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Understanding Capacities on a Finite Lattice

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This article summarizes the results obtained by the author [4] who explored a combinatorial approach when capacities are defined over a finite lattice. Let L be a finite lattice with partial ordering $\leq$, and let $\hat{0}$ and $\hat{1}$ denote the minimum and the maximum element of L. A monotone function φ on L is called a *capacity* if $\varphi(\hat{0})=0$ and $\varphi(\hat{1})=1$. Let $\mathcal{L}$ denote the collection of nonempty dual order ideals in L, and let $\mathcal{X}$ be an $\mathcal{L}$-valued random variable on some probability space $(\Omega, \mathbb{P})$, distributed as $\mathbb{P}(\mathcal{X}=V)=f(V)$. If $\mathbb{P}(\hat{0}\in\mathcal{X})=0$ then

$$\varphi(x)=\mathbb{P}(x\in\mathcal{X}) \tag{1}$$

gives a capacity, which is viewed as a *marginal condition* for $\mathcal{X}$. From another viewpoint, the collection of capacities on L is a convex polytope, every element of which can be represented as the convex combination

$$\varphi(x)=\sum_{V\in\mathcal{L}} f(V)\chi_{V}(x), \quad x\in L, \tag{2}$$

where χ_{V} denotes an indicator function of V. It should be noted, however, that the choice of f is not necessarily unique. In the way of formulating (2), the weight $f(V)$ determines a *probability mass function* (pmf) for $\mathcal{X}$, in which (2) is deemed to be (1). This probabilistic interpretation of a capacity was first considered by Choquet [1] and independently by Murofushi and Sugeno [6].

For $a_{1}, a_{2}, \ldots\in L$, we define the *difference operator* $\nabla_{a_{1}}$ by

$$\nabla_{a_{1}}\varphi(x)=\varphi(x)-\varphi(x\wedge a_{1}), \quad x\in L, \tag{3}$$

and the *successive difference operator* $\nabla_{a_{1},\ldots,a_{n}}$ recursively by

$$\nabla_{a_{1},\ldots,a_{n}}\varphi=\nabla_{a_{n}}(\nabla_{a_{1},\ldots,a_{n-1}}\varphi), \quad n=2,3,\ldots. \tag{4}$$

The monotonicity of φ is characterized by $\nabla_{a}\varphi\geq 0$ for any $a\in L$; furthermore, φ is called *completely monotone* (or monotone of order ∞; see [1]) if $\nabla_{a_{1},\ldots,a_{n}}\varphi\geq 0$ for any $a_{1}, \ldots, a_{n}\in L$ and for any $n\geq 1$.

Let X be an L-valued random variable with pmf $f(x) = \mathbb{P}(X = x)$. If $f(\hat{0}) = 0$ then

$$\varphi(x) = \sum_{y \leq x} f(y), \quad x \in L, \tag{5}$$

gives a capacity, which is viewed as a *cumulative distribution function* (cdf), also known as a *belief function* in [2]. The existence of the cdf (5) for a capacity φ is necessary and sufficient for the completely monotonicity of φ. This crucial observation, known as Choquet's theorem, was made by Choquet [1] for the class of compact sets in a topological space, and it has been instrumental in the studies of random sets. See [5] for a comprehensive review on random sets on topological spaces. This result in case of lattices was due to Norberg [7] who studied measures on continuous posets.

The function f in (5) is called the *Möbius inverse* of φ, by which the successive difference operators are fully characterized as follows.

Theorem 1. *The Möbius inverse f of φ satisfies*

$$\nabla_{a_1,\ldots,a_n}\varphi(x) = \sum\{f(y) : y \leq x, y \not\leq a_i \text{ for all } i = 1,\ldots,n\}. \tag{6}$$

Particularly we can show the Choquet's theorem for a finite lattice via combinatorial techniques.

Corollary 2. *Assume $\varphi(\hat{0}) \geq 0$. Then the Möbius inverse f of φ is nonnegative if and only if φ is completely monotone.*

The collection $\mathcal{L}$ is itself a distributive lattice when it is equipped with the order relation $U \preceq V$ by $U \supseteq V$. The lattice L is embedded as the subposet $\mathcal{L}_0 := \{\langle a\rangle^* : a \in L\}$ of principal dual order ideals. Here we introduce a completely monotone capacity Φ on $\mathcal{L}$, and call it a *completely monotone extension* of φ if it satisfies the marginal condition

$$\varphi(x) = \Phi(\langle x\rangle^*), \quad x \in L. \tag{7}$$

The marginal condition (7) is equivalent to (2), in which the weight $f(V)$ determines the *Möbius inverse* of Φ. By the same token, (1) and (7) are the same when we express $\Phi(U) = \mathbb{P}(\mathcal{X} \preceq U)$ as a cdf for $\mathcal{L}$-valued random variable $\mathcal{X}$.

Kellerer [3] and Rüschendorf [8] investigated the optimal bounds analogous to the classical Fréchet bounds systematically for various marginal problems. Let $R(\mathcal{L})$ be the space of real-valued functions on $\mathcal{L}$. Given $\Phi \in M_\infty(\mathcal{L})$ we can formulate the nonnegative linear functional

$$\Phi(g) = \sum_{V \in \mathcal{L}} f(V) g(V), \quad g \in R(\mathcal{L}),$$

where f is the Möbius inverse of Φ. Assuming $\varphi \in M_1(L)$, we can define the Fréchet bound

$$B_\varphi(g) = \min\{\Phi(g) : \Pi(\Phi) = \varphi\} \tag{8}$$

for any $g \in R(\mathcal{L})$. Duality follows from the relationship between primal and dual problem of linear programming, but it is also viewed as a straightforward application of the Hahn-Banach theorem (cf. Kellerer [3]).

Theorem 3. *The dual problem*

$$S^\varphi(g) = \max\left\{\sum_{x\in L} r_x\varphi(x) : \sum_{x\in V} r_x \le g(V), V \in \mathcal{L}\right\}. \tag{9}$$

satisfies $B_\varphi(g) = S^\varphi(g)$ *for any* $g \in R(\mathcal{L})$.

In particular we formulate the optimal lower bound $\lambda(\varphi; a, b) = B_\varphi(\langle a, b\rangle^*)$ at the dual order ideal $\langle a, b\rangle^*$ generated by a pair $\{a, b\}$ of L. Then we apply the value $\lambda(\varphi; a, x)$ to replace $\varphi(a \wedge x)$ in (3)–(4), and propose the λ-difference operator Λ_a by

$$\Lambda_a\varphi(x) = \varphi(x) - \lambda(\varphi; a, x), \quad x \in L, \tag{10}$$

and the *successive* λ*-difference operator* recursively by

$$\Lambda_{a_1,\dots,a_n}\varphi = \Lambda_{a_n}(\Lambda_{a_1,\dots,a_{n-1}}\varphi), \quad n = 2, 3, \dots. \tag{11}$$

Then we consider a stochastic comparison between $\varphi(x) = \mathbb{P}(x \in \mathcal{X})$ and $\psi(y) = \mathbb{P}(Y \le y)$, and obtain a sufficient condition for $\mathbb{P}(Y \in \mathcal{X}) = 1$.

Theorem 4. *If*

$$\Lambda_{a_1,\dots,a_k}\varphi(\hat{1}) \le \nabla_{a_1,\dots,a_k}\psi(\hat{1}) \quad \text{for every monotone path } (a_1, \dots, a_k), \tag{12}$$

then there exists a joint cdf Γ *for* $(\mathcal{X}, Y)$ *satisfying* $\mathbb{P}(Y \in \mathcal{X}) = 1$ *given the marginal conditions.*

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