SIMPLE NORMAL CROSSING FANO VARIETIES AND LOG FANO MANIFOLDS

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ABSTRACT. A projective log variety (X, D) is called "a log Fano manifold" if X is smooth and if D is a reduced simple normal crossing divisor on X with $-(K_X + D)$ ample. The n-dimensional log Fano manifolds (X, D) with nonzero D are classified in this article when the log Fano index r of (X, D) satisfies either $r \geq n/2$ with $\rho(X) \geq 2$ or $r \geq n - 2$. This result is a partial generalization of the classification of logarithmic Fano threefolds by Maeda.

1. Introduction

As is well known, Fano manifolds play an essential role in various situations. Fano manifolds have been classified up to dimension three. It is also known that the anti-canonical degree of n-dimensional Fano manifolds is bounded for an arbitrary n. For a Fano manifold X, the *Fano index* is, by definition, the largest positive integer $r = r(X)$ such that $-K_X$ is r times a Cartier divisor, and the *Fano pseudoindex* is the minimum $\iota = \iota(X)$ of the intersection number ι of $-K_X$ with a rational curve. We note that n-dimensional Fano manifolds X with $r(X) \geq n - 2$ have been classified ([KO73, Fjt90, Isk77, MM81, Muk89, Wiś90a, Wiś90b, Wiś91a]).

The *Mukai conjecture* [Muk88, Conjecture 4] asserts that $\rho(X)(r(X) - 1) \leq n$ and *the generalized Mukai conjecture* asserts that $\rho(X)(\iota(X) - 1) \leq n$ for any Fano manifold X, where $\rho(X)$ is the Picard number of X. The generalized Mukai conjecture is important in the classification theory of Fano manifolds and is still open even now except for the case $n \leq 5$ or $\rho(X) \leq 3$ (see [NO10] and references therein). One of the most significant results related to the Mukai conjecture is due to Wiśniewski [Wiś90b, Wiś91a]; he has classified n-dimensional Fano manifolds with $r(X) > n/2$ and $\rho(X) \geq 2$.

In this article, we consider *snc Fano varieties*, that is, projective simple normal crossing varieties whose dualizing sheaves are dual of ample invertible sheaves. In order to investigate snc Fano varieties, it is natural to consider all of their irreducible components with the conductor divisors. The component with the conductor was considered by Maeda [Mae86] in his study of a logarithmic Fano varieties, that is, a *log Fano manifold* in this article, which is a pair (X, D) consisting of a smooth projective variety X and a reduced snc divisor D on X with $-(K_X + D)$ ample. Maeda classified such pairs with $\dim X \leq 3$

and pointed out that the degree $(-(K_X + D)^{\cdot n})$ is unbounded for log Fano manifolds (X, D) of fixed dimension $n \geq 3$.

We introduce the *snc Fano indices* (resp. *log Fano indices*) and *snc Fano pseudoindices* (resp. *log Fano pseudoindices*) for snc Fano varieties (resp. for log Fano manifolds) similarly to the case of Fano manifolds (see Definition 2.8). It is expected that snc Fano varieties with large snc Fano indices have analogous applications (for example, see [Kol11]).

The following is the main idea to investigate n-dimensional log Fano manifolds (X, D) with $D \neq 0$. Study the contraction morphism associated to an extremal ray intersecting D positively. Moreover, D is an $(n-1)$-dimensional snc Fano variety with $r(X, D)|r(D)$ (i.e., $r(X, D)$ divides $r(D)$) and $\iota(D) \geq \iota(X, D)$. Hence we can use inductive arguments.

Now we explain how this article is organized. Section 2 is a preliminary section. We define snc Fano varieties and log Fano manifolds in Sections 2.1 and 2.2 and give some properties in Section 2.3. We show results on bundle structures (Section 2.4), extremal contractions (Section 2.5) and the special projective bundles, the so-called rational scrolls (Section 2.6). In Section 3, we give various examples of log Fano manifolds with large log Fano indices, which occur in the theorems in Section 4. In Section 4, we state the main results of this article. In Section 4.1, we treat an n-dimensional log Fano manifold (X, D) with $D \neq 0$ such that $r(X, D) = n - 1$ (Proposition 4.2). The main purpose of this article, which we discuss in Section 4.2, is to classify n-dimensional log Fano manifolds (X, D) with $D \neq 0$, $r(X, D) \geq n/2$ and $\rho(X) \geq 2$ (Theorems 4.3 and 4.5), which is a log version of the treatment of the Mukai conjecture by Wiśniewski [Wiś90b, Wiś91a]. We prove Theorem 4.5 in Section 5. Wiśniewski argued the case $r(X) > n/2$ and we treat the case $r(X, D) \geq n/2$ with $D \neq 0$. We remark that we do not treat Maeda's case $n = 3$ and $r(X, D) = 1$; some of the techniques of the proof are similar to Maeda's but the objects of our study are completely different from those of Maeda.

Theorem 1.1 (= Theorem 4.3)**.** *If (X, D) is an n-dimensional log Fano manifold with $\iota := \iota(X, D) > n/2$, $D \neq 0$ and $\rho(X) \geq 2$, then $n = 2\iota - 1$ and $(X, D) \simeq (\mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}, m], \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota-1})$ with $m \geq 0$, where the embedding $D \subset X$ is the canonical embedding $\mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}] \subset_{\mathrm{can}} \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}, m]$. (This is exactly the case in Example* O *in Section 3.)*

Theorem 1.2 (= Main Theorem 4.5)**.** *Let (X, D) be a $2r$-dimensional log Fano manifold with $r(X, D) = r \geq 2$, $D \neq 0$ and $\rho(X) \geq 2$. Then (X, D) is in exactly one of Examples* I*,* II*,* III*,* IV*,* V*,* VI*,* VII*,* VIII*,* IX*,* X*,* XI.

As a consequence of Theorems 4.3, 4.5, together with Maeda's result, we have classified n-dimensional log Fano manifolds (X, D) with $r(X, D) \geq n - 2$ and $D \neq 0$, which we discuss in Section 4.3 (Corollary 4.6).

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Notation and terminology. We always work in the category of algebraic (separated and finite type) schemes over a fixed algebraically closed field $\Bbbk$ of characteristic zero. A *variety* means a connected and reduced algebraic scheme. For the theory of extremal contraction, we refer the readers to [KM98]. For a complete variety X, the Picard number of X is denoted by $\rho(X)$. For a smooth projective variety X, we define $\mathrm{Eff}(X)$ (resp. $\mathrm{Nef}(X)$) to be the effective (resp. nef) cone, which is defined as the cone in $\mathrm{N}^1(X)$ spanned by the classes of effective (resp. nef) divisors on X. For a smooth projective variety X, let $\mathrm{NE}(X)$ be the cone in $\mathrm{N}_1(X)$ spanned by effective 1-cycles on X, and $\overline{\mathrm{NE}}(X)$ the closure of $\mathrm{NE}(X)$ in $\mathrm{N}_1(X)$. For a smooth projective variety X and a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$, let $l(R) := \min\{(-K_X \cdot C) \mid C \text{ is a rational curve with } [C] \in R\}$. This is called the *length* $l(R)$ of R. A rational curve $C \subset X$ with $[C] \in R$ and $(-K_X \cdot C) = l(R)$ is called *a minimal rational curve of* R.

For a morphism of algebraic schemes $f\colon X \to Y$, we define the *exceptional locus* $\mathrm{Exc}(f)$ *of* f by $\mathrm{Exc}(f) := \{x \in X \mid f \text{ is not isomorphism around } x\}$.

For a complete variety X, an invertible sheaf $\mathcal{L}$ on X, and $i \in \mathbb{Z}_{\geq 0}$, $\dim_{\Bbbk} H^i(X, \mathcal{L})$ is denoted by $h^i(X, \mathcal{L})$ (or simply by $h^i(X, L)$ if $\mathcal{L} = \mathcal{O}_X(L)$), where $\mathbb{Z}_{\geq 0} := \{r \in \mathbb{Z} \mid r \geq 0\}$.

For algebraic schemes (or coherent sheaves on a fixed algebraic scheme) $X_1, \dots, X_m$, the projection is denoted by $p_{i_1,\dots,i_k}\colon \prod_{i=1}^m X_i \to \prod_{j=1}^k X_{i_j}$ for any $1 \leq i_1 < \cdots < i_k \leq m$.

For an algebraic scheme X and a locally free sheaf of finite rank $\mathcal{E}$ on X, let $\mathbb{P}_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck and $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological invertible sheaf. We usually denote the projection by $p\colon \mathbb{P}_X(\mathcal{E}) \to X$. For locally free sheaves $\mathcal{E}_1, \dots, \mathcal{E}_m$ of finite rank on X and $1 \leq i_1 < \cdots < i_k \leq m$, we sometimes denote the embedding obtained by the natural projection $p_{i_1,\dots,i_k}\colon \bigoplus_{i=1}^m \mathcal{E}_i \to \bigoplus_{j=1}^k \mathcal{E}_{i_j}$ by

$$\mathbb{P}_X\left(\bigoplus_{j=1}^k \mathcal{E}_{i_j}\right) \quad \subset_{\mathrm{can}} \quad \mathbb{P}_X\left(\bigoplus_{i=1}^m \mathcal{E}_i\right)$$

and we call that this embedding *the canonical embedding.*

The symbol $\mathbb{Q}^n$ (resp. $\mathcal{Q}^n$) means a smooth (resp. possibly non-smooth or reducible) hyperquadric in $\mathbb{P}^{n+1}$ for $n \geq 2$. We write $\mathcal{O}_{\mathbb{Q}^n}(1)$ (resp. $\mathcal{O}_{\mathcal{Q}^n}(1)$) for the invertible sheaf which is the restriction of $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$ under the natural embedding. We sometimes write $\mathcal{O}(m)$ instead of $\mathcal{O}_{\mathbb{Q}^n}(m)$ (or $\mathcal{O}_{\mathcal{Q}^n}(m)$, $\mathcal{O}_{\mathbb{P}^n}(m)$) for simplicity.

For an irreducible projective variety V with $\mathrm{Pic}(V) = \mathbb{Z}$, the ample generator $\mathcal{O}_V(1)$ of $\mathrm{Pic}(V)$, a nonnegative integer t and integers $a_0, \dots, a_t$, we denote the projective space bundle

$$\mathbb{P}_V(\mathcal{O}_V(a_0) \oplus \cdots \oplus \mathcal{O}_V(a_t)) \quad \text{by} \quad \mathbb{P}[V; a_0, \dots, a_t]$$

for simplicity. (We often denote

$$\mathbb{P}[V; \underbrace{b_0, \dots, b_0}_{n_0 \text{ times}}, \dots, \underbrace{b_u, \dots, b_u}_{n_u \text{ times}}] \quad \text{by} \quad \mathbb{P}[V; b_0^{n_0}, \dots, b_u^{n_u}]$$

for any integers $b_0, \dots, b_u$ and positive integers $n_0, \dots, n_u$.) We also denote by $\mathcal{O}(m; n)$ the invertible sheaf

$$p^*\mathcal{O}_V(m) \otimes \mathcal{O}_{\mathbb{P}}(n) \quad \text{on} \quad \mathbb{P}[V; a_0, \dots, a_t]$$

for any integers m and n, where $p\colon \mathbb{P}[V; a_0, \dots, a_t] \to V$ is the projection and $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological invertible sheaf with respect to p. For any $0 \le i_1 < \cdots < i_k \le t$, we denote the canonical embedding

$$\mathbb{P}_V(\mathcal{O}_V(a_{i_1}) \oplus \cdots \oplus \mathcal{O}_V(a_{i_k})) \subset_{\mathrm{can}} \mathbb{P}_V(\mathcal{O}_V(a_0) \oplus \cdots \oplus \mathcal{O}_V(a_t))$$

by $\mathbb{P}[V; a_{i_1}, \dots, a_{i_k}] \subset_{\mathrm{can}} \mathbb{P}[V; a_0, \dots, a_t]$, and we call this *the canonical embedding.*

2. Preliminaries

2.1. **Snc varieties and log manifolds.** First, we define snc varieties and log manifolds.

Definition 2.1. Let X be a variety and $x \in X$ be a closed point. We say that X has *normal crossing singularity* at x if the completion of the local ring $\mathcal{O}_{X,x}$ is isomorphic to $\Bbbk[[x_1, \dots, x_{n+1}]]/(x_1 \cdots x_k)$ for some $1 \le k \le n+1$.

Definition 2.2. (1) An *snc variety* is a variety $\mathcal{X}$ having normal crossing singularities at any closed points $x \in \mathcal{X}$ and each irreducible component of $\mathcal{X}$ is a smooth variety.

(2) A *log manifold* is a pair (X, D) such that X is a smooth variety and D is an snc divisor on X, that is, D is a reduced divisor in X which is an snc variety.

Definition 2.3. Let $\mathcal{X}$ be an snc variety with the irreducible decomposition $\mathcal{X} = \bigcup_{1 \le i \le m} X_i$. For any distinct $1 \le i, j \le m$, the intersection $X_i \cap X_j$ can be seen as a smooth divisor D_{ij} in X_i. We define $D_i := \sum_{j \ne i} D_{ij}$ and call it the *conductor divisor* in X_i (with respect to $\mathcal{X}$). We often write that $(X_i, D_i) \subset \mathcal{X}$ is an irreducible component for the sake of simplicity. We also write $\mathcal{X} = \bigcup_{1 \le i \le m}(X_i, D_i)$.

Remark 2.4. If $\mathcal{X}$ is an snc variety, then $\mathcal{X}$ has an invertible dualizing sheaf $\omega_{\mathcal{X}}$ since $\mathcal{X}$ has only Gorenstein singularities (see [Mat86, Theorems 21.2 iii), 21.3 and Exercise 18.2]). Furthermore, if $(X, D) \subset \mathcal{X}$ is an irreducible component with its conductor divisor, then (X, D) is a log manifold and $\omega_{\mathcal{X}}|_X \simeq \mathcal{O}_X(K_X + D)$ by the adjunction formula.

Definition 2.5 ([Fjn09])**.** Let $\mathcal{X}$ be an snc variety with the irreducible decomposition $\mathcal{X} = \bigcup_{1\leq i\leq m} X_i$. A *stratum* of $\mathcal{X}$ is an irreducible component of $\bigcap_{i\in I} X_i$ with the reduced scheme structure for a subset $I \subset \{1, \dots, m\}$. A *minimal stratum* of $\mathcal{X}$ is a stratum of $\mathcal{X}$ which is a minimal in the set of strata of $\mathcal{X}$ under the partial order of the inclusion.

Now, we consider the descent of invertible sheaves.

Proposition 2.6. *Let $\mathcal{X}$ be an n-dimensional snc variety with irreducible decomposition $\mathcal{X} = \bigcup_{i=1}^m X_i$, which has a unique minimal stratum. We also let $X_{ij} := X_i \cap X_j$ (scheme theoretic intersection) for any $1 \leq i < j \leq m$. Then we have an exact sequence*

$$0 \to \operatorname{Pic}(\mathcal{X}) \xrightarrow{\eta} \bigoplus_{i=1}^m \operatorname{Pic}(X_i) \xrightarrow{\mu} \bigoplus_{1\leq i<j\leq m} \operatorname{Pic}(X_{ij}),$$

where η is the restriction homomorphism and $\mu\big((\mathcal{H}_i)_i\big) := (\mathcal{H}_i|_{X_{ij}} \otimes \mathcal{H}_j^\vee|_{X_{ij}})_{i<j}$.

Proof. Let $\mathcal{X}_i := \bigcup_{j=1}^i X_j \subset \mathcal{X}$. Then both $\mathcal{X}_i$ and $\mathcal{X}_i \cap X_{i+1}$ are snc varieties and have a unique minimal stratum. Since units of structure sheaves form an exact sequence

$$1 \to \mathcal{O}^*_{\mathcal{X}_{i+1}} \to \mathcal{O}^*_{X_{i+1}} \times \mathcal{O}^*_{\mathcal{X}_i} \to \mathcal{O}^*_{\mathcal{X}_i\cap X_{i+1}} \to 1$$

of sheaves of Abelian groups, there is a long exact sequence

$$\Bbbk^* \times \Bbbk^* \xrightarrow{\upsilon} \Bbbk^* \to \operatorname{Pic}(\mathcal{X}_{i+1}) \xrightarrow{\lambda} \operatorname{Pic}(X_{i+1}) \oplus \operatorname{Pic}(\mathcal{X}_i) \to \operatorname{Pic}(\mathcal{X}_i \cap X_{i+1}).$$

The map λ above is injective since υ is surjective. In particular, η is injective. It is obvious that $\mu \circ \eta = 0$. Assume that $(\mathcal{H}_i)_i \in \bigoplus_{i=1}^m \operatorname{Pic}(X_i)$ satisfies $\mu\big((\mathcal{H}_i)_i\big) = 0$. We will show that there exist invertible sheaves $\mathcal{L}_i \in \operatorname{Pic}(\mathcal{X}_i)$ for all $1 \leq i \leq m$, which satisfy $\mathcal{L}_i|_{X_i} \simeq \mathcal{H}_i$ and $\mathcal{L}_i|_{\mathcal{X}_{i-1}} \simeq \mathcal{L}_{i-1}$ (if $i \geq 2$). If $i = 1$, then $\mathcal{L}_1$ must be (isomorphic to) $\mathcal{H}_1$. Assume that we have constructed $\mathcal{L}_1, \dots, \mathcal{L}_i$. Since

$$\operatorname{Pic}(\mathcal{X}_{i+1}) \to \operatorname{Pic}(X_{i+1}) \oplus \operatorname{Pic}(\mathcal{X}_i) \to \operatorname{Pic}(\mathcal{X}_i \cap X_{i+1})$$

is exact, it is enough to show $\mathcal{L}_i|_{\mathcal{X}_i\cap X_{i+1}} \simeq \mathcal{H}_{i+1}|_{\mathcal{X}_i\cap X_{i+1}}$ to construct $\mathcal{L}_{i+1}$. We know that the map $\kappa\colon \operatorname{Pic}(\mathcal{X}_i \cap X_{i+1}) \to \bigoplus_{j=1}^i \operatorname{Pic}(X_{j,i+1})$ is injective since $\mathcal{X}_i \cap X_{i+1}$ has a unique minimal stratum. Both $\mathcal{L}_i|_{\mathcal{X}_i\cap X_{i+1}}$ and $\mathcal{H}_{i+1}|_{\mathcal{X}_i\cap X_{i+1}}$ map κ to $(\mathcal{H}_j|_{X_{j,i+1}})_j$, thus we can construct $\mathcal{L}_{i+1}$. Hence we construct $\mathcal{L}_i \in \operatorname{Pic}(\mathcal{X}_i)$ for all $1 \leq i \leq m$ by induction. For any $1 \leq i \leq m$, $\mathcal{L}_m|_{X_i} \simeq \mathcal{H}_i$ holds by construction. Thus $\eta(\mathcal{L}_m) = (\mathcal{H}_i)_i$. $\square$

2.2. **Snc Fano varieties and log Fano manifolds.**

Definition 2.7. (1) A projective snc variety $\mathcal{X}$ is said to be an *snc Fano variety* if the dual of the dualizing sheaf $\omega_{\mathcal{X}}^{\vee}$ is ample. (2) A projective log manifold (X, D) is said to be a *log Fano manifold* if $-(K_X + D)$ is ample.

We define the *index* and *pseudoindex* of an snc Fano variety and also of a log Fano manifold; whose notion is essential in the paper.

Definition 2.8. Let $\mathcal{X}$ be an snc Fano variety. We define the *snc Fano index* $r(\mathcal{X})$ (resp. the *snc Fano pseudoindex* $\iota(\mathcal{X})$) of $\mathcal{X}$ as $\max\{r \in \mathbb{Z}_{>0} \mid \omega_{\mathcal{X}}^{\vee} \simeq \mathcal{L}^{\otimes r}$ for some $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})\}$ (resp. $\min\{\deg_C(\omega_{\mathcal{X}}^{\vee}|_C) \mid C \subset \mathcal{X}$ rational curve$\}$). For a log Fano manifold (X, D), the *log Fano index* $r(X, D)$ and the *log Fano pseudoindex* $\iota(X, D)$ are similarly defined by replacing $\omega_{\mathcal{X}}$ by $\mathcal{O}_X(K_X + D)$.

Remark 2.9. For an snc Fano variety $\mathcal{X}$, we have $r(\mathcal{X})|\iota(\mathcal{X})$. For a log Fano manifold (X, D), we have $r(X, D)|\iota(X, D)$.

Remark 2.10. Let $\mathcal{X}$ be an n-dimensional snc Fano variety and $(X, D) \subset \mathcal{X}$ be an irreducible component with its conductor. Then (X, D) is an n-dimensional log Fano manifold such that $r(\mathcal{X})|r(X, D)$ and $\iota(X, D) \geq \iota(\mathcal{X})$ by Remark 2.4.

2.3. **First properties of log Fano manifolds.**

Theorem 2.11 ([KM98, Theorem 3.35]). *Let (X, D) be a log Fano manifold. Then* $\operatorname{NE}(X)$ *is spanned by a finite number of extremal rays. Furthermore, for any extremal ray $R \subset \operatorname{NE}(X)$, the ray R is spanned by a class of rational curve C on X and there exists a contraction morphism* $\operatorname{cont}_R : X \to Y$ *associated to R. Moreover, if $\mathcal{L} \in \operatorname{Pic}(X)$ satisfies $(\mathcal{L} \cdot C) = 0$ then there exists $\mathcal{M} \in \operatorname{Pic}(Y)$ such that* $\operatorname{cont}_R^* \mathcal{M} \simeq \mathcal{L}$.

Lemma 2.12 ([Mae86, Corollary 2.2, Lemma 2.3]). *Let (X, D) be a log Fano manifold. Then* $\operatorname{Pic}(X)$ *is torsion free and is isomorphic to $H^2(X^{\mathrm{an}}; \mathbb{Z})$ if $\Bbbk = \mathbb{C}$.*

Proposition 2.13. *Let (X, D) be a log Fano manifold with $\rho(X) = 1$ and $D \neq 0$. Then X is a Fano manifold such that $r(X) > r(X, D)$ and $\iota(X) > \iota(X, D)$.*

Theorem 2.14 (cf. [Mae86, Lemma 2.4]). (1) *Let (X, D) be a log Fano manifold. Then D is a* (*connected*) *snc Fano variety such that $r(X, D)|r(D)$ and $\iota(D) \geq \iota(X, D)$ holds.*

(2) *Let $\mathcal{X}$ be an snc Fano variety. Then there is a unique minimal stratum of $\mathcal{X}$. In particular, any two irreducible components of $\mathcal{X}$ intersect each other.*

Proof. (1) We know that D is connected by [Mae86, Lemma 2.4 (a)]. In modern terms, the connectedness of D is obtained by Shokurov's connectedness theorem [K+92, Theorem 17.4]. The other assertions follow from adjunction.

(2) We can prove that by using the same idea in [Mae86, Lemma 2.4 (a′)]. We remark that this is directly shown by [Amb03, Theorem 6.6 (ii)] and [Fjn09, Theorem 3.47 (ii)]. $\square$

Now, we give the theorem that combines log Fano manifolds into an snc Fano variety. By Theorem 2.14 (2), for an snc Fano variety $\mathcal{X} = \bigcup_{i=1}^{m}(X_i, D_i)$, any two components X_i, X_j intersect each other. Thus there exist isomorphisms $\phi_{ij}\colon D_{ij} \to D_{ji}$ for all distinct i, j such that $\phi_{ji} = \phi_{ij}^{-1}$ and $\phi_{jk}|_{D_{ji}\cap D_{jk}} \circ \phi_{ij}|_{D_{ij}\cap D_{ik}} = \phi_{ik}|_{D_{ij}\cap D_{ik}}$ hold. Conversely, we have the following result. The proof follows from Kollár's gluing theory [Kol13, Theorem 23, §3] and Proposition 2.6, Lemma 2.12 and Theorem 2.14 (2).

Theorem 2.15. *Fix n, r, $m \in \mathbb{Z}_{>0}$. Let (X_i, D_i) be an n-dimensional log Fano manifold such that $r|r(X_i, D_i)$ for any $1 \leq i \leq m$. Assume that the irreducible decomposition is written as $D_i = \sum_{j\neq i, 1\leq j\leq m} D_{ij}$ for any $1 \leq i \leq m$, and there exist isomorphisms $\phi_{ij}\colon D_{ij} \to D_{ji}$ for all distinct $1 \leq i$, $j \leq m$, which satisfy*

(1) *$\phi_{ji} = \phi_{ij}^{-1}$ for distinct i, j,*

(2) *$\phi_{ij}(D_{ij} \cap D_{ik}) = D_{ji} \cap D_{jk}$ and $\phi_{jk} \circ \phi_{ij}|_{D_{ij}\cap D_{ik}} = \phi_{ik}|_{D_{ij}\cap D_{ik}}$ for distinct i, j, k.*

Then there exists an n-dimensional snc Fano variety $\mathcal{X}$ such that $r|r(\mathcal{X})$ whose irreducible decomposition can be written as $\mathcal{X} = \bigcup_{i=1}^{m}(X_i, D_i)$.

2.4. **Bundles and subbundles.** In this section, we recall some bundle structures. The following lemma is well-known.

Lemma 2.16. *Let X be an irreducible variety, $D \subset X$ be an effective Cartier divisor and c be a nonnegative integer. Let $\pi\colon X \to Y$ be a $\mathbb{P}^c$-bundle such that $\pi|_D\colon D \to Y$ is a $\mathbb{P}^{c-1}$-subbundle. That is, π is a proper and smooth morphism such that $\pi^{-1}(y) \simeq \mathbb{P}^c$ and $(\pi|_D)^{-1}(y)$ is isomorphic to a hyperplane section under this isomorphism for any closed point $y \in Y$. Then X is isomorphic to $\mathbb{P}_Y(\pi_*\mathcal{O}_X(D))$ over Y. Moreover, under the isomorphism, D is isomorphic to $\mathbb{P}_Y((\pi|_D)_*\mathcal{N}_{D/X})$ and the embedding is induced by the natural surjection $\pi_*\mathcal{O}_X(D) \to (\pi|_D)_*\mathcal{N}_{D/X}$, where $\mathcal{N}_{D/X}$ is the normal sheaf $\mathcal{O}_D(D)$. Furthermore, we have $D \in |\mathcal{O}_{\mathbb{P}}(1)|$ under the isomorphism, where $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological invertible sheaf on $\mathbb{P}_Y(\pi_*\mathcal{O}_X(D))$.*

Next, we consider $\mathcal{Q}^{c+1}$-bundles and $\mathcal{Q}^c$-subbundles.

Definition 2.17. Let $\pi\colon X \to Y$ be a morphism between irreducible varieties and c be a positive integer. We say that $\pi\colon X \to Y$ is a $\mathcal{Q}^{c+1}$-*bundle* if π is a proper, flat morphism and $\pi^{-1}(y)$ is (scheme theoretically) isomorphic to a hyperquadric in $\mathbb{P}^{c+2}$. For a $\mathcal{Q}^{c+1}$-bundle $\pi\colon X \to Y$ and an effective Cartier divisor D on X, we say that $\pi|_D\colon D \to Y$ is a $\mathcal{Q}^c$-*subbundle* of π if $(\pi|_D)^{-1}(y)$ is isomorphic to a hyperplane section under the isomorphism $\pi^{-1}(y) \simeq \mathcal{Q}^{c+1}$ for any closed point $y \in Y$. We note the morphisms π and $\pi|_D$ need not be smooth. (That is why we do not use the symbol $\mathbb{Q}$ but $\mathcal{Q}$.)

The following lemma is proved similarly to Lemma 2.16 and is well-known.

Lemma 2.18. *Let X be an irreducible variety, $D \subset X$ be an effective Cartier divisor, Y be a smooth variety and c be a positive integer. Suppose that $\pi\colon X \to Y$ is a $\mathcal{Q}^{c+1}$-bundle and $\pi|_D\colon D \to Y$ is a $\mathcal{Q}^c$-subbundle. Then we have:*

(i) *The natural sequence $0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_X(D) \to (\pi|_D)_*\mathcal{N}_{D/X} \to 0$ is exact.*

(ii) *$\pi_*\mathcal{O}_X(D)$ and $(\pi|_D)_*\mathcal{N}_{D/X}$ are locally free of rank $c+3$ and $c+2$, respectively. In particular, $P := \mathbb{P}_Y(\pi_*\mathcal{O}_X(D))$ is a $\mathbb{P}^{c+2}$-bundle over Y and $H := \mathbb{P}_Y((\pi|_D)_*\mathcal{N}_{D/X})$ is a $\mathbb{P}^{c+1}$-subbundle.*

(iii) *The natural homomorphism $\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective, and it induces a relative quadric embedding $X \hookrightarrow P$ over Y.*

(iv) *D is isomorphic to the complete intersection $X \cap H$ in P under these embeddings.*

Lemma 2.19. *Let X be an irreducible variety such that $h^1(X, \mathcal{O}_X) = 0$.*

(1) *Let $c \in \mathbb{Z}_{\geq 0}$ and $p_1\colon X \times \mathbb{P}^c \to X$, $p_2\colon X \times \mathbb{P}^c \to \mathbb{P}^c$ be the projections. Then $(p_1)_*(p_2^*\mathcal{O}_{\mathbb{P}^c}(1)) \simeq \mathcal{O}_X^{\oplus c+1}$.*

(2) *Let $c \geq 2$ and $p_1\colon X \times \mathbb{Q}^c \to X$, $p_2\colon X \times \mathbb{Q}^c \to \mathbb{Q}^c$ be the projections. Then $(p_1)_*(p_2^*\mathcal{O}_{\mathbb{Q}^c}(1)) \simeq \mathcal{O}_X^{\oplus c+2}$.*

Proof. We prove both assertions by induction on c.

(1) The case $c = 0$ is trivial. We assume that the assertion holds for the case: $c - 1$. There has the canonical exact sequence

$$0 \to \mathcal{O}_{X\times\mathbb{P}^c} \to p_2^*\mathcal{O}_{\mathbb{P}^c}(1) \to (p_2|_{X\times\mathbb{P}^{c-1}})^*\mathcal{O}_{\mathbb{P}^{c-1}}(1) \to 0.$$

After taking $(p_1)_*$, the sequence

$$0 \to \mathcal{O}_X \to (p_1)_*(p_2^*\mathcal{O}_{\mathbb{P}^c}(1)) \to (p_1|_{X\times\mathbb{P}^{c-1}})_*\big((p_2|_{X\times\mathbb{P}^{c-1}})^*\mathcal{O}_{\mathbb{P}^{c-1}}(1)\big) \to 0$$

is exact. We note that $(p_1|_{X\times\mathbb{P}^{c-1}})_*((p_2|_{X\times\mathbb{P}^{c-1}})^*\mathcal{O}_{\mathbb{P}^{c-1}}(1)) \simeq \mathcal{O}_X^{\oplus c}$ by the induction step. The sequence always splits since $h^1(X, \mathcal{O}_X) = 0$. Hence we have proved (1).

(2) The case $c = 2$ is a direct consequence of (1) since $\mathbb{Q}^2$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We assume that the assertion holds for the case: $c - 1$. There is the exact sequence

$$0 \to \mathcal{O}_{X\times\mathbb{Q}^c} \to p_2^*\mathcal{O}_{\mathbb{Q}^c}(1) \to (p_2|_{X\times\mathbb{Q}^{c-1}})^*\mathcal{O}_{\mathbb{Q}^{c-1}}(1) \to 0.$$

After taking $(p_1)_*$, we have the splitting exact sequence

$$0 \to \mathcal{O}_X \to (p_1)_*(p_2^*\mathcal{O}_{\mathbb{Q}^c}(1)) \to \mathcal{O}_X^{\oplus c+1} \to 0$$

by repeating the argument in the proof of (1). Hence we have proved (2). $\square$

2.5. **Facts on extremal contractions and its applications.** In this section, we describe the structure of the contraction morphism associated to a special ray.

We recall Wiśniewski's inequality, which plays an essential role in this section.

Theorem 2.20 ([Wiś91b]). *Let X be a smooth projective variety and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray with the associated contraction morphism $\pi\colon X \to Y$. Then $\dim \mathrm{Exc}(\pi) + \dim F \geq \dim X + l(R) - 1$ for any nontrivial fiber F of π.*

We give a criterion for a variety X to have $\rho(X) = 1$ using Theorem 2.20.

Lemma 2.21. *Let X be a smooth projective variety, $D \subset X$ be a prime divisor and $R \subset \overline{\mathrm{NE}}(X)$ be a K_X-negative extremal ray with associated contraction morphism $\pi\colon X \to Y$ such that $(D \cdot R) > 0$.*

(1) *If the restriction morphism $\pi|_D : D \to \pi(D)$ is not birational, then π is of fiber type, i.e., $\dim Y < \dim X$ holds.*

(2) *If $l(R) \geq 3$, then $\pi|_D : D \to Y$ is not a finite morphism. Furthermore, if $\rho(D) = 1$ holds in addition, then X is a Fano manifold with $\rho(X) = 1$.*

Proof. (1) If π is birational, then it is a divisorial contraction and the exceptional divisor is exactly D, since $\pi|_D : D \to \pi(D)$ is not birational. However, we get a contradiction since $(D \cdot R) > 0$. Hence π is of fiber type.

(2) Let us choose an arbitrary nontrivial fiber F of π. We have $D \cap F \neq \emptyset$ since $(D \cdot R) > 0$. Then $\dim(F \cap D) \geq \dim F - 1 \geq l(R) - 2 \geq 1$ by Theorem 2.20. Hence $F \cap D$ contains a curve. Now, we assume $\rho(D) = 1$. Then $\pi(D)$ must be a point since all curves in D are numerically proportional. Therefore π is of fiber type by (1) and Y must be a point since $(D \cdot R) > 0$. In particular, $\rho(X) = 1$. Thus X is a Fano manifold. □

We show that there exists a 'special' K_X-negative extremal ray for a log Fano manifold with nonzero boundary, which is essential for classifying some special log Fano manifolds.

Lemma 2.22. *Let (X, D) be a log Fano manifold with $r := r(X, D)$ and $\iota := \iota(X, D)$, L be a divisor on X such that $-(K_X + D) \sim rL$ holds, and assume that $D \neq 0$. Then there exists an extremal ray $R \subset \mathrm{NE}(X)$ such that $(D \cdot R) > 0$. Let R be an extremal ray satisfying $(D \cdot R) > 0$ and $\pi\colon X \to Y$ be the contraction morphism associated to R. Then R is always K_X-negative and $l(R) \geq \iota + 1$. Moreover, the restriction morphism $\pi|_{D_1}\colon D_1 \to \pi(D_1)$ to its image is an algebraic fiber space, that is, $(\pi|_{D_1})_*\mathcal{O}_{D_1} = \mathcal{O}_{\pi(D_1)}$, for any irreducible component $D_1 \subset D$. Furthermore, for a minimal rational curve $C \subset X$ of R, we have the following properties: (1) If $l(R) = \iota + 1$, then $(D \cdot C) = 1$. (2) If $l(R) = r + 2$ and $r \geq 2$, then $(L \cdot C) = 1$ and $(D \cdot C) = 2$.*

Proof. Such an extremal ray exists, since D is a nonzero effective divisor and $\mathrm{NE}(X)$ is spanned by a finite number of extremal rays. Let $R \subset \mathrm{NE}(X)$ be an extremal ray such that $(D\cdot R)>0$. Then R is K_X-negative since $(-K_X\cdot R)=(-(K_X+D)\cdot R)+(D\cdot R)>0$. To see that $\pi|_{D_1}\colon D_1\to\pi(D_1)$ is an algebraic fiber space, it is enough to show that the homomorphism $\pi_*\mathcal{O}_X\to(\pi|_{D_1})_*\mathcal{O}_{D_1}$ is surjective. We know that $R^1\pi_*\mathcal{O}_X(-D_1)=0$ by a vanishing theorem (see for example [Fjn09, Theorem 2.42]). Hence $\pi|_{D_1}\colon D_1\to\pi(D_1)$ is an algebraic fiber space. Let $C\subset X$ be a minimal rational curve of R. Then we have $l(R)=(-K_X\cdot C)=(-(K_X+D)\cdot C)+(D\cdot C)\geq\iota+1$. If $l(R)=\iota+1$, then the above inequality is in fact equality. Hence $(D\cdot C)=1$ holds. If $l(R)=r+2$ and $r\geq 2$, then $r+2=l(R)=r(L\cdot C)+(D\cdot C)\geq r+1$. Therefore $(L\cdot C)=1$ and $(D\cdot C)=2$ holds. □

Using Lemma 2.22, we show some delicate structure properties of certain log Fano manifolds.

Proposition 2.23. *Let (X,D) be a log Fano manifold with $r:=r(X,D)$, $\iota:=\iota(X,D)$ and $D\neq 0$. Pick an arbitrary extremal ray $R\subset\mathrm{NE}(X)$ such that $(D\cdot R)>0$ and let $\pi\colon X\to Y$ be the contraction morphism associated to R. Let F be an arbitrary nontrivial fiber of π. Then $\dim(D\cap F)\geq\iota-1$ holds. Furthermore, we have the following results.*

(i) *If $\dim(D\cap F)=\iota-1$ for any nontrivial fiber F, then $\pi\colon X\to Y$ is a $\mathbb{P}^\iota$-bundle and $\pi|_D\colon D\to Y$ is a $\mathbb{P}^{\iota-1}$-subbundle.*

(ii) *If $r\geq 2$ and there exists an irreducible component D_1 of D such that $\dim(D_1\cap F)=r$ for any F, then one of the following holds.*

(a) *Y is a smooth projective variety and π is the blowup along a smooth projective subvariety $W\subset Y$ of codimension $r+2$.*

(b) *Y is smooth, $\pi\colon X\to Y$ is a $\mathcal{Q}^{r+1}$-bundle and $\pi|_{D_1}\colon D_1\to Y$ is a $\mathcal{Q}^r$-subbundle.*

(c) *$\pi\colon X\to Y$ is a $\mathbb{P}^{r+1}$-bundle and $\pi|_{D_1}\colon D_1\to Y$ is a $\mathbb{P}^r$-subbundle.*

(d) *$\pi_*\mathcal{O}_X(L)$ is locally free of rank $r+2$, where L is any divisor on X such that $-(K_X+D)\sim rL$. Furthermore, $\pi\colon X\to Y$ is isomorphic to the projection $p\colon\mathbb{P}_Y(\pi_*L)\to Y$ and $(\pi|_{D_1})^{-1}(y)$ is a hyperquadric section under the isomorphism $\pi^{-1}(y)\simeq\mathbb{P}^{r+1}$ for any closed point $y\in Y$. Moreover, $\pi_*\mathcal{O}_X(L)\simeq(p|_{D_1})_*(\mathcal{O}_{\mathbb{P}}(1)|_{D_1})$ under the isomorphism.*

Proof. Let L be a divisor on X with $-(K_X+D)\sim rL$ and C be a minimal rational curve of R. We note that D and F intersect each other since $(D\cdot R)>0$. Hence

(1) $\dim(D\cap F)\geq\dim F-1\geq\dim X-\dim\mathrm{Exc}(\pi)+l(R)-2\geq l(R)-2\geq\iota-1\geq r-1$

by Theorem 2.20 and by Lemma 2.22.

First, we consider the case (i). Then $\dim\mathrm{Exc}(\pi)=\dim X$ and $l(R)=\iota+1$. Hence π is of fiber type, all fibers of π are of dimension ι, and the equalities $(D\cdot C)=1$

and $(-K_X \cdot C) = \iota + 1$ hold by Lemma 2.22. Therefore $\pi\colon X \to Y$ is a $\mathbb{P}^\iota$-bundle and $\pi|_D\colon D \to Y$ is a $\mathbb{P}^{\iota-1}$-subbundle by [Fjt87, Lemma 2.12].

Next, we consider the case (ii). We first show that $(D_1 \cdot R) > 0$. If not, any nontrivial fiber F is included in D_1 (in particular, π is of birational type). Then Theorem 2.20 and Lemma 2.22 show that $\dim \mathrm{Exc}(\pi) + r \geq \dim X + l(R) - 1 \geq \dim X + \iota$. Hence π is of fiber type. This leads to a contradiction. Consequently, we have $(D_1 \cdot R) > 0$.

We firstly assume that $\dim \mathrm{Exc}(\pi) < \dim X$. Then $\dim \mathrm{Exc}(\pi) = \dim X - 1$ and $l(R) = r + 1$ by substituting D_1 for (1). Hence π is a divisorial contraction such that $\dim F = r + 1$ for any F, and the equality $(D \cdot C) = 1$ holds by Lemma 2.22 (1). Thus, Y is a smooth projective variety and π is the blowup whose center $W \subset Y$ is a smooth projective subvariety of codimension $r + 2$ by [AW93, Theorem 4.1 (iii)]. Therefore the condition (iia) is satisfied in the case $\dim \mathrm{Exc}(\pi) < \dim X$.

We secondly assume that $\dim \mathrm{Exc}(\pi) = \dim X$, that is, π is of fiber type. We note that $l(R) = r + 1$ or $r + 2$ by (1).

Assume that π is of fiber type and $l(R) = r+1$. Then $\dim F = r+1$ for any fiber and the equalities $(D_1 \cdot C) = 1$ and $(-K_X \cdot C) = r+1$ hold by (1) and Lemma 2.22 (1). Thus $\pi_*\mathcal{O}_X(D_1)$ is locally free of rank $r+3$ and X is embedded over Y into $\mathbb{P}_Y(\pi_*\mathcal{O}_X(D_1))$ as a divisor of relative degree 2 by [ABW93, Theorem B]. Therefore the condition (iib) is satisfied in the case $\dim \mathrm{Exc}(\pi) = \dim X$ and $l(R) = r + 1$.

Assume that π is of fiber type and $l(R) = r + 2$. Then $(L \cdot C) = 1$ and either $(D_1 \cdot C) = 1$ or 2 holds by Lemma 2.22. Thus $\pi\colon X \to Y$ is isomorphic to the $\mathbb{P}^{r+1}$-bundle $\mathbb{P}_Y(\pi_*\mathcal{O}_X(L))$ by [Fjt87, Lemma 2.12]. If $(D_1 \cdot C) = 1$, then $\pi|_{D_1}\colon D_1 \to Y$ is a $\mathbb{P}^r$-subbundle. Therefore the condition (iic) satisfied in the case $\dim \mathrm{Exc}(\pi) = \dim X$, $l(R) = r + 2$ and $(D_1 \cdot C) = 1$.

Assume that π is of fiber type, $l(R) = r + 2$ and $(D_1 \cdot C) = 2$. Under the isomorphism $X \simeq \mathbb{P}_Y(\pi_*\mathcal{O}_X(L))$, we have a natural exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(1)(-D_1) \to \mathcal{O}_{\mathbb{P}}(1) \to \mathcal{O}_{\mathbb{P}}(1)|_{D_1} \to 0,$$

where $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological invertible sheaf on $\mathbb{P}_Y(\pi_*\mathcal{O}_X(L))$. After taking p_*, we also obtain an exact sequence

$$0 \to 0 \to \pi_*\mathcal{O}_X(L) \to (p|_{D_1})_*(\mathcal{O}_{\mathbb{P}}(1)|_{D_1}) \to 0$$

by cohomology and base change theorem, since $h^i(\mathbb{P}^{r+1}, \mathcal{O}(-1)) = 0$ $(i = 0, 1)$. Hence the condition (iid) holds in the case $\dim \mathrm{Exc}(\pi) = \dim X$, $l(R) = r + 2$ and $(D_1 \cdot C) = 2$. $\square$

2.6. **Properties on scrolls.** In Section 2.6, we consider special toric varieties, so-called rational scrolls. In this section, we fix notation.

Notation 2.24. Let s, t be positive integers and $a_0, \dots, a_t$ be integers with $0 = a_0 \le a_1 \le \cdots \le a_t$. Let $X := \mathbb{P}[\mathbb{P}^s; a_0, \dots, a_t]$, that is, $X = \mathbb{P}_{\mathbb{P}^s}(\mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_t))$. We also let $D_i := \mathbb{P}[\mathbb{P}^s; a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_t] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{P}^s; a_0, \dots, a_t]$, that is, the canonical embedding, for any $0 \le i \le t$. (See Notation and terminology in Section 1.)

The following lemma is well-known.

Lemma 2.25. *We have the following properties.*

(1) $\mathrm{Pic}(X) = \mathbb{Z}[\mathcal{O}(1; 0)] \oplus \mathbb{Z}[\mathcal{O}(0; 1)]$.

(2) $\mathcal{O}_X(-K_X) \simeq \mathcal{O}(s + 1 - \sum_{i=1}^t a_i;\ t + 1)$.

(3) $D_i \in |\mathcal{O}(-a_i; 1)|$ *for any* $0 \le i \le t$.

(4) $\deg_{C_f}(\mathcal{O}(u; v)|_{C_f}) = v$ *and* $\deg_{C_h}(\mathcal{O}(u; v)|_{C_h}) = u$*, where* C_f *is a line in a fiber of* $X \to \mathbb{P}^s$ *and* C_h *is a line in* $\mathbb{P}[\mathbb{P}^s; a_0] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{P}^s; a_0, \dots, a_t]$.

(5) $\mathrm{Nef}(X) = \mathbb{R}_{\ge 0}[\mathcal{O}(1; 0)] + \mathbb{R}_{\ge 0}[\mathcal{O}(0; 1)]$ *and* $\mathrm{Eff}(X) = \mathbb{R}_{\ge 0}[\mathcal{O}(1; 0)] + \mathbb{R}_{\ge 0}[\mathcal{O}(-a_t; 1)]$.

(6) *For a divisor* $D = \sum_{i=1}^t c_i D_i + dH$ *with* c_i, $d \in \mathbb{Z}$*, where* H *is the pullback of a hyperplane in* $\mathbb{P}^s$, $h^0(X, \mathcal{O}_X(D))$ *is equal to*

$$\#\Big\{(P_1, \dots, P_s, Q_1, \dots, Q_t) \in \mathbb{Z}^{\oplus s+t} \,\Big|\, Q_j \ge -c_j \ (1 \le j \le t),$$
$$\sum_{1 \le j \le t} Q_j \le 0,\ P_i \ge 0 \ (1 \le i \le s),\ \sum_{1 \le i \le s} P_i - \sum_{1 \le j \le t} a_j Q_j \le d\Big\}.$$

(7) *If there exists* $D \in |\mathcal{O}(k; 1)|$ *such that* $k < -a_{t-1}$*, then* $\mathrm{Supp}\, D \supset D_t$.

(8) *If a member* $D \in |\mathcal{O}(k; 2)|$ *is reduced, then* $k \ge -a_t - a_{t-1}$.

(9) *Assume that* $a_{t-2} < a_t$*. Then any effective and reduced divisor* D *on* X *with* $D \in |\mathcal{O}(-a_t - a_{t-1}; 2)|$ *decomposes into two irreducible components* D^t *and* D^{t-1} *such that* $D^t \sim D_t$ *and* $D^{t-1} \sim D_{t-1}$*: here* $D^t = D_t$ *if* $a_{t-1} < a_t$*. Furthermore, there exists* $\sigma \in \mathrm{Aut}(X/\mathbb{P}^s)$ *such that* $\sigma(D^t) = D_t$ *and* $\sigma(D^{t-1}) = D_{t-1}$ *holds.*

Corollary 2.26. *Let* D *be a member of* $D \in |\mathcal{O}(c; d)|$ *for some* $d > 0$*. Assume that* (X, D) *is a log Fano manifold. Set* $r := r(X, D)$ *and* $\iota := \iota(X, D)$.

(1) *If* $\iota \ge t$*, then* $d = 1$, $t = \iota$ *and* $s \ge \iota - 1$ *holds. Furthermore, if* $s = \iota - 1$*, then* $a_1 = \cdots = a_{\iota-1} = 0$ *and* $c = -a_\iota$.

(2) *If* $r \ge t$ *(hence* $\iota \ge t$ *holds),* $s = r$ *and* $r \ge 2$*, then we have* $r = \iota$ *and either* $(a_1, \dots, a_{r-2}, a_{r-1}, c) = (0, \dots, 0, 0, 1 - a_r)$ *or* $(0, \dots, 0, 1, -a_r)$ *holds.*

Proof. By Lemma 2.25 (2), $\mathcal{O}_X(-(K_X + D)) \simeq \mathcal{O}(s + 1 - \sum_{i=1}^t a_i - c;\ t + 1 - d)$. Hence $t + 1 - d \ge \iota$ by Lemma 2.25 (4). Thus $d = 1$ and $t = \iota$ if $\iota \ge t$ holds (resp. $d = 1$ and $t = \iota = r$ if $r \ge t$ holds). We also note that $s + 1 - \sum_{i=1}^\iota a_i - c$ is at least ι and is a positive multiple of r and $c \ge -a_\iota$. Hence $s \ge \iota - 1 + \sum_{i=1}^{\iota-1} a_i \ge \iota - 1$.

(1) If $s = \iota - 1$, then $\iota - \sum_{i=1}^\iota a_i \ge \iota + c \ge \iota - a_\iota$. Therefore $\sum_{i=1}^{\iota-1} a_i = 0$ and $c = -a_\iota$.

(2) If $s = r$ and $r \geq 2$, then $r + 1 - \sum_{i=1}^{r} a_i - c$ is divisible by r and $\sum_{i=1}^{r} a_i + c \geq \sum_{i=1}^{r-1} a_i \geq 0$. Hence $\sum_{i=1}^{r} a_i + c = 1$. Therefore either $(a_1, \dots, a_{r-2}, a_{r-1}, c) = (0, \dots, 0, 0, 1 - a_r)$ or $(0, \dots, 0, 1, -a_r)$ holds. $\square$

Corollary 2.27. *Let $r := t-1$ with $r \geq 2$ and $D \in |\mathcal{O}(c; d)|$ for some $d > 0$. Assume that (X, D) is a log Fano manifold with $r|r(X, D)$. Then $d = 2$ and $s \geq r - 1$. Furthermore, if $s = r - 1$, then $a_1 = \cdots = a_{r-1} = 0$ and $c = -a_r - a_{r+1}$.*

Proof. We repeat the argument for Corollary 2.26. By Lemma 2.25 (2), $\mathcal{O}_X(-(K_X + D)) \simeq \mathcal{O}(s+1-\sum_{i=1}^{r+1} a_i - c;\ \ r+2-d)$. Thus we have $d = 2$ since $r+2-d$ is a positive multiple of r and $r \geq 2$. We also know that $s \geq r-1+\sum_{i=1}^{r+1} a_i + c \geq r-1+\sum_{i=1}^{r-1} a_i \geq r-1$ by Lemma 2.25 (8). Furthermore, if $s = r-1$, then $r-\sum_{i=1}^{r+1} a_i \geq r+c \geq r-a_r-a_{r+1}$. $\square$

3. Examples

In this section, we give some examples of log Fano manifolds with large log Fano indices.

3.1. **Example of dimension $2\iota-1$ and log Fano (pseudo)index ι.** First, we consider the case (1) in Corollary 2.26, which is the important example of $(2\iota-1)$-dimensional log Fano manifolds with the log Fano (pseudo)index ι (See Theorem 4.3).

Example O. Let $\iota \geq 2$, $m \geq 0$, $X = \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}, m]$ and $D \in |\mathcal{O}(-m; 1)|$. We know that $\mathcal{O}(1; 1)$ is ample and $\mathcal{O}_X(-(K_X + D)) \simeq \mathcal{O}(1; 1)^{\otimes \iota}$. If $m > 0$, then D is unique and $D = \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}] \subset_{\text{can}} X = \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}, m]$ by Lemma 2.25 (7). If $m = 0$, then $X = \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota}$ and $D \in |\mathcal{O}_{\mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota}}(0, 1)|$. Hence any member $D \in |\mathcal{O}_{\mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota}}(0, 1)|$ is always an irreducible smooth divisor and we may assume that $D = \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}] \subset_{\text{can}} \mathbb{P}[\mathbb{P}^{\iota-1}; 0^{\iota}, m]$. (We note that $\dim |D| = \iota$ if $m = 0$.) Thus (X, D) is a $(2\iota-1)$-dimensional log Fano manifold with $r(X, D) = \iota(X, D) = \iota$ for any $D \in |\mathcal{O}(-m; 1)|$.

3.2. **Examples of dimension $2r$ and log Fano index r.** Next, we give examples of $2r$-dimensional log Fano manifolds with the log Fano indices r (See Theorem 4.5).

Example I. Let $X := \mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r} \xrightarrow{\mathrm{Bl}} \mathbb{P}^{2r}$, that is, the blowup of $\mathbb{P}^{2r}$ along an $(r-2)$-dimensional linear subspace $\mathbb{P}^{r-2}$, and $E \subset X$ be the exceptional divisor. Take any $D \in |\mathrm{Bl}^* \mathcal{O}_{\mathbb{P}^{2r}}(1) \otimes \mathcal{O}_X(-E)|$. We have $\dim |D| = r+1$ and any D is the strict transform of a hyperplane in $\mathbb{P}^{2r}$ containing the center of the blowup. The invertible sheaf $\mathcal{H} := \mathrm{Bl}^* \mathcal{O}_{\mathbb{P}^{2r}}(2) \otimes \mathcal{O}_X(-E)$ is ample. We know that $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$ for any $D \in |\mathrm{Bl}^* \mathcal{O}_{\mathbb{P}^{2r}}(1) \otimes \mathcal{O}_X(-E)|$.

Example II. Let $X := \mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$ and let D be an effective divisor on X such that $D \in |\mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r+1}}(0, 2)|$. Then $\dim |D| = (r+2)(r+3)/2 - 1$ and D is an snc divisor if and only if D is the pullback of a smooth or reducible hyperquadric in $\mathbb{P}^{r+1}$. In particular, a general element in the linear system is an snc divisor. Let $\mathcal{H} := \mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{P}^{r+1}}(1,1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X,D) is a $2r$-dimensional log Fano manifold with $r(X,D)=r$ for any snc $D \in |\mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{P}^{r+1}}(0,2)|$.

Example III. Let $X := \mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_1, m_2]$ with $0 \leq m_1 \leq m_2$ and $1 \leq m_2$, and $D \in |\mathcal{O}(-m_1-m_2; 2)|$. All reduced elements in $|D|$ can be seen in the sum of $\mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_1]$ and $\mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_2] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_1, m_2]$ by Lemma 2.25 (9). We note that

$$\dim|D| = \begin{cases} 2 & (m_1 = m_2) \\ \binom{m_2+r-1}{r-1} + r - 1 & (m_1 = 0) \\ \binom{m_2-m_1+r-1}{r-1} & (0 < m_1 < m_2) \end{cases}$$

by Lemma 2.25 (6). Let $\mathcal{H} := \mathcal{O}(1;1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X,D) is a $2r$-dimensional log Fano manifold with $r(X,D)=r$ for any reduced $D \in |\mathcal{O}(-m_1-m_2;2)|$.

Example IV (See also Remark 3.1)**.** Let $E := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+1}] \subset_{\mathrm{can}} X' := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+1}, m]$ with $m \geq 0$. We note that $E \simeq \mathbb{P}^{r-1}\times\mathbb{P}^r$. Consider a smooth divisor B in X' with $B \in |\mathcal{O}(0;2)|$ such that the intersection $B \cap E$ is also smooth. We note that $H^0(X', \mathcal{O}(0;2)) \to H^0(E, \mathcal{O}(0;2)|_E)$ is surjective since $H^1(X', \mathcal{O}(0;2)(-E)) = 0$ by Kodaira's vanishing theorem. Hence general $B \in |\mathcal{O}(0;2)|$ satisfies this property. We note that

$$\dim|\mathcal{O}(0;2)| = \binom{2m+r-1}{r-1} + (r+1)\binom{m+r-1}{r-1} + \frac{(r+1)(r+2)}{2} - 1$$

by Lemma 2.25 (6). Let $\tau\colon X \to X'$ be the double cover of X' with the branch divisor B, and D be the strict transform of E on X. Then X is smooth and $D \simeq \mathbb{P}^{r-1}\times\mathbb{Q}^r$ by construction. We know that $\mathcal{O}_X(-K_X) \simeq \tau^*(\mathcal{O}_{X'}(-K_{X'})\otimes\mathcal{O}(0;-1)) \simeq \tau^*\mathcal{O}(r-m; r+2)$ and $\mathcal{O}_X(D) \simeq \tau^*\mathcal{O}(-m;1)$. Let $\mathcal{H} := \tau^*\mathcal{O}(1;1)$, an ample invertible sheaf on X. Then $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. We note that $\mathcal{H}$ cannot be divisible anymore by Remark 3.1. Thus (X,D) is a $2r$-dimensional log Fano manifold with $r(X,D)=r$.

Example V (See also Remark 3.2)**.** In this example, we consider the case $r \geq 3$. Let $D := \mathbb{P}[\mathbb{Q}^r; 0^r] \subset_{\mathrm{can}} X := \mathbb{P}[\mathbb{Q}^r; 0^r, m]$ with $m \geq 0$. We note that $D \simeq \mathbb{P}^{r-1}\times\mathbb{Q}^r$. We also note that $\dim|D| = 0$ if $m > 0$. If $m = 0$ then $X = \mathbb{P}^r \times \mathbb{Q}^r$ and $D \in |\mathcal{O}_{\mathbb{P}^r\times\mathbb{Q}^r}(1,0)|$ hence $\dim|D| = r$ and D is smooth. Let $\mathcal{H} := \mathcal{O}(1;1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X,D) is a $2r$-dimensional log Fano manifold with $r(X,D)=r$.

Example VI. In this example, we only consider the case $r=2$. Let $D := \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}^{\oplus 2}) \subset_{\mathrm{can}} X := \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}^{\oplus 2}\oplus\mathcal{O}(m_1,m_2))$ with $0 \leq m_1 \leq m_2$. If $m_2 > 0$, then $\dim|D| = 0$. If $m_1 = m_2 = 0$ then $X = \mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2$ and $D \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^2}(0,0,1)|$, hence $\dim|D| = 2$; any element in $|D|$ defines a smooth divisor. Let $\mathcal{H} := p^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\otimes\mathcal{O}_{\mathbb{P}}(1)$, where

$p\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ is the projection and $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological invertible sheaf with respect to the projection p. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes 2}$. Thus (X, D) is a 4-dimensional log Fano manifold with $r(X, D) = 2$.

Example VII (See also Remark 3.3)**.** Let $D := \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}) \subset_{\text{can}} X := \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r} \oplus \mathcal{O}(m))$ with $m \geq 1$. If $m \geq 2$, then $\dim |D| = 0$. If $m = 1$, then $\dim |D| = r + 1$. This follows from the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{N}_{D/X} \to 0$$

and the fact $\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(1-m, 1)|_D$ under an embedding $D \subset \mathbb{P}^r \times \mathbb{P}^r$ of bidegree $(1, 1)$. We note that there exists an embedding $X \subset X_1 := \mathbb{P}[\mathbb{P}^r; 1^{r+1}, m]$ obtained by the surjection α in the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}(1)^{\oplus r+1} \xrightarrow{\alpha} T_{\mathbb{P}^r} \to 0.$$

Let $\mathcal{H} := \mathcal{O}(0; 1)$ on X_1. Then $\mathcal{H}$ is ample and $(\mathcal{H}|_X)^{\otimes r} \simeq \mathcal{O}_X(-(K_X+D))$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$.

Example VIII**.** Let $X := \mathbb{P}^r \times \mathbb{P}^r$ and $D \in |\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(1, 1)|$. Then $\dim |D| = r(r+2)$, any smooth D is isomorphic to $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$, and any non-smooth D is the union of the first and second pullbacks of hyperplanes. Let $\mathcal{H} := \mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(1, 1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$ for any $D \in |\mathcal{O}_{\mathbb{P}^r \times \mathbb{P}^r}(1, 1)|$.

Example IX**.** Let $X := \mathbb{P}[\mathbb{P}^r; 0^r, 1]$. We can view X as the blowup of $P := \mathbb{P}^{2r}$ along an $(r-1)$-dimensional linear subspace $H \subset P$. Let E be the exceptional divisor of the blowup. Then $E = \mathbb{P}[\mathbb{P}^r; 0^r] \subset_{\text{can}} X = \mathbb{P}[\mathbb{P}^r; 0^r, 1]$. Let $D \in |\mathcal{O}(0; 1)|$. Any smooth D is the strict transform of a hyperplane in P which does not contain H. Any non-smooth D can be written as $E + D_0$, where D_0 is the strict transform of a hyperplane in P which contains H. Note that $\dim |D| = 2r$. Let $\mathcal{H} := \mathcal{O}(1; 1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$ for any $D \in |\mathcal{O}(0; 1)|$.

Example X**.** Let $X := \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1, m]$ with $m \geq 1$ and $D \in |\mathcal{O}(-m; 1)|$. If $m \geq 2$, then D is unique in $|\mathcal{O}(-m; 1)|$ and $D = \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1] \subset_{\text{can}} X = \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1, m]$. If $m = 1$, then $\dim |D| = 1$ by Lemma 2.25 (6) and we may assume that $D = \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1] \subset_{\text{can}} X = \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1, m]$. In particular, any $D \in |\mathcal{O}(-m; 1)|$ is a smooth divisor. Let $\mathcal{H} := \mathcal{O}(1; 1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X+D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$ for any $D \in |\mathcal{O}(-m; 1)|$.

Example XI. Let $X := \mathbb{P}[\mathbb{P}^r; 0^r, m]$ with $m \geq 2$ and $D \in |\mathcal{O}(-m+1; 1)|$. We note that $\operatorname{Supp} D$ always contains $D_0 := \mathbb{P}[\mathbb{P}^r; 0^r] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{P}^r; 0^r, m]$ by Lemma 2.25 (7). Furthermore, $D - D_0$ is the pullback of a hyperplane in $\mathbb{P}^r$ under the projection $p\colon X \to \mathbb{P}^r$. We also note that $\dim |\mathcal{O}(-m+1; 1)| = r$ by Lemma 2.25 (6). Let $\mathcal{H} := \mathcal{O}(1; 1)$. Then $\mathcal{H}$ is ample and $\mathcal{O}_X(-(K_X + D)) \simeq \mathcal{H}^{\otimes r}$. Thus (X, D) is a $2r$-dimensional log Fano manifold with $r(X, D) = r$ for any $D \in |\mathcal{O}(-m+1; 1)|$.

Now, we state some remarks about these examples.

Remark 3.1. In Example IV, the homomorphism $\tau^*\colon \operatorname{Pic}(X') \to \operatorname{Pic}(X)$ is an isomorphism. In particular, $\rho(X) = 2$.

Proof. We can assume $\Bbbk = \mathbb{C}$. If $m = 0$, then $X' \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$ and $X \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^{r+1}$. Thus the homomorphism τ^* is an isomorphism. We consider the case $m > 0$. Let $R \subset X$ be the ramification divisor of τ. We know that the linear system $|\mathcal{O}(0; 1)|$ on X' gives a divisorial contraction morphism $f\colon X' \to Q$ contracting $E \simeq \mathbb{P}^{r-1} \times \mathbb{P}^r$ to $\mathbb{P}^r$. We note that $B \subset X'$ is the pullback of some ample divisor $A \subset Q$. Thus $H_i((X' \setminus B)^{\mathrm{an}}; \mathbb{Z}) = 0$ for all $i > 2r + r - 2$ by [GM88, p.25, (2.3) Theorem] for the proper morphism $f|_{X' \setminus B}\colon X' \setminus B \to Q \setminus A$ to an affine variety. Thus $H^i_c((X' \setminus B)^{\mathrm{an}}; \mathbb{Z}) = 0$ for all $i < r + 2$ by Poincaré's duality. We know that there exists an exact sequence

$$H^2_c((X' \setminus B)^{\mathrm{an}}; \mathbb{Z}) \to H^2((X')^{\mathrm{an}}; \mathbb{Z}) \xrightarrow{\alpha} H^2(B^{\mathrm{an}}; \mathbb{Z}) \to H^3_c((X' \setminus B)^{\mathrm{an}}; \mathbb{Z}).$$

Thus α is an isomorphism. Applying the same argument to the composition $f \circ \tau\colon X \to Q$, we obtain the isomorphism $H^2(X^{\mathrm{an}}; \mathbb{Z}) \xrightarrow{\sim} H^2(R^{\mathrm{an}}; \mathbb{Z}) \simeq H^2(B^{\mathrm{an}}; \mathbb{Z})$. Therefore $H^2((X')^{\mathrm{an}}; \mathbb{Z}) \simeq H^2(X^{\mathrm{an}}; \mathbb{Z})$. Therefore $\operatorname{Pic}(X') \simeq \operatorname{Pic}(X)$ by Lemma 2.12. $\square$

Remark 3.2. If $m < 0$, then (X, D) in Example V is never a log Fano manifold.

Proof. Let $S := \mathbb{P}[\mathbb{Q}^r; m] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{Q}^r; 0^r, m]$, the section of the projection $p\colon X \to \mathbb{Q}^r$. Then $\mathcal{O}_X(-(K_X + D))|_S \simeq \mathcal{O}_{\mathbb{Q}^r}(r(m+1))$. Therefore $-(K_X + D)$ is never ample. $\square$

Remark 3.3. If $m < 1$, then (X, D) in Example VII is never a log Fano manifold.

Proof. Let $S := \mathbb{P}_{\mathbb{P}^r}(\mathcal{O}(m)) \subset_{\mathrm{can}} X = \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r} \oplus \mathcal{O}(m))$. Then $\mathcal{O}_X(-(K_X + D))|_S \simeq \mathcal{O}_{\mathbb{P}^r}(mr)$. Therefore $-(K_X + D)$ is never ample. $\square$

Remark 3.4. In Examples I, II, III, IV, V, VI, VII, VIII, IX, X, XI, if (X_1, D_1) and (X_2, D_2) are from different Examples, or are from the same Example but their discrete parameters are not equal, then $X_1 \not\simeq X_2$. In particular distinct X's are non-isomorphic to each other except for those in Example IV.

4. Theorems

In this section, we state the main classification results.

4.1. **Log Fano manifolds of del Pezzo type.** We classify n-dimensional log Fano manifolds (X, D) with $r(X, D) \geq n-1$. The case $D = 0$ is the well-known case of del Pezzo manifolds (see for example [Fjt90, I §8]), hyperquadrics and projective spaces (see [KO73]). Hence we consider the case $D \neq 0$. We note that the case $(n, r(X, D)) = (2, 1)$ has been treated by Maeda [Mae86, §3]. We treat $\iota(X, D)$ instead of $r(X, D)$.

Proposition 4.1. *Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$. Then $\iota(X, D) \leq n$. If $\iota(X, D) = n$, then $X \simeq \mathbb{P}^n$ and $D \in |\mathcal{O}(1)|$ under this isomorphism.*

Proof. Assume that $\iota := \iota(X, D) \geq n$. Choose an extremal ray R such that $(D \cdot R) > 0$. Let $\pi \colon X \to Y$ be the contraction morphism associated to R. Then $\dim \mathrm{Exc}(\pi) + \dim F \geq n + (\iota + 1) - 1 \geq 2n$ for any nontrivial fiber F of π by Theorems 2.20 and 2.22. Hence $\rho(X) = 1$, $\iota = n$ and $-K_X \sim (n+1)D$. Then one can apply [KO73]. □

Proposition 4.2. *Let (X, D) be an n-dimensional log Fano manifold with $n \geq 3$, $D \neq 0$ and $\iota(X, D) = n-1$. Then X is isomorphic to $\mathbb{P}^n$ or $\mathbb{Q}^n$ unless $n = 3$ and (X, D) is isomorphic to the case $\iota = 2$ in Example* O *(cf. Section 3.1). Moreover:* (1) *If $X = \mathbb{P}^n$, then $D \in |\mathcal{O}(2)|$, i.e., D is a smooth or reducible hyperquadric.* (2) *If $X = \mathbb{Q}^n$, then $D \in |\mathcal{O}(1)|$, i.e., D is a smooth hyperplane section.*

Proof. Let R be an extremal ray with a minimal rational curve $[C] \in R$ such that $(D \cdot R) > 0$. Let $\pi \colon X \to Y$ be the contraction morphism associated to R. By Theorem 2.20, $\dim \mathrm{Exc}(\pi) + \dim F \geq n + l(R) - 1 \geq 2n - 1$ holds for any nontrivial fiber F of π. Hence π is of fiber type, that is, $X = \mathrm{Exc}(\pi)$ holds.

If $\rho(X) = 1$, then X itself is a Fano manifold and $\iota(X) > n-1$ by Proposition 2.13. If $\rho(X) = 1$ and $\iota(X) \geq n+1$, then $X \simeq \mathbb{P}^n$ by [CMSB02] and $D \in |\mathcal{O}(2)|$. If $\rho(X) = 1$ and $\iota(X) = n$, then $(D \cdot C) = 1$ by Lemma 2.22. Thus $X \simeq \mathbb{Q}^n$ and $D \in |\mathcal{O}(1)|$ by [KO73].

We consider the remaining case $\rho(X) \geq 2$. Then $\dim F = n-1$ for any fiber F of π, and $l(R) = n$. Hence $(D \cdot C) = 1$ by Lemma 2.22. Therefore π is a $\mathbb{P}^{n-1}$-bundle over a smooth projective curve Y by [Fjt87, Theorem 2]. Then $Y \simeq \mathbb{P}^1$ since any extremal ray of X is spanned by the class of a rational curve. We can assume $X = \mathbb{P}[\mathbb{P}^1; a_0, \ldots, a_{n-1}]$, where $0 = a_0 \leq a_1 \leq \cdots \leq a_{n-1}$. Thus $1 \geq n-2$ by Corollary 2.26. Since $n \geq 3$, we have $n = 3$, $a_1 = 0$ and $D \in |\mathcal{O}(-a_2; 1)|$ by Corollary 2.26 (1). That is exactly the case which we have considered in Example O for the case $\iota = 2$. □

4.2. **Log Fano manifolds related to the Mukai conjecture.** We consider the log version of Wiśniewski's resluts [Wiś90b, Wiś91a] related to the Mukai conjecture [Muk88, Conjecture 4]. These are the main results in this article.

Theorem 4.3. *Let (X, D) be an n-dimensional log Fano manifold with $\iota := \iota(X, D) > n/2$, $D \neq 0$ and $\rho(X) \geq 2$. Then $n = 2\iota - 1$, and (X, D) is in Example* O *in Section 3.*

Proof. We use induction on n. We may assume $n \geq 5$ by Proposition 4.2. Choose an extremal ray R with a minimal rational curve $[C] \in R$ as in Lemma 2.22 and let $\pi\colon X \to Y$ be the associated contraction. Then $l(R) \geq \iota + 1 \geq 4$. We also choose an irreducible component with the conductor divisor $(D_1, E_1) \subset D$ such that $(D_1 \cdot R) > 0$. We know that $\rho(D_1) \geq 2$ by Lemma 2.21 (2), and (D_1, E_1) is an $(n-1)$-dimensional log Fano manifold with $\iota(D_1, E_1) \geq \iota$, $\rho(D_1) \geq 2$ and $\iota > n/2 > (n-1)/2$. Hence $E_1 = 0$ (hence $D = D_1$) by induction step. Applying [NO10, Theorem 3] to D, we have $n - 1 = 2(\iota - 1)$ and $D \simeq \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota-1}$. We know by Lemma 2.21 that $\pi|_D$ contracts a curve. Since $D \simeq \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota-1}$, $\pi|_D\colon D \to \pi(D)$ is not birational. Thus $\pi\colon X \to Y$ is of fiber type by Lemma 2.21, and $\pi|_D\colon D \to Y$ is surjective since $(D \cdot R) > 0$. We know that $\pi|_D\colon D \to Y$ is an algebraic fiber space by Lemma 2.22. Hence $\pi|_D$ is isomorphic to the first projection $p_1\colon \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota-1} \to \mathbb{P}^{\iota-1}$. In particular, $\dim(\pi^{-1}(y) \cap D) = \iota - 1$ for any closed point $y \in Y \simeq \mathbb{P}^{\iota-1}$. Therefore, $\pi\colon X \to Y$ is a $\mathbb{P}^{\iota}$-bundle and $\pi|_D\colon D \to Y$ is a $\mathbb{P}^{\iota-1}$-subbundle by Proposition 2.23 (i). Since $D \simeq \mathbb{P}^{\iota-1} \times \mathbb{P}^{\iota-1}$, there exists an integer $m \in \mathbb{Z}$ such that $(\pi|_D)_*\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^{\iota-1}}(-m)^{\oplus \iota}$ by Lemma 2.16. We also know by Lemma 2.16 that $X \simeq \mathbb{P}_{\mathbb{P}^{\iota-1}}(\pi_*\mathcal{O}_X(D))$ and the embedding $D \subset X$ is induced by the surjection α in the natural exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{\iota-1}} \to \pi_*\mathcal{O}_X(D) \xrightarrow{\alpha} (\pi|_D)_*\mathcal{N}_{D/X} \to 0.$$

Since $\iota - 1 = (n+1)/2 - 1 \geq 2$, this sequence always splits. Hence $\pi_*\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^{\iota-1}} \oplus \mathcal{O}_{\mathbb{P}^{\iota-1}}(-m)^{\oplus \iota}$ and $D \subset X$ is the canonical embedding obtained by the projection $\mathcal{O}_{\mathbb{P}^{\iota-1}} \oplus \mathcal{O}_{\mathbb{P}^{\iota-1}}(-m)^{\oplus \iota} \to \mathcal{O}_{\mathbb{P}^{\iota-1}}(-m)^{\oplus \iota}$. This case has been already considered in Corollary 2.26 (1); $m \geq 0$ holds. This is exactly the case which we have considered in Example O. □

We recall Wiśniewski's classification result.

Theorem 4.4 ([Wiś91a]). *If X is an $(2r-1)$-dimensional Fano manifold with $r(X) = r$ and $\rho(X) \geq 2$, then $X \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r$, $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$ or $\mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1]$.*

Using Theorems 4.3, 4.4, we classify log Fano manifolds (X, D) with $r(X, D) = r \geq 2$, $\dim X = 2r$ and $D \neq 0$. Note the case $r = 1$ has been treated by Maeda [Mae86, §3].

Theorem 4.5 (Main Theorem)**.** *Let (X, D) be a $2r$-dimensional log Fano manifold with $r(X, D) = r \geq 2$, $D \neq 0$ and $\rho(X) \geq 2$. Then (X, D) is in exactly one of Examples* I, II, III, IV, V, VI, VII, VIII, IX, X, XI*.*

We prove Theorem 4.5 in Section 5.

4.3. **Classification of Mukai-type log Fano manifolds.** As an immediate consequence, we can classify n-dimensional log Fano manifolds (X, D) with $r(X, D) \geq n - 2$. Those with $D = 0$ are well-known, and called Mukai manifolds (see [Isk77, MM81, Muk89, Wiś90a, Wiś90b, Wiś91a]).

Corollary 4.6. *Let (X, D) be an n-dimensional log Fano manifold with $D \neq 0$ and $r := r(X, D) \geq n - 2$.*

(1) *If $n \leq 3$, then (X, D) belongs to one in the list of* [Mae86]*.*

(2) *If $n = 4$ and $\rho(X) \geq 2$, then $r = 2$ and (X, D) belongs to one in the list of Theorem 4.5.*

(3) *If $n \geq 5$ and $\rho(X) \geq 2$, then $n = 5$, $r = 2$ and (X, D) belongs to the case in Example* O*.*

(4) *If $n \geq 4$, $\rho(X) = 1$ and $r \geq n - 1$, then (X, D) belongs to one in the list of Propositions 4.1 and 4.2.*

(5) *If $n \geq 4$, $\rho(X) = 1$ and $r = n - 2$, then (X, D) is one of:* (i) *$X \simeq \mathbb{P}^n$ and $D \in |\mathcal{O}(3)|$.* (ii) *$X \simeq \mathbb{Q}^n$ and $D \in |\mathcal{O}(2)|$.* (iii) *$X \simeq V_d$ and $D \in |\mathcal{O}(1)|$ with $1 \leq d \leq 5$, where V_d is a del Pezzo manifold of degree d in the sense of Takao Fujita* [Fjt90, Theorem 8.11, 1)–5)]*, and $\mathcal{O}(1)$ is the ample generator of* $\mathrm{Pic}(V_d)$*.* (*Conversely, general $D \in |\mathcal{O}(1)|$ in* (5i)–(5iii) *are smooth. Hence the cases* (5i)–(5iii) *actually occur.*)

5. Proof of Main Theorem 4.5

Let L be an ample divisor on X such that $-(K_X + D) \sim rL$. Pick an extremal ray R with a minimal rational curve $[C] \in R$ such that $(D \cdot R) > 0$ and let $\pi\colon X \to Y$ be the associated contraction morphism. Let $(D_1, E_1) \subset D$ be an irreducible component of D with its conductor divisor such that $(D_1 \cdot R) > 0$. By the assumption $\rho(X) \geq 2$ and Lemma 2.21 (2), the morphism $\pi|_{D_1}\colon D_1 \to \pi(D_1)$ is not a finite morphism and $\rho(D_1) \geq 2$ holds. Since (D_1, E_1) is a $(2r-1)$-dimensional log Fano manifold with $r|r(D_1, E_1)$, the possibility of $\pi|_{D_1}\colon D_1 \to \pi(D_1)$ (which is an algebraic fiber space by Lemma 2.22) is isomorphic to exactly one of the following list by Theorems 4.3 and 4.4.

(1) $\mathbb{P}^{r-1} \times \mathbb{Q}^r \xrightarrow{p_1} \mathbb{P}^{r-1}$, where $E_1 = 0$.

(2) $\mathbb{P}[\mathbb{P}^{r-1}; 0^r, m] \xrightarrow{p} \mathbb{P}^{r-1}$, where $E_1 \in |\mathcal{O}(-m; 1)|$ with $m \geq 0$.

(3) $\mathbb{P}^{r-1} \times \mathbb{Q}^r \xrightarrow{p_2} \mathbb{Q}^r$, where $E_1 = 0$.

(4) $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}) \xrightarrow{p} \mathbb{P}^r$, where $E_1 = 0$.

(5) $\mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1] \xrightarrow{p} \mathbb{P}^r$, where $E_1 = 0$.

(6) $\mathbb{P}^r \times \mathbb{P}^{r-1} \xrightarrow{p_1} \mathbb{P}^r$, where $E_1 \in |\mathcal{O}_{\mathbb{P}^r\times\mathbb{P}^{r-1}}(1,0)|$ (Theorem 4.3 with $m = 0$).

(7) $\mathbb{P}[\mathbb{P}^{r-1}; 0^r, m] \xrightarrow{\phi} Z$, the divisorial contraction morphism contracting $E_1 \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$ to $\mathbb{P}^{r-1}$, where $m > 0$.

(8) $\mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1} \xrightarrow{\mathrm{Bl}} \mathbb{P}^{2r-1}$, the blowup of $\mathbb{P}^{2r-1}$ along a linear subspace $\mathbb{P}^{r-2}$, where $E_1 = 0$.

Remark 5.1. For the cases (1), (2) and (8), $\dim(F \cap D_1) = r$ for any nontrivial fiber F of π. For the cases (3)–(7), $\dim(F \cap D_1) = r - 1$ for any nontrivial fiber F of π.

5.1. **Fiber type case.** Here, we consider the case where π is of fiber type. Since $\dim F \geq l(R) - 1 \geq r \geq 2$ for any fiber F of π, we have $\dim D_1 > \dim Y$. Hence $\pi|_{D_1}$ is surjective and belongs to one of the cases (1)–(6) (we note $\pi|_{D_1}$ is an algebraic fiber space by Lemma 2.22).

The cases (1) and (2). Since $\dim(\pi|_{D_1})^{-1}(y) = r$ for any $y \in Y \simeq \mathbb{P}^{r-1}$, one of (iib), (iic) or (iid) in Proposition 2.23 holds.

The case (1) Since $\pi|_D$ is a $\mathcal{Q}^r$-bundle, only cases (iib) and (iid) can occur.

First, we consider the case (iib). Since $D \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r$, $\pi|_D$ is isomorphic to the first projection and we can write $\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{Q}^r}(-m, 1)$ for some integer $m \in \mathbb{Z}$. Then $(\pi|_D)_*\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-m)^{\oplus r+2}$ by Lemma 2.19, and the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{r-1}} \to \pi_*\mathcal{O}_X(D) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^{r-1}}(-m)^{\oplus r+2} \to 0$$

is exact. Furthermore, X is obtained as a smooth divisor belonging to $|p^*\mathcal{O}_{\mathbb{P}^{r-1}}(s) \otimes \mathcal{O}_{\mathbb{P}}(2)|$ in $\mathbb{P} := \mathbb{P}_{\mathbb{P}^{r-1}}(\pi_*\mathcal{O}_X(D))$ for some $s \in \mathbb{Z}$, where $p\colon \mathbb{P} \to \mathbb{P}^{r-1}$ is the projection, D is the complete intersection of X with $H := \mathbb{P}[\mathbb{P}^{r-1}; (-m)^{r+2}]$ in $\mathbb{P}$. Here $H \subset \mathbb{P}$ is the subbundle of p obtained by the surjection α in the above exact sequence, by Lemma 2.18. Under the isomorphism $H \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$, the divisor $D \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r$ belongs to $|\mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{P}^{r+1}}(s - 2m, 2)|$. Thus $s \geq 2m$ since $h^0(\mathbb{P}^{r-1} \times \mathbb{P}^{r+1}, \mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{P}^{r+1}}(t, 2)) = 0$ for any $t < 0$. If $s > 2m$, then the restriction homomorphism $\mathrm{Pic}(\mathbb{P}^{r-1} \times \mathbb{P}^{r+1}) \to \mathrm{Pic}(D)$ is an isomorphism by the Lefschetz theorem and $\mathcal{O}_D(-K_D) \simeq \mathcal{O}_{\mathbb{P}^{r-1}\times\mathbb{P}^{r+1}}(r - (s - 2m), r)|_D$ by adjunction, but $-K_D$ is divisible by r, which leads to a contradiction. Thus $s = 2m$.

Claim 5.2. *$m \geq 0$ holds.*

Proof. We first consider the case $r = 2$. Since $\rho(X) = 2$, we can write $\mathrm{NE}(X) = R + R'$ and let the contraction morphism associated to R' be $\pi'\colon X \to Y'$. We note that any nontrivial fiber F' of π' satisfies $\dim F' = 1$ since no curve in F' can be contracted by π. If $(D \cdot R') > 0$, then any nontrivial fiber F' of π' satisfies $\dim F' \geq 2$ by the same argument used in Proposition 2.23, which is a contradiction. If $(D \cdot R') < 0$, then $\mathrm{Exc}(\pi') \subset D$. Hence $m > 0$. If $(D \cdot R') = 0$, then R' is a K_X-negative extremal ray and $l(R') \geq 2$ by the same argument in Proposition 2.23. Hence π' is of fiber type by Theorem 2.20. Thus $\pi'|_D$ is not a finite morphism since $(D \cdot R') = 0$. Therefore $m \geq 0$.

Now we consider the case $r \geq 3$. The above exact sequence always splits, hence $H = \mathbb{P}[\mathbb{P}^{r-1}; (-m)^{r+2}] \subset_{\mathrm{can}} \mathbb{P} = \mathbb{P}[\mathbb{P}^{r-1}; (-m)^{r+2}, 0]$. Assume that $m < 0$ holds. The total coordinate ring of $\mathbb{P}$ is the $\mathbb{Z}^{\oplus 2}$-graded polynomial ring $\Bbbk[x_0, \dots, x_{r-1}, y_0, y_1, \dots, y_{r+2}]$ with the grading $\deg x_i = (1, 0)$ $(1 \leq i \leq r-1)$, $\deg y_0 = (0, 1)$, $\deg y_i = (m, 1)$ $(1 \leq i \leq r+2)$. X is obtained by a graded equation of bidegree $(2m, 1)$. Since $m < 0$, any bidegree $(2m, 1)$ polynomial is obtained by linear combinations of the elements in $\{y_i y_j\}_{1 \leq i \leq j \leq r+2}$. Then any divisor obtained by graded equations of bidegree $(2m, 1)$ have singular points along the points defined by the graded equations $y_1 = \cdots = y_{r+2} = 0$ by the Jacobian criterion. This is a contradiction since X must be a smooth divisor. Therefore $m \geq 0$. □

Hence the above exact sequence splits. We now normalize the bundle structures for simplicity. That is, we rewrite $H := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+2}] \subset_{\mathrm{can}} \mathbb{P} := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+2}, m]$ with $m \geq 0$, X is a smooth divisor on $\mathbb{P}$ with $X \in |\mathcal{O}(0; 2)|$ and $D = X \cap H$ and D is smooth. Since $H \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$ and $D \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r$, we can take the pullback of a point, $S(\simeq \mathbb{P}^{r-1}) \subset H \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r+1} \xrightarrow{p_2} \mathbb{P}^{r+1}$ in $\mathbb{P}^{r+1}$ such that $S \cap D = \emptyset$. We can assume that S is the section of $p \colon \mathbb{P} \to \mathbb{P}^{r-1}$ obtained by the canonical first projection, that is, $S = \mathbb{P}[\mathbb{P}^{r-1}; 0] \subset_{\mathrm{can}} \mathbb{P} = \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+2}, m]$. Then the relative linear projection from S over $\mathbb{P}^{r-1} \simeq Y$ yields a morphism $\sigma \colon \mathbb{P} \setminus S \to X' := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+1}, m]$ over $\mathbb{P}^{r-1} \simeq Y$. The restriction of σ to X gives a double cover $\tau \colon X \to X'$. The branch divisor $B \subset X'$ of τ is smooth with $B \in |\mathcal{O}(0; 2)|$. Since the strict transform of the divisor $D' := \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+1}] \subset_{\mathrm{can}} X' = \mathbb{P}[\mathbb{P}^{r-1}; 0^{r+1}, m]$ in X' is exactly D, the intersection $B \cap D'$ is also smooth. This is Example IV.

Now, we consider the case (iid). We write $\mathcal{E} := \pi_* \mathcal{O}_X(L)$, then $X \simeq \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{E}) \xrightarrow{p} \mathbb{P}^{r-1}$. We can write $\mathcal{O}_{\mathbb{P}}(1)|_D \simeq \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{Q}^r}(-m, 1)$ for some integer $m \in \mathbb{Z}$, where $\mathcal{O}_{\mathbb{P}}(1)$ is the tautological invertible sheaf on X with respect to the projection p. Hence $\mathcal{E} \simeq (p|_D)_* (\mathcal{O}_{\mathbb{P}}(1)|_D) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-m)^{\oplus r+2}$ by Lemma 2.19. Therefore $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$, which is Example II. Thus we have completed to distinguish the case (1).

The case (2) For convenience, let $m_1 := m$, where m is in (2). Then only the case (iic) can occur since $\pi|_{D_1}$ is a $\mathbb{P}^r$-bundle. We note that D has two irreducible components D_1 and D_2 since E_1 is irreducible. We note $(D_2 \cdot R) > 0$ since π is of fiber type and $\pi|_{E_1}$ is surjective. Hence $\rho(D_2) \geq 2$ by the previous argument. Therefore $\pi|_{D_2} \colon D_2 \to Y$ is isomorphic to $\mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_2] \xrightarrow{p} \mathbb{P}^{r-1}$ with $m_2 \geq 0$. That is, D_2 also satisfies the hypothesis of the case (2) by repeating the same argument. We can assume $0 \leq m_1 \leq m_2$.

Under the isomorphism $D_1 \simeq \mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_1]$, we can write $\mathcal{N}_{D_1/X} \simeq \mathcal{O}(u; 1)$ with $u \in \mathbb{Z}$. We have $u = -m_2$ since $\mathcal{N}_{D_1/X}|_{D_1 \cap D_2} \simeq \mathcal{N}_{D_1 \cap D_2/D_2}$ and $\mathcal{N}_{D_1 \cap D_2/D_2} \simeq \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{P}^{r-1}}(-m_2, 1)$. Hence

$$p_*\mathcal{N}_{D_1/X} \simeq p_*(p^*\mathcal{O}_{\mathbb{P}^{r-1}}(-m_2) \otimes \mathcal{O}_{\mathbb{P}}(1)) \simeq \mathcal{O}_{\mathbb{P}^{r-1}}(-m_2)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(m_1 - m_2).$$

Thus the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{r-1}} \to \pi_*\mathcal{O}_X(D_1) \to \mathcal{O}_{\mathbb{P}^{r-1}}(-m_2)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^{r-1}}(m_1 - m_2) \to 0$$

splits since $m_1 \leq m_2$. Therefore $X \simeq \mathbb{P}[\mathbb{P}^{r-1}; 0^r, m_1, m_2]$ with $0 \leq m_1 \leq m_2$. We note that $D \in |\mathcal{O}(-m_1 - m_2; 2)|$ by Corollary 2.27. This is Example III.

The cases (3)–(6) Next, we consider the cases (3)–(6). Then $\dim(\pi|_{D_1})^{-1}(y) = r - 1$ for any closed point $y \in Y$. Hence only the case (i) in Proposition 2.23 occurs.

The case (3) In this case, Y is isomorphic to $\mathbb{Q}^r$.

First, we consider the case $r = 2$. $\pi|_D$ is isomorphic to $p_{23} \colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ and we can write $\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}(1, -m_1, -m_2)$ with $m_1, m_2 \in \mathbb{Z}$.

Claim 5.3. m_1, $m_2 \geq 0$.

Proof. It is enough to show $m_1 \geq 0$. Let $f = \{t\} \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \simeq Y$ be a fiber of $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, where $t \in \mathbb{P}^1$. Let $X_f := \pi^{-1}(f)$ and $D_f := \pi^{-1}(f) \cap D$. Then $X_f \to f$ is a $\mathbb{P}^2$-bundle, D_f is a smooth divisor in X_f with $D_f \neq 0$ and $\mathcal{O}_{X_f}(-(K_{X_f} + D_f)) \simeq \mathcal{O}_X(-(K_X + D))|_{X_f} \simeq \mathcal{O}_X(2L)|_{X_f}$. Thus (X_f, D_f) is a log Fano manifold such that $r(X_f, D_f)$ is an even number and $\rho(X_f) = 2$. Hence $D_f = \mathbb{P}[\mathbb{P}^1; 0^2](\simeq \mathbb{P}^1 \times \mathbb{P}^1) \subset_{\mathrm{can}} X_f = \mathbb{P}[\mathbb{P}^1; 0^2, m]$ with $m \geq 0$ by Proposition 4.2. Thus $\mathcal{N}_{D_f/X_f} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -m)$. Since $\mathcal{N}_{D_f/X_f} \simeq \mathcal{N}_{D/X}|_{D_f} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -m_1)$, we have $m_1 = m \geq 0$. □

We know that $(p_{23})_*\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-m_1, -m_2)^{\oplus 2}$ by Lemma 2.19. Hence we can show that the exact sequence obtained by Lemma 2.16

$$0 \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \pi_*\mathcal{O}_X(D) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-m_1, -m_2)^{\oplus 2} \to 0$$

splits. Hence we we can show that $D = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus 2}) \subset_{\mathrm{can}} X = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m_1, m_2))$ with $0 \leq m_1 \leq m_2$ by Lemma 2.16. This is Example VI.

We now consider the remaining case $r \geq 3$. We can write the normal sheaf $\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{P}^{r-1} \times \mathbb{Q}^r}(1, -m)$ with $m \in \mathbb{Z}$. Then $(\pi|_D)_*\mathcal{N}_{D/X} \simeq \mathcal{O}_{\mathbb{Q}^r}(-m)^{\oplus r}$ by Lemma 2.19. Hence we can see that the exact sequence obtained by Lemma 2.16

$$0 \to \mathcal{O}_{\mathbb{Q}^r} \to \pi_*\mathcal{O}_X(D) \to \mathcal{O}_{\mathbb{Q}^r}(-m)^{\oplus r} \to 0$$

splits. Hence $D = \mathbb{P}[\mathbb{Q}^r; 0^r] \subset_{\mathrm{can}} X = \mathbb{P}[\mathbb{Q}^r; 0^r, m]$ by Lemma 2.16. This is Example V; the divisor $-(K_X + D)$ is ample if and only if $m \geq 0$ by Remark 3.2. Thus we have completed to distinguish the case (3).

The case (4) We can write $(\pi|_D)_*\mathcal{N}_{D/X} \simeq T_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(-m)$ with $m \in \mathbb{Z}$ by Lemma 2.16. Hence we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \pi_*\mathcal{O}_X(D) \to T_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(-m) \to 0$$

with surjectivity following from Lemma 2.16. It is well known that

$$\mathrm{Ext}^1_{\mathbb{P}^r}(T_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(-m), \mathcal{O}_{\mathbb{P}^r}) \simeq \begin{cases} 0 & (m \neq 0) \\ \Bbbk & (m = 0). \end{cases}$$

We also know that all nonsplit exact sequences for the case $m = 0$ are obtained by

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to T_{\mathbb{P}^r} \to 0.$$

If the exact sequence is not split, then $X \simeq \mathbb{P}^r \times \mathbb{P}^r$ by the above argument. This case has been considered in Example VIII. If the exact sequence splits, then we can show that $D = \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}) \subset_{\mathrm{can}} X = \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r} \oplus \mathcal{O}_{\mathbb{P}^r}(m))$. This case has been considered in Example VII; the divisor $-(K_X + D)$ is ample if and only if $m \geq 1$ by Remark 3.3.

The case (5) We can write $(\pi|_D)_*\mathcal{N}_{D/X} \simeq (\mathcal{O}_{\mathbb{P}^r}^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^r}(1)) \otimes \mathcal{O}_{\mathbb{P}^r}(-m)$ with $m \in \mathbb{Z}$ by Lemma 2.16. Since $r \geq 2$, the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \pi_*\mathcal{O}_X(D) \to (\mathcal{O}_{\mathbb{P}^r}^{\oplus r-1} \oplus \mathcal{O}_{\mathbb{P}^r}(1)) \otimes \mathcal{O}_{\mathbb{P}^r}(-m) \to 0$$

splits. Thus $X \simeq \mathbb{P}[\mathbb{P}^r; 0^{r-1}, 1, m]$ and $D \in |\mathcal{O}(-m; 1)|$. Since $\mathcal{O}_X(-K_X) \simeq \mathcal{O}(r - m; r + 1)$, we have $\mathcal{O}_X(L) \simeq \mathcal{O}(1; 1)$. We know in Corollary 2.26 (2) such that $m \geq 0$; this case has been considered in Examples IX and X.

The case (6) We can write $(\pi|_{D_1})_*\mathcal{N}_{D_1/X} \simeq \mathcal{O}_{\mathbb{P}^r}(-m)^{\oplus r}$ with $m \in \mathbb{Z}$ by Lemma 2.16. Since $r \geq 2$, the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \pi_*\mathcal{O}_X(D) \to \mathcal{O}_{\mathbb{P}^r}(-m)^{\oplus r} \to 0$$

splits. Thus $X \simeq \mathbb{P}[\mathbb{P}^r; 0^r, m]$. We know in Corollary 2.26 (2) such that $m \geq 0$ holds; this case has been considered in Examples VIII, IX and XI.

5.2. **Birational type case.** Here, we consider the case where π is birational. We know that $\pi|_{D_1} \colon D_1 \to \pi(D_1)$ is a birational morphism by Lemma 2.21 (1) and an algebraic fiber space by Lemma 2.22. Hence $\pi|_{D_1} \colon D_1 \to \pi(D_1)$ belongs to the case (7) or (8). However, we have $\dim(D_1 \cap F) = r - 1$ for any nontrivial fiber F of π for the case (7); this contradicts to Proposition 2.23 (i). For the case (8), we have $\dim(D \cap F) = r$ for any nontrivial fiber F of π. Thus only the case (iia) in Proposition 2.23 occurs. That is, Y is smooth and π is the blowup along a smooth projective subvariety $W \subset Y$ of dimension $r - 2$. Let $D_Y := \pi(D) \subset Y$. Then $D_Y \simeq \mathbb{P}^{2r-1}$, and $W \subset D_Y$ is a linear subspace of dimension $r - 2$ under the isomorphism $D_Y \simeq \mathbb{P}^{2r-1}$. Let $E \subset X$ be the exceptional divisor of π. Then $\pi^*D_Y = D + E$. We note that there exists a divisor L_Y on Y such that

$\pi^*\mathcal{O}_Y(L_Y) \simeq \mathcal{O}_X(L+E)$ by Theorem 2.11 since $(E \cdot C) = -1$ and $(L \cdot C) = 1$. Therefore $\mathcal{O}_Y(rL_Y) \simeq \mathcal{O}_Y(-(K_Y + D_Y))$ by Theorem 2.11 since $\pi^*\mathcal{O}_Y(rL_Y) \simeq \mathcal{O}_X(rL + rE) \simeq \mathcal{O}_X(-(K_X + D) + rE) \simeq \mathcal{O}_X(-\pi^* K_Y - D - E) \simeq \pi^*\mathcal{O}_Y(-(K_Y + D_Y))$.

Claim 5.4. *(Y, D_Y) is also a log Fano manifold with $r|r(Y, D_Y)$.*

Proof. It is enough to show that L_Y is an ample divisor on Y. We know that $\mathrm{NE}(Y)$ is a closed convex cone since $\mathrm{NE}(X)$ is. Hence it is enough to show that $(L_Y \cdot C_Y) > 0$ for any irreducible curve $C_Y \subset Y$. If $C_Y \not\subset W$, taking the strict transform $\widehat{C}_Y$ of C_Y in X, then $(L_Y \cdot C_Y) = (L \cdot \widehat{C}_Y) + (E \cdot \widehat{C}_Y) > 0$. Hence it is enough to treat the case $C_Y \subset W$. We note that $W \subset D_Y$ and all curves in D_Y are numerically proportional since $D_Y \simeq \mathbb{P}^{2r-1}$. Therefore we can reduce to the case $C_Y \not\subset W$. $\square$

Since $D_Y \simeq \mathbb{P}^{2r-1}$, we have $\rho(Y) = 1$ by Lemma 2.21 (2). Thus $Y \simeq \mathbb{P}^{2r}$ and $D_Y \in |\mathcal{O}(1)|$ by [Fjt90, Theorem 7.18]. This is Example I.

Therefore we have completed the proof of Theorem 4.5.

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