Algebraic and Statistical Approach to Infinite Quantum Systems

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This thesis is arranged in four chapters.

- Chapter I: The organization of this thesis and the author's studies

In the thesis, quantum statistical inference is discussed from the veiwpoint of axiomatic quantum theory based on the author's studies. Quantum theory has played a remarkable role in the history of physics, but its conceptual and mathematical bases have not established yet. It is not deniable that many studies pay little attention to their conceptual bases which guarantee logical consistencies of a body of theory. What the author would like to stress in this context is that many results from the viewpoint of mathematics and mathematical physics are not always efficiently used in conceptual studies. Quantum technology is now developing and a trend to discuss foundations and applications of quantum theory containing quantum measurement theory is reaching a peak, but many fundamental problems are not clarified conceptually and mathematically. We are going to present a mathematical framework of quantum theory, which gives us a new perspective on its probablistic and statistical aspects.

Let us explain the author's studies in master and doctoral courses. His studies are classified into the following four topics:

1) A formulation of statistical inference applicable to infinite degrees of freedom. Especially, large deviation type estimates and quantum version of information criteria [6],

2) An extension of the result of 1) into situations that involve measuring processes [9],

3) The proof of Born statistical formula based on the concept of sector and measuring processes [8],

4) A justification of the mathematical aspect of the definition of sector by noncommutative Gel'fand-Naimark theorem and Tomita decomposition theorem.

2) and 4) are the main topics in this thesis. 1) is already reported in my master thesis.

The topic 2) is a generalization of 1). A difficulty the author(K.O.) of [9] encountered in this study is that we need choose measurements appropriate for the structure inside of the system and for statistical methods and aims. Physically speaking, for the purpose of estimating and evaluating an "intra-sectorial" structure of the system under consideration, we should be couple it with another one, outside of the system, which is nothing but a measurement. This action is a universal physical process and is also called a measuring process. The author defined a special measurement which can compare two states of the system in the following meaning: one central measure of the composite system of the system and an appartus is absolutely continuous with respect to another one. The existence is, of course, not trivial. Also, by using such a measurement, he proved a quantum version of Stein's lemma and showed that the framework of model selection is applicable to the situations focusing on an intra-sectorial structure of the system.

The topic 3) is one of the author's projects in collaboration with Prof. I. Ojima and Dr. H. Saigo. In this collaboration, Born statistical formula is proved in the mathematical formalism of general quantum theory including systems of infinite degrees of freedom.

The purpose of the study 4) is to prove the mathematical validity of the definition of sector, according to the previous studies 1)∼3). The definition of sector is given by [5], and its physical validity is sufficiently discussed in these papers and established. On the other hand, there is much room for the validity of its mathematical definition, which is partially justified by a noncommutative version of Gel'fand-Naimark theorem and of Riesz-Markov-Kakutani theorem.

- Chapter II: Preliminaries: C^*-algebras and Gel'fand-Naimark Theorem

In Chapter II, we remind you of the basic knowledge of operator algebras. Especally, we focus on C^*-algebras, representation theorems and integral decompositions of states. Gel'fand-Naimark theorem and its noncommutative generalizarion are the main theorems in this chapter. The latter is proved by Takesaki and Bichteler and, therefore, is called Takesaki-Bichteler theorem. It is hard to say that this theorem is widely known among users of operator algebras, but the author believes that we should deepen its implication to understand the category of general C^*-algebras.

- Chapter III: Sector Theory and Measuring Processes

In Chapter III, we discuss physical and mathematical aspects of sector theory, and reformulate the basis of quantum probability theory in this context. In quantum probability theory, an algebraic probability space $(\mathcal{X}, \omega)$ of a *-algebra $\mathcal{X}$ and of a state ω on $\mathcal{X}$ is a standard starting point. However, the relation between algebraic probability spaces and measure-theoretical description of probability are often nontrivial. Sector theory is essential for resolving this problem. The concept of sector itself was considered in the context of quantum theory at the beginning, but in the thesis we use it here in order to reformulate quantum probability theory. This attempt is nothing but an introduction of sectors to mathematics. Consequently, we can reach our goal naturally.

We aim at starting with a new axiomatic system for quantum theory [8]. They are very powerful for discussing probablistic and statistical aspects of quantum theory. Furthermore, Born statistical formula can be proved in the axiomatic system[1].

- Chapter IV: Statistical Aspects of Quantum Theory

We show that some of classical statistical methods are applicable to quantum systems, owing to the universality of statistics and information theory. As a part of this attempt, we show that the quantum relative entropy is a rate function. In the concrete, a quantum version of Stein's lemma and of Sanov's theorem are proved in section IV.2.

Next in section IV.3, quantum model selection is discussed. We try to apply information criteria for quantum states. Information criteria for probability distributions are usually described in terms of the relative entropy, but we derive those for quantum states

[1]The original form of this formulation was reported in [7]. The idea of this investigation can be traced back to the efficient use of sector in large deviation strategy [6].

from the quantum relative entropy. In preceding investigations [12], information criteria are applied to probability distributions calculated from POVM (positive operator-valued measures). Accordingly, the accuracy of the estimation by the use of information criteria for quantum states has not been discussed.

Finally, in section IV.4, we discuss the quantum α-divergence [2] as a quasi-entropy[2] formulated by D. Petz [10]. See [2] for its fundamental properties. We will show that the quantum α-divergence is applicable in the (C*-)algebraic setting. First, for this purpose, the α-analogue of Hiai-Ohya-Tsukada theorem is proved. Secondly, a quantum version of Chernoff bound is proved, which is different from that in [3]. This is a large deviation type estimate, too. Quantum hypothesis testing for Chernoff and Hoeffding bounds, in which many researchers are interested next to a quantum version of Stein's lemma, has recently been developed (see [4] for instance). Lastly, we define a Bayesian α-predictive state and prove that this minimizes a risk function constructed of the quantum α-divergence. This result generalizes those in [1].

References

[1] J. Aitchison, Biometrika **62**, 547-554 (1975); F. Tanaka, Generalized Bayesian predictive density operator; In *Proceedings of 14th Quantum Information Technology Symposium(QIT14)*, Tokyo, Japan, May 29-30, 2006, 107-110: F. Tanaka and F. Komaki, Phys. Rev. A **71**, 052323 (2005).

[2] S. Amari and H. Nagaoka, *Methods of Information Geometry*, Translations of mathematical monographs; v. 191, Amer. Math. Soc. & Oxford Univ. Press (2000).

[3] K.M.R. Audenaert, J. Calsamiglia, R. Munoz-Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete, Phys. Rev. Lett. **98**, 160501 (2007); M. Hayashi, Phys. Rev. A **76**, 062301 (2007); H. Nagaoka, The converse part of the theorem for quantum Hoeffding bound, (2006), arXiv:quant-ph/0611289; M. Nussbaum and A. Szkola, Ann. Statist. **37** (2009), 1040-1057.

[4] V. Jaksic, Y. Ogata, Y. Pautrat, and C.-A. Pillet, Entropic fluctuations in quantum statistical mechanics. An introduction, In *Quantum Theory from Small to Large Scales*, Lecture Notes of the Les Houches Summer School Volume 95, August 2010, (Oxford University Press, USA, 2012); V. Jaksic, Y. Ogata, C.-A. Pillet, and R. Seiringer, Rev. Math. Phys. **24**, 1230002 (2012).

[5] I. Ojima, Open Sys. Inf. Dyn. **10**, 235-279 (2003); Publ. RIMS **40**, 731-756 (2004).

[6] I. Ojima and K. Okamura, Large deviation strategy for inverse problem I & II, Open Sys. Inf. Dyn. **19**, (2012), 1250021, 1250022.

[7] I. Ojima, K. Okamura and H. Saigo, From Born rule to large deviations (in Japanese), Soryûshiron-kenkyû **13**, No. 1 (2012); K. Okamura, From Born rule to large deviations, AIP Conf. Proc. **1508**, 433-437, (2012).

[8] I. Ojima, K. Okamura and H. Saigo, Derivation of Born Rule from Algebraic and Statistical Axioms (2013), arXiv:1304.6618.

[2] In probabiliy theory and classical information theory, theory of generalized entropies is started by A. Rényi [11].

[9] K. Okamura, The quantum relative entropy as a rate function and information criteria, Quantum Info. Processing **12**, 2551-2575 (2013).

[10] D. Petz, Publ. RIMS, Kyoto Univ. **21** (1985), 787-800.

[11] A. Rényi, *Wahrscheinlichkeitsrechnung*, (VED Deutsch, Ver der Wiss., Berlin, 1962); Rev. Int. Stat. Inst. **33**, (1965), 1-14.

[12] K. Usami, Y. Nambu, Y. Tsuda, K. Matsumoto and K. Nakamura, Phys. Rev. A **68**, 022314 (2003); J.O.S. Yin and S.J. van Enk, Phys. Rev. A **83**, 062110 (2011).