zeta values if $a+b\geq 8$, except for the case $a=b$ or $a=1$ (see [1, Section 4]).

§ 4. Root Systems

Now we start to describe our general theory. First, for reader's convenience, we give the definition and several examples of root systems.

§ 4.1. Definitions

Let V be an r dimensional real vector space equipped with the inner product $\langle\cdot,\cdot\rangle$. A root system $\Delta\subset V$ is a set of vectors (roots) satisfying

1. $|\Delta|<\infty$ and $0\notin\Delta$,
2. $\sigma_\alpha\Delta=\Delta$ for all $\alpha\in\Delta$,
3. $\langle\alpha^\vee,\beta\rangle\in\mathbb{Z}$ for all $\alpha,\beta\in\Delta$,
4. $\alpha, c\alpha\in\Delta$ for $c\in\mathbb{R}\Longrightarrow c=\pm1$,

where σ_α denotes the reflection with respect to the hyperplane H_α orthogonal to α and $\alpha^\vee=2\alpha/\langle\alpha,\alpha\rangle$ (coroot). A root system Δ is called irreducible if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other.

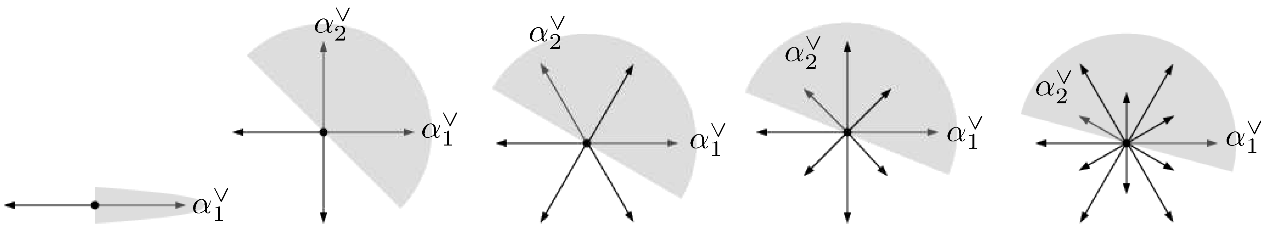
Let W be the Weyl group (the group generated by all σ_α). Let $\{\alpha_1, \dots, \alpha_r\}$ be the set of all fundamental roots (a basis by which any $\alpha \in \Delta$ can be written as $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta$, $c_i \in \mathbb{Z}$ with all $c_i \geq 0$ or all $c_i \leq 0$). Let Δ_+ be the set of all positive roots (all roots $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r \in \Delta$, $c_i \in \mathbb{Z}$ with all $c_i \geq 0$), $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. Let $Q = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ be the root lattice, Let $P = \bigoplus_{i=1}^r \mathbb{Z}\lambda_i$ (the weight lattice) and $P_+ = \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\lambda_i$, where $\{\lambda_1, \dots, \lambda_r\}$ is the dual basis of $\{\alpha_1^\vee, \dots, \alpha_r^\vee\}$.

§ 4.2. Examples

Since we mainly treat coroots in this paper, we give examples of root systems in terms of coroots. Note that if Δ is a root system, then $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ is also a root system.

There is only one root system of rank 1, that is, of type A_1 and there are four root systems of rank 2, that is, of type $A_1 \times A_1$, A_2, C_2 (or B_2) and G_2 (roots in the shaded region are positive):

A_1 $\quad$ $A_1 \times A_1$ $\quad$ A_2 $\quad$ C_2 (or B_2) $\quad$ G_2



$$\Delta_+^\vee = \{\alpha_1^\vee\} \qquad \{\alpha_1^\vee, \alpha_2^\vee\} \qquad \left\{\begin{matrix} \alpha_1^\vee, \alpha_2^\vee \\ \alpha_1^\vee + \alpha_2^\vee \end{matrix}\right\} \qquad \left\{\begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \end{matrix}\right\} \qquad \left\{\begin{matrix} \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee \\ \alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee \\ \alpha_1^\vee + 3\alpha_2^\vee \\ 2\alpha_1^\vee + 3\alpha_2^\vee \end{matrix}\right\}$$

§ 5. Zeta-Functions of Root Systems

§ 5.1. Witten Zeta-Functions

As prototypes of zeta-functions of root systems, we give the definition of Witten zeta-functions.

Definition 5.1 (Witten zeta-functions [36, 37]). For a complex simple Lie algebra $\mathfrak{g}$ with the root system Δ,

$$\zeta_W(s;\Delta) = \sum_{\varphi} (\dim\varphi)^{-s} = K(\Delta)^s \sum_{\lambda\in P_+} \prod_{\alpha\in\Delta_+} \frac{1}{\langle\alpha^\vee, \lambda+\rho\rangle^s}, \tag{5.1}$$

where the summation runs over all finite dimensional irreducible representations φ on the second member of the above and $K(\Delta) \in \mathbb{Z}_{\geq 1}$ is a constant.

Note that in the second equality in Definition 5.1, we have used Weyl's dimension formula.

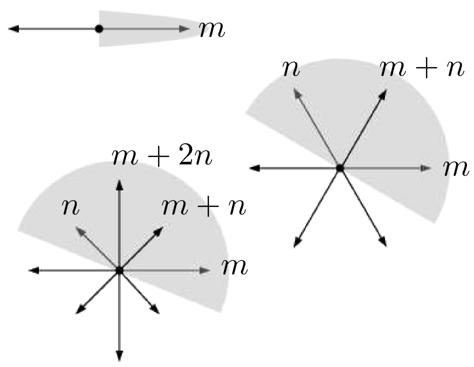
We also use the notation

$$\zeta_W(s;X_r)=\zeta_W(s;\Delta) \tag{5.2}$$

if Δ is of type X_r.

Example 5.2. From the second expression of Definition 5.1, we obtain the explicit forms of Witten zeta-functions as follows in the A_1, A_2, C_2 cases:

$$\begin{aligned}
\zeta_W(s;A_1)&=\sum_{m=1}^{\infty}\frac{1}{m^s}=\zeta(s),\\
\zeta_W(s;A_2)&=2^s\sum_{m,n=1}^{\infty}\frac{1}{m^s n^s (m+n)^s},\\
\zeta_W(s;C_2)&=6^s\sum_{m,n=1}^{\infty}\frac{1}{m^s n^s (m+n)^s (m+2n)^s}.
\end{aligned}$$



Comparing these with Section 4.2, we observe that each factor of the form $am+bn$ $(a,b\in\mathbb{Z}_{\geq 0})$ in the denominators corresponds to the coroot of the form $a\alpha_1^\vee+b\alpha_2^\vee$.

§ 5.2. Zeta-Functions of Root Systems

Definition 5.3 (Zeta-functions of root systems [11,15,17,28])**.** For a root system Δ, define

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta)=\sum_{\lambda\in P_+}e^{2\pi i\langle \mathbf{y},\lambda+\rho\rangle}\prod_{\alpha\in\Delta_+}\frac{1}{\langle\alpha^\vee,\lambda+\rho\rangle^{s_\alpha}}, \tag{5.3}$$

where $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}\in\mathbb{C}^{|\Delta_+|}$ and $\mathbf{y}\in V$.

As in the case of Witten zeta-functions, we may write

$$\zeta_r(\mathbf{s},\mathbf{y};\Delta)=\zeta_r(\mathbf{s},\mathbf{y};X_r) \tag{5.4}$$

if Δ is of type X_r. It is easy to see that (5.3) with $\mathbf{y}=\mathbf{0}$ is essentially a multi-variable version of Witten zeta-functions. Indeed we see that $\zeta_W(s;\Delta)=K(\Delta)^s\zeta_r((s,\ldots,s),\mathbf{0};\Delta)$.

To define an action of the Weyl group, we extend $\mathbf{s}=(s_\alpha)_{\alpha\in\Delta_+}$ to $(s_\alpha)_{\alpha\in\Delta}$ by $s_\alpha=s_{-\alpha}$ and define $(w\mathbf{s})_\alpha=s_{w^{-1}\alpha}$. Then we have our first theorem.

Theorem 5.4 ([17]). *For* $\mathbf{s}=\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in\mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

(5.5)
$$\sum_{w\in W}\Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-}(-1)^{k_\alpha}\Bigr)\zeta_r(w^{-1}\mathbf{k},w^{-1}\mathbf{y};\Delta)=(-1)^{|\Delta_+|}\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr),$$

where $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ *is a multiple periodic Bernoulli function (defined later).*

Example 5.5. If $X_r=A_1$, noting that $W=\{\mathrm{id},\sigma_\alpha\}$, we have (2.7).

§ 6. Special Zeta-Values

Theorem 5.4 immediately implies the following theorem:

Theorem 6.1 ([17]). *For* $\mathbf{k}=(k_\alpha)_{\alpha\in\Delta_+}\in(2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ *satisfying* $w^{-1}\mathbf{k}=\mathbf{k}$ *for all* $w\in W$ *(i.e.* $k_\alpha=k_\beta$ *if* α *and* β *are of the same length),*

$$\zeta_r(\mathbf{k},\mathbf{0};\Delta)=\frac{(-1)^{|\Delta_+|}}{|W|}\mathcal{P}(\mathbf{k},\mathbf{0};\Delta)\Bigl(\prod_{\alpha\in\Delta_+}\frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr)\in\mathbb{Q}\pi^{|\mathbf{k}|} \tag{6.1}$$

where $|\mathbf{k}|=\sum_{\alpha\in\Delta_+}k_\alpha$.

Example 6.2. If $X_r=A_1$, we have

$$\zeta(k)=\frac{-1}{2}B_k\frac{(2\pi i)^k}{k!}\in\mathbb{Q}\pi^k \qquad (k\in 2\mathbb{Z}_{\geq 1}). \tag{6.2}$$

In particular, $\mathbf{k}=(k)_{\alpha\in\Delta_+}$ with $k\in 2\mathbb{Z}_{\geq 1}$ (that is, all $k_\alpha=k$) satisfies the condition in Theorem 2. In this case, $\zeta_r(\mathbf{k},\mathbf{0};\Delta)\in\mathbb{Q}\pi^{|\Delta_+|k}$ was shown by Witten and Zagier. In our method, the rational factor is explicitly evaluated via the generating function. Our statement is indeed a non-trivial generalization of their results since we also have for example,

$$\begin{aligned}\zeta_2((2,4,4,2),\mathbf{0};C_2)&=\sum_{m,n=1}^{\infty}\frac{1}{m^2n^4(m+n)^4(m+2n)^2}\\&=\frac{(-1)^4}{2^2 2!}\frac{53}{1513512000}\Bigl(\frac{(2\pi i)^2}{2!}\Bigr)^2\Bigl(\frac{(2\pi i)^4}{4!}\Bigr)^2\\&=\frac{53\pi^{12}}{6810804000}.\end{aligned} \tag{6.3}$$

§ 7. Multiple Periodic Bernoulli Functions

In this section, we give the definition of generating functions of multiple periodic Bernoulli functions. Let $\mathscr{V}$ be the set of all $\mathbb{R}$-bases $\mathbf{V}\subset\Delta_+$ and let $\mathbf{V}^\vee=\{\beta^\vee\}_{\beta\in\mathbf{V}}$. Let

$\mathbf{V}^* = \{\mu_\beta^{\mathbf{V}}\}_{\beta\in\mathbf{V}}$ be the dual basis of $\mathbf{V}^\vee$. Let $Q^\vee = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $L(\mathbf{V}^\vee) = \bigoplus_{\beta\in\mathbf{V}} \mathbb{Z}\beta^\vee$, which is a sublattice of $Q^\vee$ with finite index ($|Q^\vee/L(\mathbf{V}^\vee)| < \infty$).

Fix a certain $\phi \in V$ and define a multiple generalization of the notion of "fractional part" of $\mathbf{y} \in V$ as

$$\{\mathbf{y}\}_{\mathbf{V},\beta} = \begin{cases} \{\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle > 0), \\ 1 - \{-\langle \mathbf{y}, \mu_\beta^{\mathbf{V}}\rangle\} & (\langle \phi, \mu_\beta^{\mathbf{V}}\rangle < 0). \end{cases} \tag{7.1}$$

Using these definitions, we have

Definition 7.1 (The generating function [12,16,17]). For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+} \in \mathbb{C}^{|\Delta_+|}$,

$$\begin{aligned} F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{V}\in\mathscr{V}} \Big(\prod_{\gamma\in\Delta_+\setminus\mathbf{V}} \frac{t_\gamma}{t_\gamma - \sum_{\beta\in\mathbf{V}} t_\beta \langle \gamma^\vee, \mu_\beta^{\mathbf{V}}\rangle} \Big) \\ \times \frac{1}{|Q^\vee/L(\mathbf{V}^\vee)|} \sum_{q\in Q^\vee/L(\mathbf{V}^\vee)} \Big(\prod_{\beta\in\mathbf{V}} \frac{t_\beta \exp(t_\beta\{\mathbf{y}+q\}_{\mathbf{V},\beta})}{e^{t_\beta}-1} \Big). \end{aligned} \tag{7.2}$$

It can be shown that the generating function $F(\mathbf{t},\mathbf{y};\Delta)$ is holomorphic in the neighborhood of the origin in $\mathbf{t}$.

Definition 7.2 (Multiple periodic Bernoulli functions [12,16,17]). We define multiple periodic Bernoulli functions $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ by the coefficients of the Taylor expansion

$$F(\mathbf{t},\mathbf{y};\Delta) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} \mathcal{P}(\mathbf{k},\mathbf{y};\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \tag{7.3}$$

Example 7.3. If $X_r = A_1$, we have

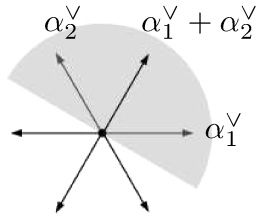
$$F(t,y) = \frac{te^{t\{y\}}}{e^t-1} = \sum_{k=0}^\infty B_k(\{y\}) \frac{t^k}{k!}. \tag{7.4}$$

From this example, we see that $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ can be regarded as natural generalizations of $B_k(\{y\})$.

§ 8. An Example: A_2 Case

We calculate a multiple periodic Bernoulli function and its generating function in the case of the root system of type A_2.

We have the basic data as follows:



$\Delta_+^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee+\alpha_2^\vee\}$, $\mathscr{V} = \{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$,
$\mathbf{t} = (t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}) = (t_1, t_2, t_3)$,
$\mathbf{y} = y_1\alpha_1^\vee + y_2\alpha_2^\vee$,

where $\mathbf{V}_1 = \{\alpha_1, \alpha_2\}$, $\mathbf{V}_2 = \{\alpha_1, \alpha_1+\alpha_2\}$ and $\mathbf{V}_3 = \{\alpha_2, \alpha_1+\alpha_2\}$. By definition, we also have $\mathbf{V}_1^\vee=\{\alpha_1^\vee,\alpha_2^\vee\}$, $\mathbf{V}_2^\vee=\{\alpha_1^\vee,\alpha_1^\vee+\alpha_2^\vee\}$, $\mathbf{V}_3^\vee=\{\alpha_2^\vee,\alpha_1^\vee+\alpha_2^\vee\}$ and $\mathbf{V}_1^*=\{\lambda_1,\lambda_2\}$, $\mathbf{V}_2^*=\{\lambda_1-\lambda_2,\lambda_2\}$, $\mathbf{V}_3^*=\{\lambda_2-\lambda_1,\lambda_1\}$. Fix a sufficiently small $\varepsilon>0$ and $\phi=\alpha_1^\vee+\varepsilon\alpha_2^\vee$. Then by Definition 7.1 and using these data, we have the generating function as

$$F(\mathbf{t},\mathbf{y};A_2)=\frac{t_3}{t_3-t_1-t_2}\frac{t_1e^{t_1\{y_1\}}}{e^{t_1}-1}\frac{t_2e^{t_2\{y_2\}}}{e^{t_2}-1} \tag{8.1a}$$

$$+\frac{t_2}{t_2+t_1-t_3}\frac{t_1e^{t_1\{y_1-y_2\}}}{e^{t_1}-1}\frac{t_3e^{t_3\{y_2\}}}{e^{t_3}-1} \tag{8.1b}$$

$$+\frac{t_1}{t_1+t_2-t_3}\frac{t_2e^{t_2(1-\{y_1-y_2\})}}{e^{t_2}-1}\frac{t_3e^{t_3\{y_1\}}}{e^{t_3}-1}, \tag{8.1c}$$

where (8.1a), (8.1b) and (8.1c) correspond to $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ respectively. Note that in this case, $L(\mathbf{V}_1^\vee)=L(\mathbf{V}_2^\vee)=L(\mathbf{V}_3^\vee)=Q^\vee$ and the second sum of (7.2) is trivial.

For $\mathbf{k}=\mathbf{2}=(2,2,2)$, expanding the right-hand side of (8.1a)–(8.1c), we find that the multiple periodic Bernoulli function is

$$\begin{aligned}\mathcal{P}(\mathbf{2},(y_1,y_2);A_2)=&\frac{1}{3780}+\frac{1}{90}(\{y_1\}-\{y_1-y_2\}-\{y_2\})\\
&+\frac{1}{90}(-\{y_1\}^2-2\{y_1-y_2\}\{y_1\}+\{y_1-y_2\}^2-\{y_2\}^2+2\{y_1-y_2\}\{y_2\})\\
&+\frac{1}{18}(-\{y_1\}^3+3\{y_1-y_2\}\{y_1\}^2+3\{y_2\}^3+3\{y_1-y_2\}\{y_2\}^2)\\
&+\frac{1}{18}(\{y_1\}^4-2\{y_1-y_2\}\{y_1\}^3-3\{y_1-y_2\}^2\{y_1\}^2\\
&\qquad-5\{y_2\}^4-10\{y_1-y_2\}\{y_2\}^3-3\{y_1-y_2\}^2\{y_2\}^2)\\
&+\frac{1}{30}(\{y_1\}^5-5\{y_1-y_2\}\{y_1\}^4+10\{y_1-y_2\}^2\{y_1\}^3\\
&\qquad+5\{y_2\}^5+15\{y_1-y_2\}\{y_2\}^4+10\{y_1-y_2\}^2\{y_2\}^3)\\
&+\frac{1}{30}(-\{y_1\}^6+4\{y_1-y_2\}\{y_1\}^5-5\{y_1-y_2\}^2\{y_1\}^4\\
&\qquad-\{y_2\}^6-4\{y_1-y_2\}\{y_2\}^5-5\{y_1-y_2\}^2\{y_2\}^4).\end{aligned} \tag{8.2}$$

By Theorem 5.4, we have a functional relation in y_1,y_2 corresponding to this multiple periodic Bernoulli function:

$$\begin{aligned}&\zeta_2(\mathbf{2},(y_1,y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,y_2);A_2)+\zeta_2(\mathbf{2},(y_1,y_1-y_2);A_2)\\
&+\zeta_2(\mathbf{2},(-y_2,y_1-y_2);A_2)+\zeta_2(\mathbf{2},(-y_1+y_2,-y_1);A_2)+\zeta_2(\mathbf{2},(-y_2,-y_1);A_2)\\
&\qquad=(-1)^3\mathcal{P}(\mathbf{2},(y_1,y_2);A_2)\frac{(2\pi i)^6}{(2!)^3}.\end{aligned} \tag{8.3}$$

In particular if $(y_1,y_2)=(0,0)$, then

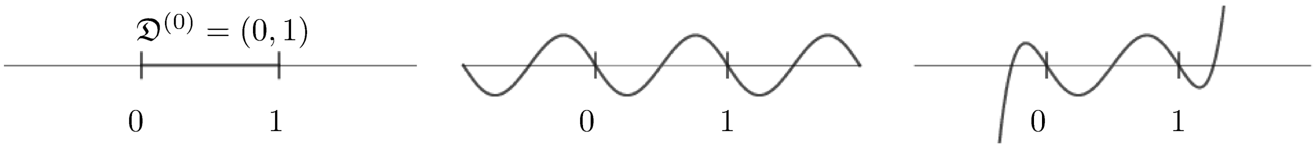
$$\zeta_2(\mathbf{2},(0,0);A_2)=\frac{1}{6}(-1)^3\frac{1}{3780}\frac{(2\pi i)^6}{(2!)^3}=\frac{\pi^6}{2835}. \tag{8.4}$$

Example 8.1. If $X_r = A_1$, we have

$$\zeta(2) = \frac{1}{2}(-1)\frac{1}{6}\frac{(2\pi i)^2}{2!} = \frac{\pi^2}{6}, \qquad B_2(\{y\}) = \frac{1}{6} - \{y\} + \{y\}^2. \tag{8.5}$$

§ 9. Multiple Bernoulli Polynomials

In the classical theory, Bernoulli polynomials can be derived by the analytic continuation of periodic Bernoulli functions. We explain this fact. Let $\mathfrak{H} = \{y \in \mathbb{R} \mid \{y\} \in \mathbb{Z}\} = \mathbb{Z}$ (discontinuous points of $\{y\}$). Let $\mathbb{R} \setminus \mathfrak{H} = \coprod_{\nu\in\mathbb{Z}} \mathfrak{D}^{(\nu)}$, where $\mathfrak{D}^{(\nu)} = (\nu, \nu+1)$. From each $\mathfrak{D}^{(\nu)}$ to $\mathbb{C}$, the function $B(\{y\})$ is analytically continued to a polynomial function $B_k^{(\nu)}(y) = B_k(y-\nu) \in \mathbb{Q}[y]$.



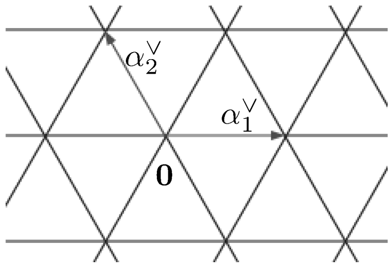
$\mathbb{R} \setminus \mathfrak{H} = \coprod_{\nu\in\mathfrak{J}} \mathfrak{D}^{(\nu)}$ $\qquad$ $B_k(\{y\})$ $\qquad$ $B_k^{(0)}(y) = B_k(y)$

A similar procedure works well in general cases and we can define multiple generalizations of Bernoulli polynomials. Let

$$\mathfrak{H} = \bigcup_{\mathbf{V}\in\mathscr{V}} \bigcup_{q\in Q^\vee} \bigcup_{\beta\in\mathbf{V}} \{\mathbf{y} \in V \mid \{\mathbf{y}+q\}_{\mathbf{V},\beta} \in \mathbb{Z}\} \tag{9.1}$$

(discontinuous points of $\{\mathbf{y}+q\}_{\mathbf{V},\beta}$ appearing in the generating function). Let

$$V \setminus \mathfrak{H} = \coprod_{\nu\in\mathfrak{J}} \mathfrak{D}^{(\nu)}, \tag{9.2}$$



A_2 case

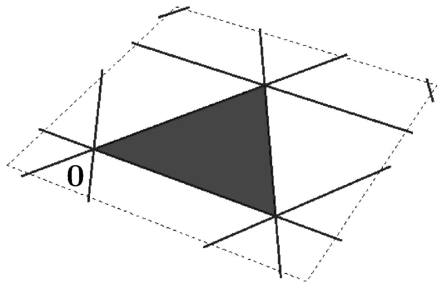
where $\mathfrak{D}^{(\nu)}$ is an open connected component and $\mathfrak{J}$ is a set of indices.

The above figure is the situation in the A_2 case, where lines are $\mathfrak{H}$ and open triangles are $\mathfrak{D}^{(\nu)}$. For example, $\{\mathbf{y}+q\}_{\mathbf{V}_1,\alpha_1} = \{\langle y_1\alpha_1^\vee + y_2\alpha_2^\vee + q, \lambda_1\rangle\} = \{y_1\} \in \mathbb{Z}$ gives the lines parallel to $\alpha_2^\vee$.

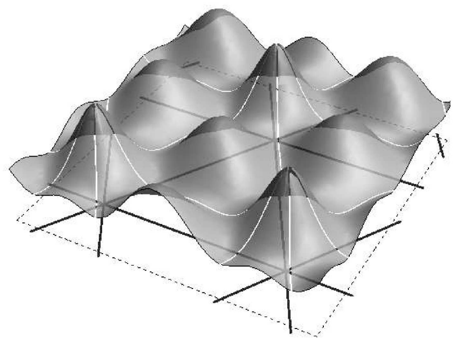
Theorem 9.1 ([12, 16, 17])**.** *From each region $\mathfrak{D}^{(\nu)}$ to the whole space $\mathbb{C}\otimes V$, $\mathcal{P}(\mathbf{k},\mathbf{y};\Delta)$ is analytically continued in $\mathbf{y}$ to a polynomial function $\mathcal{B}_{\mathbf{k}}^{(\nu)}(\mathbf{y};\Delta) \in \mathbb{Q}[\mathbf{y}]$ of total degree at most $|\mathbf{k}|$, where $\mathbf{y} = \sum_{n=1}^r y_n\alpha_n^\vee$.*

§ 9.1. An Example: A_2 Case

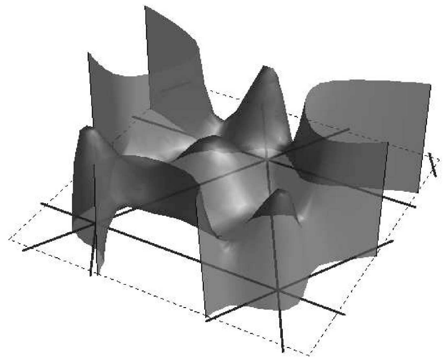
The Bernoulli polynomial $\mathcal{B}_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y};A_2)$ is obtained by the analytic continuation of the periodic Bernoulli function $\mathcal{P}(\mathbf{2},\mathbf{y};A_2)$ from the region $\mathfrak{D}^{(\mathbf{0})}$, which is the shaded triangle region in the figure below.



$V \setminus \mathfrak{H} = \coprod_{\nu \in \mathfrak{J}} \mathfrak{D}^{(\nu)}$



$\mathcal{P}(\mathbf{2}, \mathbf{y}; A_2)$
(Periodic Bernoulli function)



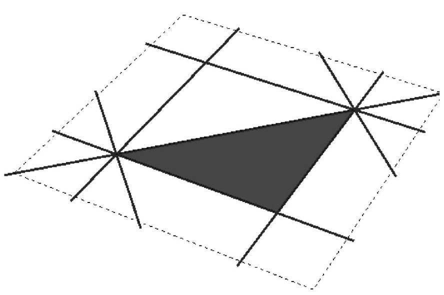
$\mathcal{B}_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y}; A_2)$
(Bernoulli polynomial)

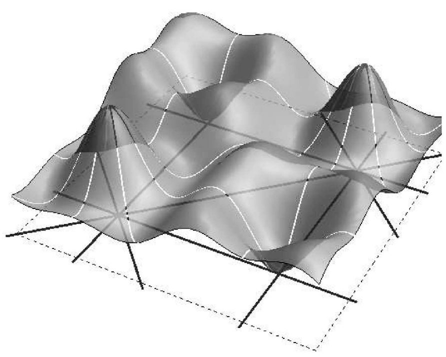
The explicit form of the Bernoulli polynomial $\mathcal{B}_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y}; A_2)$ is given, simply by removing all curly brackets from (8.2), as follows:

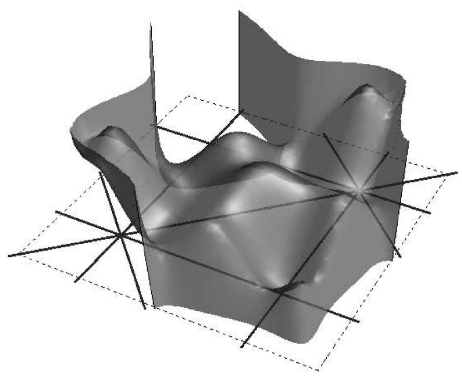
$$
\begin{aligned}
(9.3)\quad \mathcal{B}_{\mathbf{2}}^{(\mathbf{0})}(\mathbf{y}; A_2) &= \frac{1}{3780} + \frac{1}{45}(y_1 y_2 - y_1^2 - y_2^2) + \frac{1}{18}(3y_1 y_2^2 - 3y_1^2 y_2 + 2y_1^3) \\
&\quad + \frac{1}{9}(-2y_1 y_2^3 - 3y_1^2 y_2^2 + 4y_1^3 y_2 - 2y_1^4 + y_2^4) \\
&\quad + \frac{1}{30}(-5y_1 y_2^4 + 10y_1^2 y_2^3 + 10y_1^3 y_2^2 - 15y_1^4 y_2 + 6y_1^5) \\
&\quad + \frac{1}{30}(6y_1 y_2^5 - 5y_1^2 y_2^4 - 5y_1^4 y_2^2 + 6y_1^5 y_2 - 2y_1^6 - 2y_2^6) \in \mathbb{Q}[\mathbf{y}].
\end{aligned}
$$

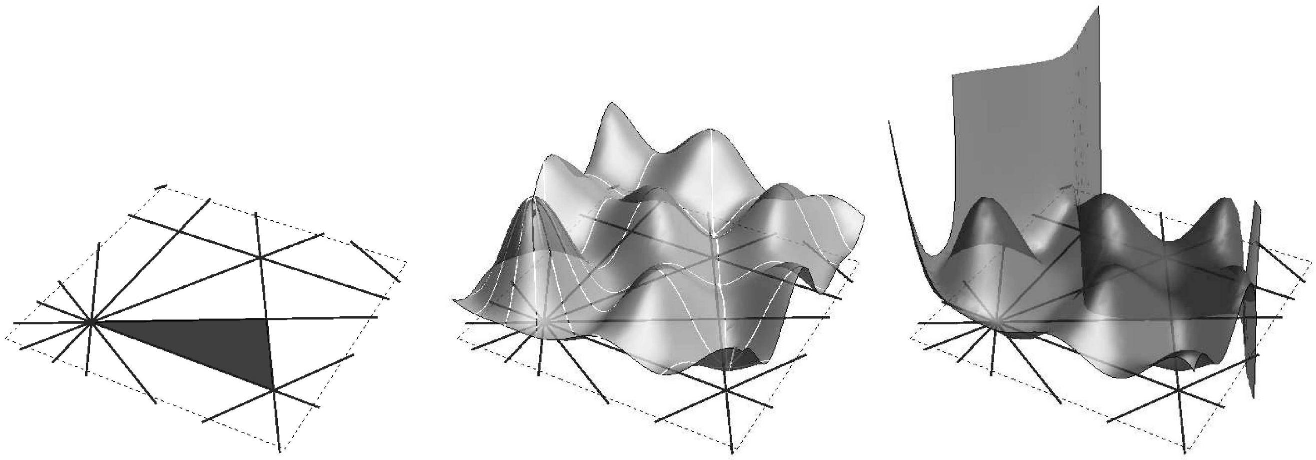
§ 9.2. Further Examples: C_2, G_2 Cases

The following graphs in the upper (resp. lower) row are of type C_2 (resp. G_2).









We summarize what we have obtained: We have constructed periodic Bernoulli functions so that they describe functional-relations in $\mathbf{y}$ of multiple zeta-functions of root systems, which can be calculated by use of the generating function; Bernoulli polynomials are obtained by the analytic continuation of periodic Bernoulli functions.

§ 10. L-Functions of Root Systems

We give another application of periodic Bernoulli functions or equivalently Bernoulli polynomials. For this purpose, we define an L-analogue of zeta-functions of root systems.

Definition 10.1 (L-functions of root systems [12, 16]). For a root system Δ, define

$$L_r(\mathbf{s}, \boldsymbol{\chi}; \Delta) = \sum_{\lambda \in P_+} \prod_{\alpha \in \Delta_+} \frac{\chi_\alpha(\langle \alpha^\vee, \lambda + \rho \rangle)}{\langle \alpha^\vee, \lambda + \rho \rangle^{s_\alpha}}, \tag{10.1}$$

where $\boldsymbol{\chi} = (\chi_\alpha)_{\alpha \in \Delta_+}$ is a set of primitive Dirichlet characters of conductors $f_\alpha \in \mathbb{Z}_{\geq 1}$.

We extend $\boldsymbol{\chi} = (\chi_\alpha)_{\alpha \in \Delta_+}$ to $(\chi_\alpha)_{\alpha \in \Delta}$ by $\chi_\alpha = \chi_{-\alpha}$ and define $(w\boldsymbol{\chi})_\alpha = \chi_{w^{-1}\alpha}$. Then we have value-relations of L-functions.

Theorem 10.2 ([12, 16]). *For* $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{Z}_{\geq 2}^{|\Delta_+|}$, *we have*

$$\begin{aligned} \sum_{w \in W} \Bigl(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1) \Bigr) L_r(w^{-1}\mathbf{k}, w^{-1}\boldsymbol{\chi}; \Delta) \\ = (-1)^{|\Delta_+|} \Bigl(\prod_{\alpha \in \Delta_+} \chi_\alpha(-1) g(\chi_\alpha) \frac{(2\pi i)^{k_\alpha}}{k_\alpha! f^{k_\alpha}} \Bigr) \mathcal{B}_{\mathbf{k}, \overline{\boldsymbol{\chi}}}(\Delta), \end{aligned} \tag{10.2}$$

where $\mathcal{B}_{\mathbf{k}, \boldsymbol{\chi}}(\Delta)$ *is a multiple generalized Bernoulli number (defined later).*

Example 10.3. If $X_r = A_1$, we have the classical result

$$L(k,\chi) + (-1)^k \chi(-1) L(k,\chi) = -\chi(-1) g(\chi) \frac{(2\pi i)^k}{k! f^k} B_{k,\overline{\chi}}, \tag{10.3}$$

where $B_{k,\overline{\chi}}$ is the genealized Bernoulli number given in (2.10). As for the traditional account of this formula, see [3, chapter 1] for example.

§ 11. Special L-Values

Theorem 10.2 immediately implies a formula for special values of L-functions:

Theorem 11.1 ([12, 16]). *For* $\mathbf{k} \in (\mathbb{Z}_{\geq 2})^{|\Delta_+|}$ *and* $\boldsymbol{\chi}$ *such that* $w^{-1}\mathbf{k} = \mathbf{k}$, $w^{-1}\boldsymbol{\chi} = \boldsymbol{\chi}$ *for all* $w \in W$ *and* $(-1)^{k_\alpha} \chi_\alpha(-1) = 1$ *for all* $\alpha \in \Delta_+$, *we have*

$$L_r(\mathbf{k}, \boldsymbol{\chi}; \Delta) = \frac{(-1)^{|\mathbf{k}|+|\Delta_+|}}{|W|} \Bigl(\prod_{\alpha \in \Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha! f_\alpha^{k_\alpha}} g(\chi_\alpha) \Bigr) \mathcal{B}_{\mathbf{k},\overline{\boldsymbol{\chi}}}(\Delta). \tag{11.1}$$

Example 11.2. If $X_r = A_1$, we have

$$L(k,\chi) = \frac{(-1)^{k+1}}{2} \frac{(2\pi i)^k}{k! f^k} g(\chi) B_{k,\overline{\chi}}. \tag{11.2}$$

Example 11.3. Let ρ_7 be the Dirichlet character of conductor 7 defined by $\rho_7(1) = \rho_7(6) = 1$, $\rho_7(2) = \rho_7(5) = e^{2\pi i/3}$, $\rho_7(3) = \rho_7(4) = e^{4\pi i/3}$. Then the associated Gauss sum is $g(\rho_7) = 2\bigl(\cos(2\pi/7) + e^{2\pi i/3}\cos(4\pi/7) + e^{4\pi i/3}\cos(6\pi/7)\bigr)$ and we have

$$\begin{aligned} L_2((2,4,4,2),(1,\rho_7,\rho_7,1);C_2) &= \sum_{m,n=1}^{\infty} \frac{\rho_7(n)\rho_7(m+n)}{m^2 n^4 (m+n)^4 (m+2n)^2} \\ &= \frac{(-1)^{12+4}}{2^2 2!} \Bigl(\frac{(2\pi i)^2}{2!}\Bigr)^2 \Bigl(\frac{(2\pi i)^4}{4! 7^4} g(\rho_7)\Bigr)^2 \Bigl(\frac{69967019}{6988350600} + \frac{102810289\sqrt{-3}}{6988350600}\Bigr) \\ &= g(\rho_7)^2 \pi^{12} \Bigl(\frac{69967019}{181289027372537700} + \frac{102810289\sqrt{-3}}{181289027372537700}\Bigr). \end{aligned} \tag{11.3}$$

Example 11.4. Let ρ_5 be the quadratic character of conductor 5 given in Theorem B. Then we have

$$L_2((2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5);C_2) = \frac{92}{29296875}\pi^8; \tag{11.4}$$

$$L_3((2,2,2,2,2,2),(\rho_5,\rho_5,\rho_5,\rho_5,\rho_5,\rho_5);A_3) = -\frac{1856}{213623046875}\pi^{12}. \tag{11.5}$$

The latter can be regarded as a character analogue of the formula in [6, Prop. 8.5].

§ 12. Multiple Generalized Bernoulli Numbers

The generating function of multiple generalized Bernoulli numbers is given in terms of that of multiple Bernoulli polynomials as in the classical theory.

Definition 12.1 (The generating function [12, 16])**.** For $\mathbf{t} = (t_\alpha)_{\alpha\in\Delta_+}$,

$$G(\mathbf{t}, \boldsymbol{\chi}; \Delta) = \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Bigl(\prod_{\alpha\in\Delta_+} \frac{\chi_\alpha(a_\alpha)}{f_\alpha}\Bigr) F(\mathbf{f}\,\mathbf{t}, \mathbf{y}(\mathbf{a};\mathbf{f}); \Delta), \tag{12.1}$$

where $F(\mathbf{t}, \mathbf{y}; \Delta)$ is the generating function of multiple periodic Bernoulli functions in Definition 7.1 and $\mathbf{f}\,\mathbf{t} = (f_\alpha t_\alpha)_{\alpha\in\Delta_+}$, $\mathbf{y}(\mathbf{a};\mathbf{f}) = \sum_{\alpha\in\Delta_+} a_\alpha \alpha^\vee / f_\alpha$.

Definition 12.2 (Multiple generalized Bernoulli numbers [12, 16])**.** We define multiple generalized Bernoulli numbers $\mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta)$ by the coefficients of the Taylor expansion

$$G(\mathbf{t}, \boldsymbol{\chi}; \Delta) = \sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^{|\Delta_+|}} \mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta) \prod_{\alpha\in\Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!}. \tag{12.2}$$

We note that $\mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta)$ can be written in terms of multiple periodic Bernoulli functions as

$$\mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta) = \Bigl(\prod_{\alpha\in\Delta_+} f_\alpha^{k_\alpha - 1}\Bigr) \sum_{\substack{a_\alpha=1\\ \alpha\in\Delta_+}}^{f_\alpha} \Bigl(\prod_{\alpha\in\Delta_+} \chi_\alpha(a_\alpha)\Bigr) \mathcal{P}(\mathbf{k}, \mathbf{y}(\mathbf{a};\mathbf{f}); \Delta). \tag{12.3}$$

Example 12.3. If $X_r = A_1$, we have the generating function

$$G(t, \chi) = \sum_{a=1}^{f} \frac{\chi(a)}{f} F(ft, a/f) = \sum_{a=1}^{f} \frac{\chi(a)}{f} \frac{fte^{ft\{a/f\}}}{e^{ft} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}. \tag{12.4}$$

Theorem 12.4 ([12, 16])**.** *Assume that Δ is irreducible. Moreover assume that $f_\alpha > 1$ if Δ is of type A_1. Then for $w \in W$, we have*

$$B_{w^{-1}\mathbf{k}, w^{-1}\boldsymbol{\chi}}(\Delta) = \Bigl(\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1)\Bigr) \mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta). \tag{12.5}$$

Hence $\mathcal{B}_{\mathbf{k},\boldsymbol{\chi}}(\Delta) = 0$ if there exists an element $w \in W_{\mathbf{k}} \cap W_{\boldsymbol{\chi}}$ such that

$$\prod_{\alpha\in\Delta_+\cap w\Delta_-} (-1)^{k_\alpha} \chi_\alpha(-1) \neq 1, \tag{12.6}$$

where $W_{\mathbf{k}}$ and $W_{\boldsymbol{\chi}}$ are the stabilizers of $\mathbf{k}$ and $\boldsymbol{\chi}$ respectively.

Example 12.5. If $X_r = A_1$, we have

$$B_{k,\chi} = 0 \quad \text{if } (-1)^k \chi(-1) \neq 1. \tag{12.7}$$

Several other properties in the classical theory such as

$$F(t,y) = F(-t,-y) \text{ for } y \in \mathbb{R} \setminus \mathbb{Z}, \quad B_k(1-y) = (-1)^k B_k(y), \quad \frac{1}{t}\frac{\partial}{\partial y} F(t,y) = F(t,y)$$

can be reinterpreted in terms of root systems and Weyl groups.

§ 13. Zeta-Functions for Lie Groups

Recall that volume formulas are associated with all connected compact semisimple Lie groups. It is known that there is one-to-one correspondence between finite dimensional representations of complex semisimple Lie algebra $\mathfrak{g}$ and those of connected simply-connected compact semisimple Lie group G. In the cases of general compact semisimple Lie groups, we need analytically integral forms L for a maximal torus of G, which satisfies $Q \subset L \subset P$.

Definition 13.1 (Zeta-functions of Lie groups)**.** For a connected compact semisimple Lie group G,

$$\zeta_r(\mathbf{s}, \mathbf{y}; G) = \sum_{\lambda \in L \cap P_+} e^{2\pi i \langle \mathbf{y}, \lambda + \rho \rangle} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda + \rho \rangle^{s_\alpha}}. \tag{13.1}$$

Lemma 13.2.

$$\zeta_r(\mathbf{s}, \mathbf{y}; G) = \sum_{\mu \in P^\vee / Q^\vee} \widehat{\iota_{L+\rho}}(\mu) \zeta_r(\mathbf{s}, \mathbf{y} + \mu; \Delta), \tag{13.2}$$

where $\widehat{\iota_{L+\rho}} : P^\vee / Q^\vee \to \mathbb{C}$ *is the Fourier transformation of the characteristic function of* $L + \rho$ *given by*

$$\widehat{\iota_{L+\rho}}(\mu) = \frac{1}{|P/Q|} \sum_{\lambda \in (L+\rho)/Q} e^{-2\pi i \langle \mu, \lambda \rangle}. \tag{13.3}$$

Note that this expression plays the same role as the finite Fourier transformation of the Dirichlet character (see [35, Lemma 4.7]) in the theory of Dirichlet L-functions, whose origin is the study of prime numbers satisfying congruence conditions. In fact, our $\zeta_r(\mathbf{s}, \mathbf{y}; G)$ is a kind of Dirichlet series with congruence conditions (see (13.8) as an example).

In the A_1 case with $L = Q$, Lemma 13.2 implies

$$\sum_{m=0}^{\infty} \frac{e^{2\pi i (2m+1) y}}{(2m+1)^s} = \sum_{m=0}^{\infty} \frac{1}{2} \frac{e^{2\pi i (m+1) y}}{(m+1)^s} + \sum_{m=0}^{\infty} \frac{-1}{2} \frac{e^{2\pi i (m+1)(y + \frac{1}{2})}}{(m+1)^s}. \tag{13.4}$$

Lemma 13.3. *For $\mu \in P^\vee/Q^\vee$, we have*

$$\widehat{\iota_{L+\rho}}(\mu) = \frac{(-1)^{\langle \mu, 2\rho\rangle}}{|\pi_1(G)|}\delta_{L^*/Q^\vee}(\mu) \in \frac{\{-1,0,1\}}{|\pi_1(G)|} \subset \mathbb{Q}, \tag{13.5}$$

where $\pi_1(G)$ denotes the fundamental group of G and

$$\delta_{L^*/Q^\vee}(\mu) = \begin{cases} 1 & (\mu \in L^*/Q^\vee), \\ 0 & (\mu \notin L^*/Q^\vee). \end{cases} \tag{13.6}$$

Noting $P/L \simeq L^*/Q^\vee \simeq \pi_1(G)$, we have the following, where G may not be simply-connected.

Theorem 13.4 ([22])**.** *For $\mathbf{k} = (k_\alpha)_{\alpha\in\Delta_+} \in (2\mathbb{Z}_{\geq 1})^{|\Delta_+|}$ satisfying $w^{-1}\mathbf{k} = \mathbf{k}$ for all $w \in W$, and $\nu \in P^\vee/Q^\vee$ (a central element of G), we have*

$$\zeta_r(\mathbf{k}, \nu; G) = \frac{(-1)^{|\Delta_+|}}{|W|}\mathcal{P}(\mathbf{k}, \nu; G)\Bigl(\prod_{\alpha\in\Delta_+} \frac{(2\pi i)^{k_\alpha}}{k_\alpha!}\Bigr) \in \mathbb{Q}\pi^{|\mathbf{k}|}. \tag{13.7}$$

As an example, we obtain for the projective unitary group $PU(3)$,

$$\begin{aligned} \zeta_2(\mathbf{2}, \mathbf{0}; PU(3)) &= \sum_{\substack{m,n=1 \\ m\equiv n \pmod 3}}^{\infty} \frac{1}{m^2n^2(m+n)^2} \\ &= \sum_{2m-n,2n-m>0} \frac{1}{(2m-n)^2(2n-m)^2(m+n)^2} \\ &= \frac{(-1)^3}{3!}\frac{187}{918540}\Bigl(\frac{(2\pi i)^2}{2!}\Bigr)^3 \\ &= \frac{187\pi^6}{688905}. \end{aligned} \tag{13.8}$$

Remark. Originally, Witten zeta-functions represent the volumes of certain moduli spaces. Introducing multi-variable generalizations, we find some new applications. For example, we give a new interpretation of the shuffle product in the theory of Euler-Zagier multiple zeta values [19] and evaluate a class of Euler-Zagier multiple zeta values [20, 23]. However the geometric meaning of special values of zeta-functions of root systems is yet to be clarified.

§ 14. An Integral Representation

This section is based on the results by the first author in [9,10]. So far, we focused on special values on the region of convergence. On the other hand, analytic continuations enable us to discuss special values on the whole space in $\mathbf{s}$.

The analytic continuations of general multiple zeta-functions were already obtained by Lichtin [24], Essouabri [4,5], Matsumoto [25,26], de Crisenoy [2], etc. (See [27] for an elaborated survey on the analytic continuations of multiple zeta-functions.) However we give yet another method which is a generalization of the formula

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(s)(e^{2\pi i s}-1)}\int_C \frac{z^{s-1}}{e^z-1}dz \qquad (C\colon \text{Hankel contour}). \tag{14.1}$$

Let N, R be positive integers. For $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_R)\in\mathbb{C}^R$, $\boldsymbol{a}=(a_1,\ldots,a_N)$, $\boldsymbol{s}=(s_1,\ldots,s_N)\in\mathbb{C}^N$ and $\boldsymbol{b}=(b_{ij})_{1\le i\le N,1\le j\le R}\in\mathbb{C}^{N\times R}$, consider the multiple series

$$\zeta(\boldsymbol{\xi},\boldsymbol{a},\boldsymbol{b},\boldsymbol{s})= \sum_{m_1=0}^{\infty}\cdots\sum_{m_R=0}^{\infty}\frac{e^{\xi_1 m_1}\cdots e^{\xi_R m_R}}{(a_1+b_{11}m_1+\cdots+b_{1R}m_R)^{s_1}\cdots(a_N+b_{N1}m_1+\cdots+b_{NR}m_R)^{s_N}}. \tag{14.2}$$

Theorem 14.1 ([9,10])**.**

$$\zeta(\boldsymbol{\xi},\boldsymbol{a},\boldsymbol{b},\boldsymbol{s})=\frac{1}{\Gamma(s_1)\cdots\Gamma(s_N)}\prod_{t\in S}\frac{1}{e^{2\pi i t(\boldsymbol{s})}-1}\times \int_\Sigma \frac{e^{(b_{11}+\cdots+b_{1R}-a_1)z_1}\cdots e^{(b_{N1}+\cdots+b_{NR}-a_N)z_N}z_1^{s_1-1}\cdots z_N^{s_N-1}}{(e^{z_1 b_{11}+\cdots+z_N b_{N1}}-e^{\xi_1})\cdots(e^{z_1 b_{1R}+\cdots+z_N b_{NR}}-e^{\xi_R})}dz_1\wedge\cdots\wedge dz_N, \tag{14.3}$$

where Σ is a union of certain surfaces and S is a set of certain linear functionals on $\mathbb{C}^N$.

If $b_{ij}>0$ for all i,j satisfying $1\le i\le N$, $1\le j\le R$, then this integral representation can be derived by use of Shintani's result [30]. In fact, Theorem 14.1 is a refinement of his integral representation.

Setting $\xi_i=0$, $a_\alpha=\langle\alpha^\vee,\rho\rangle$ and $b_{\alpha i}=\langle\alpha^\vee,\lambda_i\rangle$ for $\alpha\in\Delta_+$ and $1\le i\le R=r$, we obtain integral representations of zeta-functions of root systems. In this setting, from the integrand, we can construct generating functions of Bernoulli numbers for nonpositive domain.

§ 15. Possibilities of Elliptic Generalizations

Lastly we give two possibilities of "elliptic" generalizations by regarding $\zeta_r(\mathbf{s},\mathbf{y};\Delta)$ as "rational" or "trigonometric" versions.

The first is an Eisenstein analogue. Let $k>2$ be an integer, $(x,y)\in\mathbb{R}^2\setminus\mathbb{Z}^2$ and $\tau\in\mathbb{C}$ with $\Im\tau>0$. The Eisenstein series is defined by

$$G_k(\tau;x,y)=\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{e^{2\pi i(mx+ny)}}{(m+n\tau)^k}. \tag{15.1}$$

We define $\mathcal{H}_k(x,y;\tau)$ by

$$e^{2\pi ixt}\frac{\theta'(0;\tau)\theta(t+x\tau-y;\tau)}{\theta(t;\tau)\theta(x\tau-y;\tau)}=\sum_{k=0}^{\infty}\mathcal{H}_k(x,y;\tau)\frac{(2\pi i)^k t^{k-1}}{k!}, \tag{15.2}$$

where $t\in\mathbb{C}$ with $|t|<\epsilon$ for sufficiently small $\epsilon>0$ and $\theta(z;\tau)$ is the Jacobi theta function defined by

$$\theta(z;\tau)=-i\sum_{n\in\mathbb{Z}}\exp\Bigl(\pi i\Bigl(n+\frac{1}{2}\Bigr)^2\tau+2\pi i\Bigl(n+\frac{1}{2}\Bigr)z+\pi in\Bigr) \tag{15.3}$$

for $z\in\mathbb{C}$. Then we have the following, which can be regarded as an elliptic analogue of the result on the zeta-function of root system of type A_1 given in (2.7).

Proposition 15.1 (Katayama [8]). *For $k\in\mathbb{Z}_{\geq 2}$, we have*

$$G_k(\tau;x,y)=-\mathcal{H}_k(x,y;\tau)\frac{(2\pi i)^k}{k!}. \tag{15.4}$$

From this viewpoint, it is desirable to develop a theory on elliptic analogues of the results on zeta-functions of root systems mentioned in the previous sections, by constructing corresponding Eisenstein series. For example, we hope to extend (15.4) to that associated with root systems.

The second is a q-analogue. Instead of Weyl's dimension formula, we employ the character formula. For $q=e^{-2\pi i/\tau}$, $s,z\in\mathbb{C}$ with $\Re z>0$ and $x\in\mathbb{R}$, define

(15.5)
$$\zeta_q(s,z;x)=\sum_{m=1}^{\infty}\frac{e^{2\pi imx}q^{mz}}{[m]_q^s},\qquad [m]_q=\frac{1-q^m}{1-q},\qquad [m]_q!=[m]_q[m-1]_q\cdots[1]_q.$$

Let

$$\psi(t)=\frac{\tau}{2\pi i}\frac{e^{2\pi it/\tau}-1}{e^{2\pi itz/\tau}}=t+O(t^2) \tag{15.6}$$

be a local coordinate around the origin. Define $\mathcal{Q}_k(x,y,z;\tau)$ by

$$e^{2\pi ixt}\frac{\theta'(0;\tau)\theta(t+x\tau-y;\tau)}{\theta(t;\tau)\theta(x\tau-y;\tau)}=\sum_{k=0}^{\infty}\mathcal{Q}_k(x,y,z;\tau)\Bigl(\frac{2\pi i/\tau}{1-q}\Bigr)^k\frac{\psi'(t)\psi(t)^{k-1}}{[k]_q!}. \tag{15.7}$$

Then

Theorem 15.2. *For $k\in\mathbb{N}$, $0<z<1$ and $x,y\in\mathbb{R}$ with $y+kz\in\mathbb{Z}$, we have*

$$\zeta_q(k,k(1-z);x)+(-1)^k\zeta_q(k,kz;-x)=-\mathcal{Q}_k(x,y,z;\tau)\frac{1}{[k]_q!}. \tag{15.8}$$

This is a q-analogue of (2.7). Not only the result, but also the proof can be done analogously. In fact, formula (2.7) can be shown by a residue calculus on the space $\mathbb{C}$. Similarly, we can prove Theorem 15.2 employing the space $\mathbb{C}/\tau\mathbb{Z}$.

In particular, from this formula, we have for $\tau = i$,

$$\zeta_q(2,1;0) = (1-e^{-2\pi})^2 \frac{\pi-3}{24\pi}, \qquad \zeta_q(4,2;0) = (1-e^{-2\pi})^4 \frac{30\pi^3 - 11\pi^4 + 3\varpi^4}{1440\pi^4}, \tag{15.9}$$

where ϖ is the lemniscate constant defined by

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}}. \tag{15.10}$$

By aid of generalizations of Eisenstein series, these special values are also calculated in [21].

We hope that generalizations of the above will be constructed in arbitrary root systems.

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