Integrable deformations of principal chiral models and the AdS/CFT correspondence

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In this thesis, we consider three kinds of integrable two-dimensional non-linear sigma models (and Wess-Zumino-Novikov-Witten (WZNW) models) . They are regarded as integrable deformations of $SU(2)$ and $SL(2,\mathbb{R})$ principal chiral models (PCMs). The undeformed PCMs have left and right descriptions and their equivalence is quite clear. The deformations here resolve the degeneracy of two descriptions and hence the realization of left-right duality becomes unclear. We discuss not only how each of the two descriptions is deformed individually but also how the left-right duality is modified.

(1) Squashed sigma models

The first is the two-dimensional non-linear sigma models on the three-dimensional squashed sphere (squashed sigma models). The deformation term breaks a part of the isometry of S^3 , $SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ to $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{R}}$. According to the target space isometry, there are two descriptions to discuss its classical dynamics: one of them is 1) the left description and the other 2) the right one.

The left description is based on the $SU(2)_{\mathrm{L}}$ symmetry and isotropic type. A couple of flat $SU(2)_{\mathrm{L}}$ conserved current are obtained. Then two Lax pairs are shown to be written in terms of the flat $SU(2)_{\mathrm{L}}$ currents. The isotropic r/s-matrices are obtained from these Lax pairs. In addition, the $SU(2)_{\mathrm{L}}$ symmetry is shown to be enhanced to a pair of $su(2)_{\mathrm{L}}$ Yangians.

On the other hand, the right description is associated with the enhanced $U(1)_{\mathrm{R}}$ symmetry and anisotropic type. The $U(1)_{\mathrm{R}}$ is enhanced to q-deformed $su(2)_{\mathrm{R}}$ with non-local symmetries. Then the q-deformed $su(2)_{\mathrm{R}}$ is enhanced to a classical analogue of quantum affine $su(2)_{\mathrm{R}}$. The quantum affine $su(2)_{\mathrm{R}}$ should be related to anisotropic integrable models. In fact, the r/s-matrices in right description have anisotropic form.

In this way, it is shown that the classical integrable structure of squashed sigma models accepts two descriptions. The two descriptions seem to be quite different: one is isotropic and the other is anisotropic. However, we show the gauge equivalence between monodromy matrices of two descriptions with a certain identification of spectral parameters and the

isomorphisms of $su(2)$ generators. In addition, the right monodromy matrix turns out to be decomposed into a pair of reduced right monodromy matrices and each of them is gauge equivalent to the corresponding left monodromy matrix.

(2) Squashed WZNW models

The second is the Wess-Zumino-Novikov-Witten models on the three-dimensional squashed sphere (squashed WZNW models) . An additional deformation parameter is introduced as the coefficient of Wess-Zumino term and hence the analysis is modified. In particular, the $U(1)_{\rm R}$ is not enhanced to quantum affine $su(2)_{\rm R}$ but a one-parameter deformation of quantum affine $su(2)_{\rm R}$.

As in case of squashed sigma models, the classical dynamics can be described in two ways: 1) the left description based on $SU(2)_{\rm L}$ and 2) the right descriptions based on enhanced $U(1)_{\rm R}$.

In the left description, it is shown that the $SU(2)_{\rm L}$ symmetry is enhanced to $su(2)_{\rm L}$ Yangians. A couple of Lax pairs are constructed from the two flat $SU(2)_{\rm L}$ currents. In addition, the associated r/s-matrices are also computed.

In the right description, it is discussed that q-deformed $su(2)_{\rm R}$ is enhanced to one-parameter deformation of quantum affine $su(2)_{\rm R}$. It does not seem to be isomorphic to the standard quantum affine $su(2)_{\rm R}$ and hence the algebra essentially contains two deformation parameters. We also construct an anisotropic Lax pair in right description. It is a one-parameter deformation of that for squashed sigma models. The effect of deformation does not seem to be eliminated by gauge transformations. This is because the deformed quantum affine $su(2)_{\rm R}$ is considered not to be isomorphic to the standard quantum affine $su(2)_{\rm R}$.

It is also shown that the left-right duality is maintained after adding the Wess-Zumino term. The argument is a generalization of the squashed sigma model case. The right monodromy matrix is shown to be gauge equivalent to a pair of the left monodromy matrices with a certain map of spectral parameters and the $su(2)$ isomorphism. A couple of reduced right Lax pair are also obtained.

In addition, we discuss two degenerate limits. They correspond to the two lines in the parameter space of squashed WZNW models. One of them corresponds to the undeformed $SU(2)$ WZNW models. As a matter of course, the target space becomes undeformed S^3

and hence the right description becomes the isotropic. In case of models on the other line, while the metric of the target space remains to be squashed the effect of the squashing is compensated by the Wess-Zumino term. As a consequence, the right description becomes the isotropic. In each of the degenerate limits, a flat $SU(2)_{\rm R}$ conserved current is constructed. By using the flat current, it is shown that $su(2)_{\rm R}$ Yangian generators are also obtained.

(3) Schrödinger sigma models

The final is the two-dimensional non-linear sigma models on the three-dimensional Schrödinger spacetimes (Schrödinger sigma models) . The model is a kind of deformed $SL(2,\mathbb{R})$ PCMs and its action is invariant under the $SL(2,\mathbb{R})_{\rm L}\times U(1)_{\rm R}$ transformation. The classical integrable structure has two descriptions: 1) the left description and 2) the right description.

The left description is based on the $SL(2,\mathbb{R})_{\rm L}$ and similar to that of the squashed sigma models. A pair of flat $SL(2,\mathbb{R})_{\rm L}$ currents are constructed. Then two Lax pairs are obtained from the flat $SL(2,\mathbb{R})_{\rm L}$ currents. The generators of $sl(2,\mathbb{R})_{\rm L}$ Yangians are also constructed. This description is shown to be isotropic.

The right description is related to the enhanced $U(1)_{\rm R}$. This $U(1)_{\rm R}$ is generated by T^{-} , which is a non-Cartan generator. As a result, the $U(1)_{\rm R}$ is enhanced to the non-standard q-deformed $sl(2,\mathbb{R})_{\rm R}$. It is a non-Cartan q-deformation of $sl(2,\mathbb{R})_{\rm R}$. Then the Lax pairs and the corresponding r/s-matrices are obtained. They can be regarded as non-Cartan deformed isotropic r/s-matrices. We also show the realization of an "exotic" symmetry. It is an affine extension of non-standard q-deformed $sl(2,\mathbb{R})_{\rm R}$. The resulting algebra has the semi-infinite tower structure and looks like a tilted $sl(2,\mathbb{R})_{\rm R}$ Yangian.

The left-right duality is also shown. The gauge equivalence of monodromy matrices in the two descriptions is shown by taking account of the relations between the spectral parameters.

In addition, it is revealed that the isotropic Lax pairs are realized even in the right description. They are written in terms of "non-local" flat conserved currents and related to the anisotropic right Lax pairs through non-local gauge transformations. It is also shown that undeformed $sl(2,\mathbb{R})_{\rm R}$ Yangians are realized in a quite non-trivial way. The homomorphism from $sl(2,\mathbb{R})_{\rm R}$ Yangians to the exotic symmetry is obtained.